§4 Additivity theorem consequences I

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1) Preliminaries

<u>Recall</u>: Given $\mathcal{C}$ exact $\rightsquigarrow$ $S\mathcal{C}[-] = \Delta([-],[-]) \longrightarrow \mathcal{C}$ s.t. $A_{ii}=0$ and $A_{ij} \to A_{in} \to A_{jn}$ exact for $i \leq j \leq n$

<u>Thm</u> | Additivity | The functor

$$S\mathcal{C}[2] \longrightarrow \mathcal{C} \times \mathcal{C}$$
$$E \longmapsto (d_2 E, d_0 E)$$

induces a homotopy equivalence

$$|s(S\mathcal{C}[2])[-]| \xrightarrow{|sS[-]|} |s\mathcal{C}[-]| \times |s\mathcal{C}[-]|$$

This is equivalent to the statement:

Let $F, F', F'' : \mathcal{A} \to \mathcal{B}$ be exact functors s.t.

$$F' \to F \to F''$$

is exact. Then

$$|sF[-]| \simeq |s(F' \oplus F'')| : |s\mathcal{A}[-]| \to |s\mathcal{B}[-]|.$$

<u>Thm</u>: $\Delta : S\mathcal{C}[n] \longrightarrow \underbrace{\mathcal{C} \times \dots \times \mathcal{C}}_{n}$

$$A \longmapsto (A_{01}, A_{12}, \dots, A_{n-1,n})$$

induces a htpy. eq.

$$|s(S\mathcal{C}[n])| \xrightarrow{\sim} |s\mathcal{C}[-]|^n.$$

<u>Def</u>: A commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d, "f"] \\ B \arrow[r] & D \end{tikzcd}$$

is said to be a homotopy fibre sequence if $\exists$

$$\begin{tikzcd} C \arrow[r] \arrow[rr, bend right, "0"'] & W \arrow[r] & D \end{tikzcd}$$

s.t.

$$A \longrightarrow B \times_D W$$

is a homotopy equivalence.

In the case

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ B \arrow[r] & 0 \end{tikzcd}$$

$A \to B \to 0$ is said to be a homotopy fibre sequence

<u>Thm:</u> 9.10 Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

of bisimplicial sets (sufficiently nice), then if

$$\begin{tikzcd} |A[m,-]| \arrow[r] \arrow[d] & |B[m,-]| \arrow[d] \\ |C[m,-]| \arrow[r] & |D[m,-]| \end{tikzcd}$$

is a HFS (+ some technical assumptions)

then

$$\begin{tikzcd} |A[-,-]| \arrow[r] \arrow[d] & |B[-,-]| \arrow[d] \\ |C[-,-]| \arrow[r] & |D[-,-]| \end{tikzcd}$$

is a HFS.

2) Lemmas

$F: \mathcal{A} \to \mathcal{B}$ exact, then we get an exact functor

$$SF[-]: SA[-] \longrightarrow SB[-]$$

<u>Def:</u> Given 2 functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{F'} \mathcal{C}'$ we define the pullback

$$\begin{tikzcd} (c \in \mathcal{C}, c' \in \mathcal{C}', \alpha_c : Fc \simeq Fc') \in \mathcal{C} \times_{\mathcal{D}} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

Given $F: \mathcal{A} \to \mathcal{B}$ we define

$$\begin{tikzcd} S(F: \mathcal{A} \to \mathcal{B})[-] \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & ? \arrow[d] \\ S\mathcal{A}[-] \arrow[r, "SF[-]"'] & S\mathcal{B}[-] \end{tikzcd}$$

Need to construct this

Here, given $K[-] \in sSet$, we write

$$PK[m] := K[0][m] = K[m+1]$$

with simplicial maps: For $\theta: [n] \to [m]$

$$PK[m] \xrightarrow{(\mathrm{id} \,\lrcorner\, \theta)^*} PK[n]$$

<u>Fact:</u> $i_1: [0] \to [0][m]$ induces a homotopy equivalence

$$|PK[-]| \to |K[0]|.$$

We further have maps

$$[m] \xrightarrow{i_2} [0][m] \xrightarrow{\theta_m} [0][0] = [1]$$

where

$$\theta_m \circ i_2 = \text{constant}$$

$$K[1] \longrightarrow PK[m] \longrightarrow K[$$

We now define:

$$\begin{tikzcd} S(F: \mathcal{A} \to \mathcal{B})[-] \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & PS\mathcal{B}[-] \arrow[d] \\ S\mathcal{A}[-] \arrow[r, "SF[-]"'] & S\mathcal{B}[-] \end{tikzcd}$$

<u>Thm:</u> $F: A \to B$ induces a homotopy fibre sequence

<u>9.15</u>

$$|sB[-]| \to |s(\underbrace{S(F: A \to B)[-]}_{SF[-]})[-]| \to |s(SA[-])[-]|.$$

<u>Pf:</u> By 9.10 it is sufficient to prove this levelwise:

$$|sB[-]| \to |s(SF[m])[-]| \to |s(SA[m])[-]|$$

Want to construct 3 endofunctors on $S(F: A \to B)[m]$.

Note that an object in $SF[m]$ has the form

$$(A \in SA[m],\ b \in PSB[-],\ \alpha : SF[m](A) \cong \pi(b))$$

So we define

$$g' : SF[m] \xrightarrow{ev_0} B \xrightarrow{\Theta_n^{\cdot}} SF[m]$$

$$b \longmapsto (0, \Theta_n^{\cdot}(b), id)$$

$$(A, b, \alpha) \longmapsto b_{01}$$

object in $PSB[m]$, so define comp. wise

$$\Theta_n^{\cdot}(b)_{ij} = \begin{cases} b & i=0, j \geq 1 \\ 0 & \text{otw} \end{cases}$$

$$g'' : SF[m] \xrightarrow{pr} SA[m] \longrightarrow SF[m]$$

$$A \longmapsto (A, B(A), \alpha_A)$$

$$B(A)_{ij} = \begin{cases} F(A_{ij}), & i \geq 1 \\ F(A_{ij}), & i=0 \end{cases}$$

<u>Claim:</u> These endofunctors g', g'', id fits into an exact square:

<u>Pf:</u>

$$(0, \Theta_n^{\cdot}(b_{01}), id) \to (A, b, \alpha) \xrightarrow{\tau_n} (A, B(A), id)$$

Check component-wise:

$$\Theta_n^{\cdot}(b_{01})_{ij} \longrightarrow b_{ij} \longrightarrow B(A)_{ij}$$

$i=0$: $b_{01} \longrightarrow b_{0j} \longrightarrow B(A)_{0j} = F(A_{0j}) = b_{0j}$. exact ✓

$i \geq 1$: $0 \longrightarrow b_{ij} \longrightarrow F(A_{ij}) \cong b_{ij}$ ✓

$0 \to A \to A$ ✓

So

$$g' \longrightarrow id \longrightarrow g'' \quad \text{is exact.}$$

We can now invoke the additivity theorem:

$$id \simeq |s(g'' \oplus g')| : |s(S(F: A \to B)[m])[-]|$$

$$\rightsquigarrow \quad s(S|F: \mathcal{A} \to \mathcal{B})[m][-] \underset{s(\theta_i \oplus \tau_n)}{\overset{(s\iota_{\mathcal{B}[-]}, s\,pr)}{\rightleftarrows}} s\mathcal{B}[-] \times s S\mathcal{A}[m][-]$$

induces a levelwise homotopy equivalence on geometric realization

$$\begin{tikzcd}
 & |s\mathcal{B}[-]| \arrow[d] & |s\mathcal{B}[-]| \arrow[d] \\
|s(S|F: \mathcal{A} \to \mathcal{B})[m][-]| \arrow[r, "\simeq"] \arrow[d] & |s\mathcal{B}[-]| \times |s S\mathcal{A}[m][-]| \\
|s(S\mathcal{A}[-])[-]| & |s(S\mathcal{A}[-])[-]|
\end{tikzcd}$$

$\Rightarrow$ homotopy fibre sequence $\square$

Consider the situation

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

w. F, G exact. Then

$$\begin{tikzcd}
\overset{A}{|s\mathcal{B}[-]|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \overset{B}{|sSF[-][-]|} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \overset{C}{|sS\mathcal{A}[-][-]|} \arrow[d, equal] \\
\underset{A'}{|s\mathcal{C}[-]|} \arrow[r] & \underset{B'}{|sSGF[-][-]|} \arrow[r] & \underset{C'}{|sS\mathcal{A}[-][-]|}
\end{tikzcd}$$

commutes, hence

<u>Cor.</u> Given $F: \mathcal{A} \to \mathcal{B}$, we have a homotopy fiber sequence:

$$|s\mathcal{A}[-]| \longrightarrow |s\mathcal{B}[-]| \longrightarrow |sSF[-][-]|$$

3) Cofinality

<u>Def.</u> $\mathcal{C} \subseteq \mathcal{D}$ exact subcategory is strictly cofinal if

i) $\mathcal{C}$ is closed under extension

ii) $\mathcal{C}$ is weakly cofinal: $\forall d \in \mathcal{D}\ \exists d' \in \mathcal{D}$ s.t. $d \oplus d' \in \mathcal{C}$

<u>Thm</u> |Cofinality| Given $\mathcal{C} \subseteq \mathcal{D}$ cofinal exact subcategory, we have a homotopy fibre sequence

$$K(\mathcal{C}) \longrightarrow K(\mathcal{D}) \longrightarrow G := K_0(\mathcal{D}) / K_0(\mathcal{C})$$

It is an exercise to show this is welldefined

<u>Pf:</u> We know

$$|S\mathcal{C}[-]| \longrightarrow |S\mathcal{D}[-]| \longrightarrow |S(\mathcal{C} \xrightarrow{i} \mathcal{D})[-])[-]|$$

is a hmpty fib. seq.

$\rightsquigarrow$ $|S\mathcal{C}[-]| \longrightarrow |S\mathcal{D}[-]| \longrightarrow BG$ htpy fib. seq.

Aim: show that

$$|S(\mathcal{C} \to \mathcal{D})[-])[-]| \simeq BG.$$

We have

$$NG[n] = G^n$$

$$BG := |NG[-]|.$$

We will need the following 2 facts:

i) $\pi_0 |S(\mathcal{C} \xrightarrow{i} \mathcal{D})[-]| \cong G$ and further, each path component of $|S(\mathcal{C} \to \mathcal{D})[-]|$ is contractible "$|G|$ many path compts"

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ii) $(S\mathcal{C}[m])[n] \cong (S\mathcal{C}[n])[m]$

$\Rightarrow$ $|S(S(\mathcal{C} \to \mathcal{D})[-])[-]| \simeq |[n] \mapsto |S(S(\mathcal{C} \to \mathcal{D})[m])[-]]|$

iii) $S\mathcal{C}[n] \xrightarrow{\Delta} \mathcal{C}^n$ induces

$A \longmapsto (A_{01}, \ldots, A_{n-1,n})$ $\qquad |SS\mathcal{C}[n][-]| \simeq |S\mathcal{C}[-]|^n$

$$\Rightarrow |[n] \mapsto |S(S(\mathcal{C} \to \mathcal{D})[n][-]|| \simeq |S(\mathcal{C} \to \mathcal{D})[-]|^n$$

$$\simeq (K_0(\mathcal{D}) / K(\mathcal{C}))^n$$

$$\simeq G^n$$

$$= NG[n]$$

One need to check this eq. is compatible w. structure maps

$\Rightarrow$ $|S(S(\mathcal{C} \to \mathcal{D})[-])[-]| \simeq |NG[-]| = BG.$

6.14

$\square$