§9: K-theory through localisation

Goal: For $\mathcal{E}$ split exact category: $K(\mathcal{E}) \simeq B((i\mathcal{E})^{-1}(s\mathcal{E}))$.

Throughout let $\mathcal{E}$ be a split exact category.

__Def:__ $\mathcal{E}(\mathcal{E})$ is the category with objects exact sequences

$$A \rightarrowtail^{i} B \xrightarrow{p} C$$

and morphisms:

$$(i,p) \longrightarrow (i',p')$$

are equivalence classes of diagrams

$$
\begin{array}{ccccc}
A' & \leftarrowtail & A & = & A \\
\downarrow{i'} & & \downarrow & & \downarrow{i} \\
B' & = & B' & \rightarrowtail & B \\
\downarrow{p'} & & \downarrow & & \downarrow{p} \\
C' & \leftarrow & D & \rightarrowtail & C
\end{array}
$$

pullback

We have a map

$$f: \mathcal{E}(\mathcal{E}) \longrightarrow Q(\mathcal{E})$$

sending the above diagram to the bottom line. For $c \in ob(Q(\mathcal{E})^{op})$ we write

$$E_c = f^{-1}(c).$$

So a map in E_c is a diagram

$$
\begin{array}{ccc}
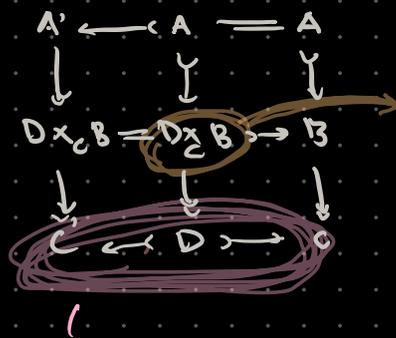A' & \leftarrowtail^{\sim} & A \\
\downarrow{i'} & & \downarrow{i} \\
B' & \rightarrowtail^{\sim} & B \\
\downarrow{p'} & & \downarrow{p} \\
C & = & C
\end{array}
$$

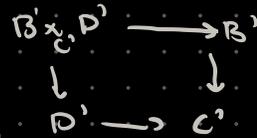In particular, $E_0 = \mathrm{sd}(i\mathcal{E})$.

__Lem:__ f is cofibered

__Pf:__ Take $A \rightarrowtail B \xrightarrow{p} C$ an object in $\mathcal{E}(\mathcal{E})$, $c' \in Q(\mathcal{E})^{op}$ and

$$\psi = (c' \leftarrow D \rightarrowtail c) \text{ map in } Q(\mathcal{E}).$$

Then the universal lift of φ is the diagram

$$
\begin{array}{ccccc}
A' & \hookleftarrow & A & = & A \\
\downarrow & & \downarrow & & \downarrow \\
D \times_C B & = & D \underset{\mathcal{E}}{\times} B & \to & B \\
\downarrow & & \downarrow & & \downarrow \\
C' & \hookleftarrow & D & \to & C
\end{array}
$$

For this map fixed, the cobase change sends right hand vertical line to left hand vertical line

Remark

Remark: before hand we have chosen for each $B' \to c'$ and $D' \to c'$ some pullback

$$
\begin{array}{ccc}
B' \times_{c'} D' & \longrightarrow & B' \\
\downarrow & & \downarrow \\
D' & \longrightarrow & c'
\end{array}
$$

Consider the action

$$
\boxed{
\begin{array}{l}
\iota G \times \xi(e) \longrightarrow \xi(e) \\[6pt]
(P, A \overset{\iota}{\rightarrowtail} B \overset{P}{\to} c) \longmapsto (P \oplus A \to P \oplus B \to c) \\[2pt]
\hspace{4cm} \left(\begin{smallmatrix} id & 0 \\ 0 & i \end{smallmatrix}\right) \quad (0 \; p)
\end{array}
}
$$

This is fibrewise w.r.t. $\mathcal{E}$

$\leadsto \quad \tilde{\mathcal{E}} : \iota(e)^{-1} \xi(e) \longrightarrow Q(c)^{op}$

what we earlier denoted by $\hat{\mathcal{E}}$

$[B, \varphi_1 : B + A \to A, \varphi_2 : B + F \to F']$

Recall:
In general

$$
\begin{array}{l}
q : \chi \longrightarrow \mathcal{E} \\
+ \quad S \times \chi \longrightarrow \chi \quad \text{fibrewise} \\
\text{w.r.t } q
\end{array}
$$

$\leadsto \quad \tilde{q} : S \cdot \chi \longrightarrow \mathcal{E}$

$$
\begin{array}{ccc}
(A, F) & \longmapsto & q(F) \\
\downarrow & & \downarrow{\scriptstyle q(\varphi_2)} \\
(A', F') & & q(F')
\end{array}
$$

$\leadsto \quad q$ cofibered $\Rightarrow \tilde{q}$ cofibered

$\leadsto \quad \mathcal{E}$ cofibered $\Rightarrow \tilde{\mathcal{E}}$ cofibered

We need 3 things:

① $B \xi(c) \simeq *$

<u>Proof:</u> Let $\tau \mathcal{E} \subseteq \mathcal{E}$ be the full subcategory spanned by admissible monics. We have

$$g: \xi(c) \longrightarrow sd(\tau G)$$

$$(A \gg B \dashrightarrow C) \mapsto (A \gg B)$$

$$\text{map} \longmapsto \text{upper two squares in diagram}$$

Note that g is fully faithful and surjective on objects.
So it suffices to note that

$$Bsd(\tau G) = |N(sd(\tau G))|$$

$$\underset{Nsd \cong sdN}{\overset{\cong}{\longrightarrow}} |sd(N \tau G)|$$

$$\cong B(\tau G)$$

$$\cong * \qquad \overset{\frown}{\quad} 0 \in \tau G \text{ is initial}$$

② $B\langle i, G, E_c\rangle$ is contractible

Proof Let $\mathcal{E} = \langle i, G, E_c\rangle$. Consider

$$*: \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$$

$$(A \gg B \to C, A' \gg B' \dashrightarrow C') \mapsto (A \oplus A' \gg \underset{c}{B \wedge B'} \dashrightarrow C')$$

Then the induced maps

$$\mu: B\mathcal{E} \times B\mathcal{E} \longrightarrow B\mathcal{E}$$

$$c = (0 \Rightarrow 0 \to c)$$

endow $B\mathcal{E}$ with structure of an H-Space $\underset{\text{This uses } G \text{ is split.}}{}$ and moreover $B\mathcal{E}$ is
(path) connected

$\mu(-,c), \mu(c,-)$ is homotopy eq.

We have that μ induces

$$\bullet: \pi_n(B\mathcal{E}, c) \times \pi_n(B_{\mathcal{E}}, c) \longrightarrow \pi_n(B\mathcal{E}, c)$$

which coincides with the usual group structure

$$a \cdot b = a + b \qquad \curvearrowright \text{ Eckmann Hilton argument}$$

So suffices: $a \cdot a = a$ for all A
Note that the canonical

$$\kappa: -(-) * \Delta \Rightarrow id$$

gives a natural transformation

$\rightsquigarrow$ $y \cdot B(\Delta) \cong id$

$\rightsquigarrow$ $a \cdot a \cong a \, \forall a$

$\rightsquigarrow$ $a > 0 \quad \forall a \in \Pi_n(B\mathcal{S}, c)$

③ For each $\varphi = (c' \leftarrow d \twoheadrightarrow c)$ in $Q(\mathcal{C})^{op}$, the cobase change

$$\varphi_* : (i\mathcal{C})^{\cdot} E_c \longrightarrow i_{\varphi_c}^{\cdot}$$

induces a hmpty equivalence on classifying spaces.

Pf. First consider

$$\varphi = (0 = 0 \twoheadrightarrow c)$$

$$E_0 = sd(i\mathcal{C})$$

$$\mathcal{C} \xrightarrow{\mathcal{F}} sd(e_1, e) \xrightarrow{\partial} i\mathcal{C}$$

$$A \longmapsto A \times_{id} A$$

$$(A \xrightarrow{\cdot}) \longmapsto A$$

Take $F = (A \xrightarrow{i} B \xrightarrow{p} c$ and consider

$$\ell_F : \mathcal{C} \longrightarrow E_c$$
$$\mathcal{Q} \longrightarrow \mathcal{Q}(\mathcal{C})^{op}$$

Now

$$i\mathcal{C} \xrightarrow{is} E_c \xrightarrow{\ell_*} E_0 \xrightarrow{\partial} i\mathcal{C}$$

which we know induces a htpy. eq. on classing spaces, but let $\langle \mathcal{C}, E_c \rangle$ is contractible $\sim \ell_*$ induces an equivalence. Now consider

$$\varphi_0 = (c' \leftarrow x \twoheadrightarrow c)$$

and take

$$\varphi_0' = \varphi_0 \cdot \varphi.$$

were x is $\ker(0 \twoheadrightarrow c')$. Then

$$(\iota e)^* E_c \longrightarrow (\iota c)^{\sim} E_c$$

$$\varphi_0^? \searrow \qquad \swarrow \tilde{\varphi}_0^?$$
$$(\iota e)^{\gamma}$$

Thm: $K(e) \simeq B(\iota e)^{-1} \times B(\iota c)$

Pf: f cofibrant $\leadsto B\hat{f}^{-1}(c) \simeq B\widetilde{(f/c)}$ we can apply ThmB

$\leadsto$ htpy fiber sequence

$$B((\iota e)^{-1} E_c) \to B[(\iota c)^{-1} \xi(e)] \xrightarrow{B[\tilde{f}]} B[(Q(e)^{op}]$$

$$\overset{12}{B((\iota e)^{-1} \iota e)} \qquad\qquad \overset{\iota}{B\xi(e)} \text{ contractible, so}$$

the action is invertible, so

$$\Rightarrow \quad B((\iota e)^{-1} \iota e) \simeq \Omega B((Q(e)^{op}) \qquad\qquad B((\iota e)^{-1} \xi(c))$$

$$\simeq K(e). \qquad\qquad\qquad\qquad \simeq B(\xi(c))$$

$$\simeq *$$