§5 Consequences of the additivity theorem II

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<u>Quillen's theorem A:</u> Let $f: \mathcal{C} \to \mathcal{D}$ be a functor. If $B(Y \backslash f) \simeq *$ for all $Y \in \mathcal{D}$. Then $Bf: B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.

<u>Thm:</u> | Resolution theorem | Let $\mathcal{D}$ be an exact category with $\mathcal{C}$ a full exact subcategory, s.t. $\mathcal{C}$ is closed under exact sequences, extensions and cokernels. Assume that for any $D \in \mathcal{D}$ there exists an exact sequence

$$D \rightarrowtail C \twoheadrightarrow C'$$

called the resolution

with $C, C' \in \mathcal{C}$. Then

$$K(\mathcal{C}) \to K(\mathcal{D})$$

is a homotopy equivalence.

<u>Pf:</u> We will show that $|s\mathcal{C}[-]| \to |s\mathcal{D}[-]|$ is a homotopy equivalence. From last time we know that

$$|s\mathcal{C}[-]| \to |s\mathcal{D}[-]| \to |s(S(\mathcal{C} \hookrightarrow \mathcal{D})[-])[-]|$$

is a homotopy fiber sequence, hence sufficient to show that $|s(S(\mathcal{C} \hookrightarrow \mathcal{D})[-])[-]|$ is contractible.

$$|s(S(\mathcal{C} \hookrightarrow \mathcal{D})[-])[-]| \simeq \left([n] \mapsto |s(S\mathcal{C}[n] \to S\mathcal{D}[n])[-]|\right)$$

By 6.14 we know it suffices to show that for any $n \geq 0$ $|s(S\mathcal{C}[n] \to S\mathcal{D}[n])[-]|$ is contractible.

$$\begin{tikzcd}
D_0 \arrow[r, tail] \arrow[d, tail] & D_1 \arrow[r, two heads] \arrow[d, tail] & D_2 \arrow[d, tail] \\
C_0 \arrow[r, tail] \arrow[d, two heads] & C_1 \arrow[r, two heads] \arrow[d] & C_2 \arrow[d, two heads] \\
C_0' \arrow[r, tail] & C_1' \arrow[r, two heads] & C_2'
\end{tikzcd}$$

$$\begin{tikzcd} D_0 \arrow[r, tail] \arrow[d, tail] & D_1 \arrow[d, tail] \\ C_0 \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & P \end{tikzcd} \rightsquigarrow \text{exact sequence } P \rightarrowtail C_1 \twoheadrightarrow \bar{C}$$

Using that $\mathcal{C}$ is extension closed we get that $P \in \mathcal{C}$. Moreover, we have another pushout diagram

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & C_1 \arrow[d, tail] \\ C_0' \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & C_1/D_e \end{tikzcd}$$

$$\wr$$

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & C_1 \arrow[r, two heads] \arrow[d, tail] & \bar{C} \\ C_0' \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & C_1/D_e \arrow[r, two heads] & \bar{C} \end{tikzcd} \quad \text{SES}$$

$$\hat{\mathcal{C}}$$

$\wr$

$S\mathcal{C}[n]$, $SD[n]$ satisfies the same condition as $\mathcal{C}$.

<u>Claim:</u> $|s(\mathcal{C} \hookrightarrow D)[-]| \simeq *$

$$s(\mathcal{C} \to D)[n] = \{ (C, D, \alpha) \mid C \in S\mathcal{C}[n],\ D \in SD[n+1],\ \alpha : C \xrightarrow{\sim} \pi D \}$$

$(D_{0,1} \rightarrowtail D_{0,2} \rightarrowtail \dots D_{0,n+1}) \in N(D^{inj})[n]$.

s.t. $D_{0,j}/D_{0,i} \in \mathcal{C}$ $\forall\, 1 \le i \le j \le n+1$. By theorem A we then get

$$|s(\mathcal{C} \hookrightarrow D)| \xrightarrow{\simeq} B(D_{\mathcal{C}}^{(n)})$$

$$\exists : \mathcal{C}^{inj} \to D_{\mathcal{C}}^{inj}.$$

$\mathcal{C}^{inj}$ has an initial object $\rightsquigarrow B(\mathcal{C}^{(n)}) \simeq *$, so sufficient to prove that

$$B(D \setminus \exists) \simeq \quad \forall\, d \in D_{\mathcal{C}}^{(n)}.$$

$\mathcal{D} \rightarrowtail C_0 \twoheadrightarrow C_0'$ SES, and $(C_0, D \rightarrowtail C_0) \in D\backslash J$.

By taking iterated pushout, we get

$$\begin{tikzcd} D \arrow[r, tail] \arrow[d] & C_0 \arrow[r, two heads] \arrow[d, tail] & C_0' \\ C \arrow[r, tail] & P \arrow[r] & C_0'' \end{tikzcd} \qquad \begin{tikzcd} C_0 \arrow[r, two heads] \arrow[d, tail] & C_0 \arrow[d, tail] \\ D \arrow[r, two heads] & D/P \end{tikzcd}$$

s.t. $\quad C_0 \rightarrowtail P/D \twoheadrightarrow P/C_0 \quad$ SES

$$(C, D \rightarrowtail C) \rightarrow (P, D \rightarrow P) \leftarrow (0,$$

identity factor

$\rightsquigarrow \quad |B(D\backslash J)| \simeq *$. $\square$

<u>Thm:</u> |Devissage| Let $\mathcal{B}$ be an abelian category and assume $\mathcal{A}$ is a full subcategory closed in $\mathcal{B}$ under direct sum, subobjects and quotient objects. i.e. $\mathcal{A}$ abel cat subcat If every object $B \in \mathcal{B}$ has a finite filtration

$$0 = B_{-1} \rightarrowtail B_0 \rightarrowtail \cdots \rightarrowtail B_p = B$$

with $B_i / B_{i-1} \in \mathcal{A}$. Then

$$K(\mathcal{A}) \xrightarrow{\sim} K(\mathcal{B})$$

is a homotopy equivalence.

<u>Pf:</u> As before it is sufficient to prove that

$$|s(S\mathcal{A}[n] \rightarrow S\mathcal{B}[n])[-]| \simeq * \quad \forall n.$$

Delicate issue/difference from before, is that $S\mathcal{A}[n]$, $S\mathcal{B}[n]$ are no longer abelian, so have to analyze the properties of the functor $S\mathcal{A}[n] \rightarrow S\mathcal{B}[n]$. Write

$$F\mathcal{C}[n] := \{ \overset{\in \mathcal{C}}{C_1} \rightarrowtail \cdots \rightarrowtail \overset{\in \mathcal{C}}{C_n} \}.$$

This can be shown to be equivalent to $S\mathcal{C}[n]$. More precisely,

an object of $FC[n]$ is a string of monomorphisms

$$C_1 \rightarrowtail C_2 \rightarrowtail \cdots \rightarrowtail C_n$$

with morphisms given by usual commutative diagrams

↳ without further assumptions. The equivalence sends any $D \in SC[n]$ to the top row

$$D_{01} \rightarrowtail D_{02} \rightarrowtail \cdots \rightarrowtail D_{0n}.$$

Under this equivalence admissible monomorphisms in $SC[n]$ corresponds to commutative diagrams

$$\begin{tikzcd}
C_1 \arrow[r, tail] \arrow[d, tail] & C_2 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & C_n \arrow[d, tail] \\
D_1 \arrow[r, tail] & D_2 \arrow[r, tail] & \cdots \arrow[r, tail] & D_n
\end{tikzcd}$$

s.t.

$$D_i \oplus_{C_i} C_{i+1} \longrightarrow D_{i+1}$$

are admissible monomorphisms for any $1 \leq i \leq n-1$.

<u>Claim:</u> $FA[n]$ has all kernels and cokernels, and they are all admissable.

<u>Claim:</u> For any commutative diagram in $FB[n]$

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d] & B \arrow[r, "f"] \arrow[d, equal] & \operatorname{coker}(a) \arrow[d, "g"] \\
X' \arrow[r, "a'"'] & B \arrow[r, "f'"] & \operatorname{coker}(a')
\end{tikzcd}$$

The map $FA[n] \to FB[n]$ is so called regular

s.t. $X/X' \in FA[n]$, then $\ker(f')/\ker(f) \in FA[n]$.

Now, given $B_0 \rightarrowtail B_1 \in FB[2]$, filter B_1 as

$$0 = B_{-1,1} \rightarrowtail B_{0,1} \rightarrowtail B_{1,1} \rightarrowtail \cdots \rightarrowtail B_{p,1} = B_1$$

so, $B_{i,1}/B_{i-1,1} \in \mathcal{A}$ for all $0 \leq i \leq p$. By taking iterative pullback we get a diagram

$$\begin{tikzcd}
0 = B_{-1,0} \arrow[r, tail] \arrow[d, tail] & B_{1,0} \arrow[r, tail] \arrow[d, tail] & B_{2,0} \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_{p,0} = B_0 \arrow[d, tail] \\
0 = B_{-1,1} \arrow[r, tail] & B_{1,1} \arrow[r, tail] & B_{2,1} \arrow[r, tail] & \cdots \arrow[r, tail] & B_{p,1} = B_1
\end{tikzcd}$$

It can be shown that this is a filtration of $B_0 \rightarrowtail B_1$ by admissible monomorphisms with $B_{i,0}/B_{i-1,0} \rightarrowtail B_{i,1}/B_{i-1,1}$ is a monomorphism for $0 \leq i \leq p$. Since $\mathcal{A}$ is closed under subobjects and $B_{i,1}/B_{i-1,1} \in \mathcal{A}$ we see that $B_{i,0}/B_{i-1,0} \in \mathcal{A}$. The result now follows from 9.22. $\square$

Lem. 9.22 Assume $\mathcal{B}$ is an exact category with all admissible kernels and cokernels. Let $\mathcal{A}$ be a full exact subcategory of $\mathcal{B}$ which is closed under direct sums, subobjects and quotient objects. Suppose that the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ is regular. If every object $B \in \mathcal{B}$ has a finite filtration

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n = B$$

w. $C_i / C_{i-1} \in \mathcal{A}$, then

$$|s(\mathcal{A} \to \mathcal{B})[-1]| \simeq *.$$