§3 The additivity Theorem

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Let $\mathcal{C}$ be an exact category.

Recall Waldhausen's S-dot construction which at $S\mathcal{C}[1]$ has objects exact sequences

$$E : A_{01} \longrightarrow A_{02} \longrightarrow A_{12}$$

and the face maps

$$d_i : S\mathcal{C}[2] \longrightarrow S\mathcal{C}[1] = \mathcal{C}$$

are given by

$$d_0(E) = A_{12}, \quad d_1(E) = A_{02}, \quad d_2(E) = A_{01}$$

Now $S\mathcal{C}[2]$ inherits an exact structure where

$$E' \longrightarrow E \longrightarrow E''$$

in $S\mathcal{C}[2]$ is exact if

$$d_i E' \longrightarrow d_i E \longrightarrow d_i E''$$

is exact in $\mathcal{C}$ for $i = 0, 1, 2$.

We have

$$\delta : S\mathcal{C}[2] \longrightarrow \mathcal{C} \times \mathcal{C}$$
$$E \longmapsto (d_2(E), d_0(E))$$

is exact.

<u>Additivity theorem</u>: δ induces a homotopy equivalence

$$K(S\mathcal{C}[2]) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathcal{C})$$

<u>Notation</u>: For $m, n \in \mathbb{N}$ we write

$$[m][n] := [m] \amalg [n] = [m+n+1]$$

with inclusions

$$i_1 : [m] \longrightarrow [m][n]$$
$$j \longmapsto j$$

$$i_2 : [n] \longrightarrow [m][n]$$
$$j \longmapsto m+1+j$$

and projections

$$p_1 : [m][n] \longrightarrow [m]$$
$$j \longmapsto \begin{cases} j & , j \in [m] \\ m \in [m], & j \in [n] \end{cases}$$

$$p_2 : [m][n] \longrightarrow [n]$$
$$j \longmapsto \begin{cases} j & , j \in [n] \\ 0 \in [n] & , j \in [m]. \end{cases}$$

<u>Note</u> $p_1 \circ i_1 \simeq id$, $p_2 \circ i_2 \simeq id$.

For $\theta : [n] \longrightarrow [1]$ $(0, \dots, 0, 1, \dots, 1)$ we define

$$\theta_L : [m][n] \longrightarrow [m][n]$$
$$j \longmapsto \begin{cases} m \in [m] & , j \in [n] \text{ and } \theta(j) = 0 \\ j & , \text{otw} \end{cases}$$

Now, for $X \in sSet$ we have an induced map

$$X[m][n] \times \Delta^1[n] \xrightarrow{h_L} X[m][n]$$ ← simplicial for fixed m

$$(y, \theta) \longmapsto \theta_L^*(y)$$

Then

$$h_L(y, 0) = (i_1 \circ p_1)^*(y)$$
$$h_L(y, 1) = y \quad \forall y$$

hence h_L is a homotopy from

$$X[m][-] \xrightarrow{i_1^*} X[m] \xrightarrow{p_1^*} X[m][-]$$

simplicial for m-fixed

bisimplicial

to the identity

Lem. 9.2 $i_1^*: X[m][-] \longrightarrow X[m]$ and $i_2^*: X[-][n] \longrightarrow X[n]$ induces a homotopy equivalences on geometric realization with homotopy inverses

$$|p_1^*| \quad \text{and} \quad |p_2^*|$$

respectively.

Now, let $P[m,n]$ be the bisimplicial set defined as the pullback

$$\begin{tikzcd}
P[m][n] \arrow[r, "\hat{\delta}"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & s\mathcal{C}[m][n] \times s\mathcal{C}[m] \arrow[d, "i_1^* \times id"] \\
s(s\mathcal{C}[-])[m] \arrow[r, "\delta"] & s\mathcal{C}[m] \times s\mathcal{C}[n]
\end{tikzcd}$$

Consider the following diagram

$$\begin{tikzcd}
(E,S) \arrow[d, mapsto] & P[m][n] \arrow[d, "\hat{\delta}"'] \arrow[dr, "\pi_2"] & (E,S) \in s(s\mathcal{C}[-])[m] \times s\mathcal{C}[m][n] \text{ s.t. } d_2(E) = i_1^*(S) \arrow[d, mapsto] \\
(S, d_0 E) & s\mathcal{C}[m][n] \times s\mathcal{C}[m] \arrow[r, "i_2^* \times id"'] & s\mathcal{C}[n] \times s\mathcal{C}[m] \quad (i_2^*(S), d_0(E))
\end{tikzcd}$$

Note: π_2 has a section

$$\sigma_2 : s\mathcal{C}[n] \times s\mathcal{C}[m] \longrightarrow P[m,n]$$

$$(A,B) \longmapsto (0 \to B \xrightarrow{id} B,\ p_2^*(A))$$

Indeed

$$\pi_2 \circ \sigma_2 = id$$

and

$$P[-,n] \xrightarrow{\pi_2} s\mathcal{C}[n] \times s\mathcal{C}[-] \xrightarrow{\sigma_2} P[-,n]$$

is homotopic to the identity (lemma 9.5)

$\rightsquigarrow$ $|\pi_2| : |P[-,n]| \longrightarrow s\mathcal{C}[n] \times |s\mathcal{C}[-]|$ homotopy eq.

By realization lemma:

$$|\pi_2| : |P[-,-]| \longrightarrow |s\mathcal{C}[-]| \times |s\mathcal{C}[-]|$$

is a homotopy equivalence

<u>Lem</u> | 9.3 & 9.4 |

$$|\pi_1| : |P[-,-]| \longrightarrow |s(S\mathcal{C}[2])[-]|$$

is a homotopy equivalence.

<u>Pf.</u> For each m we have a pullback

$$\begin{tikzcd}
s(S\mathcal{C}[2])[m] \arrow[dr, "\exists!\,\sigma_1"] \arrow[drr, bend left, "(p_1^* \times id)\circ\delta"] & & \\
& P[m,-] \arrow[r, "\delta"] \arrow[d, "\pi_1"'] & s\mathcal{C}[m][-] \times s\mathcal{C}[m] \arrow[d, "i_1^* \times id"] \\
& s(S\mathcal{C}[2])[m] \arrow[r] \arrow[uu, bend left, "="] & s\mathcal{C}[m] \times s\mathcal{C}[m] \arrow[u, bend right, "p_1^* \times id"']
\end{tikzcd}$$

Have section σ_1

$$\pi_1 \circ \sigma_1 = id$$

and

$$\hat{\delta} \circ \sigma_1 \circ \pi_1 = (p_1^* \times id) \circ \delta \circ \pi_1$$
$$= (p_1^* \times id) \circ (i_1^* \times id) \circ \hat{\delta}$$

$$|\hat{\delta}| \circ |\sigma_1| \circ |\pi_1| = (p_1^* \times id) \circ \delta \circ \pi_1$$
$$= |(p_1^* \times id)| \circ |(i_1^* \times id)| \circ |\hat{\delta}|$$
$$\simeq |\hat{\delta}|$$
$$\Rightarrow |\sigma_1| \circ |\pi_1| \simeq id$$

so

$$|\pi_1| : |P[m,-]| \to s(S\mathcal{C}[2])[m]$$

homotopy equivalence.

$$|\pi_1| : |P[-,-]| \xrightarrow{\sim} |s(S\mathcal{C}[2])[-]|$$

<u>Cor:</u> δ induces a homotopy equivalence

$$K(\mathcal{C}[2]) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathcal{C})$$
$$(A \to B \to C) \longmapsto (A, C)$$