# §7: Monoidal categories & localisation

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Outline:

1) Monoidal categories
2) Cofibered functors
3) Quillen's theorem B for cofibered functors

## §1 Monoidal categories

<u>Def.</u> A monoidal category is a category $\mathcal{S}$ together with

$$0 \in \mathcal{S}$$
$$+ : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$$
$$\alpha_{A,B,C} : A + (B + C) \longrightarrow (A + B) + C \quad + \text{ commutative diagrams}$$
$$0 + A \cong A \cong A + 0$$

It is further called symmetric if

$$\gamma_{A,B} : A + B \longrightarrow B + A$$

We say $\mathcal{S}$ acts on some category $\mathcal{X}$ if

$$+ : \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$$

s.t. $(A + B) + C \cong A + (B + C)$, $0 + C \cong C$

<u>Def.</u> We define the orbit category $\langle \mathcal{S}, \mathcal{X} \rangle$:

$$\mathrm{ob}\langle \mathcal{S}, \mathcal{X} \rangle = \mathrm{ob}\, \mathcal{X}$$

Morphism $A \longrightarrow B$ = equivalence $[A, \varphi]$ w.

$A \in \mathrm{ob}\, \mathcal{S}$, $\varphi : A + F \longrightarrow G$ in $\mathcal{X}$

and $(A, \varphi) \sim (A', \varphi')$ if $\exists\, a : A \xrightarrow{\sim} A'$ in $\mathcal{S}$

s.t.

$$\begin{tikzcd} A + F \arrow[rr, "a + \mathrm{id}_F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\circlearrowleft"] & A' + F \arrow[dl, "\varphi'"] \\ & G & \end{tikzcd}$$

**Def.** $\mathcal{S}^{-1}\mathcal{X} = \langle \mathcal{S}, \mathcal{S}\times\mathcal{X} \rangle$ localization

$$(F,c) \xrightarrow[(A,\varphi_1,\varphi_2)]{} (F',c'), \quad \varphi_1 : A+F \to F', \quad \varphi_2 : A+c \to c'.$$

## §2: Cofibered functors

**Def.** A functor $f : \mathcal{C} \to \mathcal{D}$ is called cofibered if for all $\varphi : f(x) \to Y$ there exists a universal lift, i.e. there exists

cofiber lift

$$\hat{\varphi} : x \to \hat{x} \quad \text{s.t. } f\hat{\varphi} = \varphi,$$

and s.t.

$$\begin{tikzcd} x \arrow[r, "\hat{\varphi}"] \arrow[d, "\hat{\varphi}'"'] & \hat{x} \\ \hat{x}' \arrow[ur, "\exists!\,\gamma"'] & \end{tikzcd} \qquad f(\gamma) = \mathrm{id}.$$

**Def.** $f : \mathcal{C} \to \mathcal{D}$ we define $f^{-1}(Y)$ to be the subcategory of $\mathcal{C}$ consisting of those $X \in \mathcal{C}$ s.t. $f(X) = Y$, and those $a : X \to X'$ satisfying $fa = \mathrm{id}_Y$.

Functoriality of cofiber lift: Given $f : \mathcal{C} \to \mathcal{D}$ cofibered, $\varphi : Y \to Y'$ in $\mathcal{D}$ $\rightsquigarrow$ coarse change functor

$$\varphi_* : f^{-1}(Y) \to f^{-1}(Y').$$

**Def.** $f : \mathcal{C} \to \mathcal{D}$, $\varphi : Y \to Y'$ in $\mathcal{D}$. Define category $f/Y$ s.t.

$$\mathrm{ob}(f/Y) = \{(x,y) \mid x \in \mathcal{C},\ y : fx \to Y\}$$

and we get

$$\varphi_\# : f/Y \to f/Y'.$$

Assume:

Let $\mathcal{S}$ be a monoidal category acting on $\mathcal{X}$ and assume

1) $\mathcal{S} \simeq {}_{\mathcal{S}}\mathcal{S}$

2) $\mathcal{S}(A, A') \hookrightarrow \mathcal{S}(A+B, A'+B)$
$\quad a \longmapsto a + id_B$

3) all morphisms in $\mathcal{X}$ are monomorphisms

4) $\mathcal{S}(A, A') \hookrightarrow \mathcal{X}(A+c, A'+c)$,

Lem: $\langle \mathcal{S}, \mathcal{S} \rangle \xleftarrow{p} \langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle \xrightarrow{q} \langle \mathcal{S}, \mathcal{X} \rangle$ are both cofibered.

$\langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle = \mathcal{S}^{-1}\mathcal{X}$

Pf: $q : \mathcal{S}^{-1}\mathcal{X} \to \langle \mathcal{S}, \mathcal{X} \rangle$ cofibered: First we note that

$$q(F, c) = (c \xrightarrow{(A,\varphi)} d) \quad , \quad \varphi : A + c \to d$$

Need lift:

$$\begin{tikzcd}
(F, c) \arrow[r, "{(A, id, \varphi)}"] \arrow[d, "{(A', \varphi_1, \varphi_2)}"'] & (A+F, d) \\
(F', d) \arrow[ur, "\text{Need to construct a unique map}"'] &
\end{tikzcd}$$

$(A', \varphi_2) \sim (A, \varphi) \rightsquigarrow \exists\ a : A \xrightarrow{\sim} A'$ s.t.

$$\begin{tikzcd}
A + c \arrow[rr, "a + id_c"] \arrow[dr, "\varphi"'] & & A' + c \arrow[dl, "\varphi_2"] \\
& d &
\end{tikzcd}$$

$$\rightsquigarrow \quad A + F \xrightarrow{a + id_F} A' + F \xrightarrow{\varphi_1} F'$$

So we choose the map to be represented by

$$(0, \varphi_1 (a + id_F), id)$$

which can be shown to be unique.

The case of $p$ is similar. $\square$

## §3) Quillen's theorem B for cofibered functors.

<u>Thm B – classical version</u>: Assume $f: \mathcal{C} \to \mathcal{D}$ s.t. $\forall\, \varphi: Y \to Y'$ the induced map

$$\varphi_{\#}: f/Y \to f/Y'$$

gives hmpty eq.

$$B(f/Y) \xrightarrow{\sim} B(f/Y'),$$

then

$$Bf: B\mathcal{C} \xrightarrow{\sim} B\mathcal{D}.$$

<u>Lem:</u> $f^{-1}(Y) \xrightarrow{\iota_Y} f/Y$, $X \longmapsto (X, id)$ admits a left adjoint $r_Y$.

<u>Pf.</u> $f/Y \to f^{-1}(Y)$, $(X, \varphi) \longmapsto \varphi_! X$

<u>Thm B – Cofibered functors</u>: Let $f: \mathcal{C} \to \mathcal{D}$ be cofibered s.t. for all $\varphi: Y \to Y'$ the induced map $f^{-1}(Y) \to f^{-1}(Y')$ induces a homotopy equivalence of classifying spaces, then

$$\underbrace{B(f^{-1}(Y))}_{\text{fiber}} \to B\mathcal{C} \xrightarrow{Bf} B\mathcal{D}$$

is a quasifibration.