1 Introduction to sheaves

Definition 1.1. Let $X$ be a topological space. Then we define the category $\mathcal{O}(X)$ to have all open subsets $U$ of $X$ as objects and defining the morphisms $V \to U$ to be the inclusion $V \subset U$.

Definition 1.2. A presheaf is a functor $P : \mathcal{O}(X)^{\text{op}} \to \text{Sets}$.

This means that every inclusion of open sets $V \subset U$ in $X$ determines a functor

$$P(V \subset U) : P(U) \to P(V)$$

which we will often denote as $t \mapsto t|_V$ for each $t \in P(U)$ (as if they were restriction of ordinary maps).
Definition 1.3. A sheaf of sets on a topological space $X$ is a presheaf $F : \mathcal{O}(X)^{op} \to \text{Sets}$ such that each open covering $U = \sqcup_i U_i, i \in I$ of an open set $U$ of $X$ yields an equalizer diagram

$$FU \xrightarrow{e} \prod_i FU_i \xrightarrow{p} \prod_{i,j} F(U_i \cap U_j)$$

where for $t \in FU$

$$e(t) = \{t|_{U_i} | i \in I\}$$

and for a family $t_i \in FU_i$,

$$p\{t_i\} = \{t_i|_{U_i \cap U_j}\}, \quad q\{t_i\} = \{t_j|_{U_i \cap U_j}\}$$

Definition 1.4. Let $X$ be a topological space. We define a new category $\text{Sh}(X)$:

- **Objects**: All sheaves $F$ of sets on $X$
- **Morphisms** $F \to G$: Natural transformations $F \Rightarrow G$.

Note that $\text{Sh}(X)$ is a full subcategory of the functor category $\mathcal{O}(X) := \text{Sets}^{\mathcal{O}(X)^{op}}$ which is the category of presheaves on $X$.

Example 1.5. Let $X$ be a topological space and $U \subseteq X$ an open subset. Define

$$CU := \{f | f : U \to \mathbb{R} \text{ is continuous}\}.$$ 

If $V \subseteq U$ then restricting each $f \mapsto f|_V$ gives us a function $CU \to CV$. This restriction is transitive in the sense that if $W \subseteq V \subseteq U$ are three nested open subsets, then $(f|_V)|_W = f|_W$. This gives us that the assignments

$$U \mapsto CU, \quad \{C \subseteq U\} \mapsto \{CU \to CV \text{ by } f \mapsto f|_V\}$$

defines a presheaf $C : \mathcal{O}(X)^{op} \to \text{Sets}$. We will prove that this is indeed a sheaf. Let $U = U_i$ be an open covering with index set $I$. Then an $I$-indexed family of continuous functions $f_i : U_i \to \mathbb{R}$ is an element of the product set $\prod_i CU_i$. The assignments

$$\{f_i\} \mapsto \{f_i|_{U_i \cap U_j}\}, \quad \text{and} \quad \{f_j\} \mapsto \{f_j|_{U_i \cap U_j}\}$$

gives two maps $p$ and $q$ from $I$-indexed set to $I \times I$-indexed set as in

$$\prod_i CU_i \xrightarrow{q} \prod_{i,j} C(U_i \cap U_j).$$

Since being continuous is a local property this gives us an equalizer diagram

$$CU \xrightarrow{e} \prod_i CU_i \xrightarrow{p} \prod_{i,j} C(U_i \cap U_j).$$

where $e$ maps $f$ to the $I$-indexed set $\{f_i|_{U_i}\}$. This proves that $C$ is a sheaf on $X$. 

2
Example 1.6. Let $X$ be a locally connected space, then a sheaf $F$ on $X$ is \textit{locally constant} if each point $x \in X$ has a basis of open neighbourhoods $\mathcal{N}_x$ such that whenever $U, V \in \mathcal{N}_x$ with $U \subset V$, the restriction $F_V \to F_U$ is a bijection.

Example 1.7. Let $X$ be a topological space, $x \in X$ a point and $S \in \text{Set}$ a set. We define the \textit{skyscraper sheaf} concentrated at $x$, denoted by $\text{Sky}_x(A) : \mathcal{O}(X)^{\text{op}} \to \text{Sets}$, by

$$\text{Sky}_x(A)(U) = \begin{cases} A, & x \in U \\ 1, & \text{otherwise} \end{cases}$$

where $1$ denotes a fixed point and $U$ an open subset of $X$. We can turn this into a functor $\text{Sky}_x : \text{Sets} \to \text{Sh}(X)$ by saying that for an inclusion $V \hookrightarrow U$ of open sets, the restriction map

$$\text{Sky}_x(A)(U) \to \text{Sky}_x(A)(V)$$

is the evident maps:

$$A \xrightarrow{id} A \text{ if } x \in V \subseteq U$$

$$A \xrightarrow{\text{unique}} 1 \text{ if } x \notin V \subseteq U.$$ 

Proposition 1.8. Let $X$ be a topological space, then the category $\text{Sh}(X)$ has all small limits and the inclusion of sheaves into presheaves $\text{Sh}(X) \to \mathcal{O}(X)$ preserve all small limits.

2 Sheaf of cross-sections

Definition 2.1. Let $X$ be a topological space. A continuous map $p : Y \to X$ is called a \textit{bundle over} $X$.

Definition 2.2. Let $p : Y \to X$ be a bundle and $U$ an open subset of $X$, then $p_U : p^{-1}U \to U$ is a bundle over $U$. Moreover the square diagram

$$
\begin{array}{ccc}
p^{-1}U & \xrightarrow{\text{incl.}} & Y \\
p_U \downarrow & s \searrow & \downarrow p \\
U & \xrightarrow{\text{incl.}} & X
\end{array}
$$

is a pullback diagram in Top. A \textit{cross-section} $s$ of the bundle $p_U$ is a continuous map $s : U \to Y$ such that $p \circ s$ is the inclusion $i : U \to X$. Let

$$\Gamma_p U = \{s | s : U \to Y \text{ and } p \circ s = \text{incl.}\}$$

denote the set of all such cross-sections over $U$. 

If we have $V \subseteq U$ in the above context, then we have a restriction $\Gamma_p U \to \Gamma_p V$ so we get a functor

$$\Gamma_p(-) : \mathcal{O}(X)^{\text{op}} \to \text{Sets}$$

This can be proven to be a sheaf since it’s sufficient to check locally if a given function is a cross-section. Hence $\Gamma_p$ is a sheaf of sets $X$, called the sheaf of cross-sections of the bundle $p$. So we have a functor

$$\Gamma : \text{Bund}(X) \to \text{Sh}(X).$$

3 Bundles of germs

**Definition 3.1.** Let $P : \mathcal{O}(X)^{\text{op}} \to \text{Sets}$ be a presheaf, $x$ a point and $U, V$ two open neighbourhoods of $x$. Further let $u \in PU$ and $v \in PV$. We say that $u$ and $v$ has the same germ if there exists an open set $W \subset U \cap V$ such that $u|_W = v|_W \in PW$. This is an equivalence. The equivalence class of any one such $u$ is called the germ of $u$ at $x$. This will be denoted $\text{germ}_x u$.

![Diagram of germs]

Further let

$$P_x = \{\text{germ}_x u \mid u \in PU, x \in U \text{ with } U \text{ open in } X\}$$

be the set of all germs at $x$. This is called the stalk of $P$ at $x$.

![Diagram of stalks]

The definition of stalks allow us to consider our sheaf locally.
Example 3.2. Let $X$ be a topological space. A constant sheaf $F$ associated to some set $A$ is a sheaf $F \in Sh(X)$ for which it is true that $F_x = A$ for all $x \in X$. A sheaf $F \in Sh(X)$ is locally constant if for each $x \in X$ there exists an open neighbourhood $U \subseteq X$ of $x$ such that the restriction $F|_U$ is a constant sheaf on $U$.

Definition 3.3.

- $\Lambda P := \bigsqcup_x P_x = \{ \text{germ}_x s \mid x \in X, s \in PU \}$
- Let $p : \Lambda P \to X$, $p(\text{germ}_x s) = x$
- Let $s \in PU$. This determines a function $\hat{s}$ by
  $$\hat{s} : U \to \Lambda P, \quad \hat{s}x = \text{germ}_x s$$
  which is a section of $p$.

Topologize $\Lambda P$ by taking as a base of open sets all the images $\hat{s}(U) \subseteq \Lambda P$, so an open set in $\Lambda P$ is a union of images of $\hat{s}$. This makes every $p$ and every $\hat{s}$ continuous - in particular is $\hat{s} : U \to \hat{s}(U)$ a homeomorphism. Using this above construction we get a functor

$$\Lambda : \widehat{\mathcal{O}(X)} \to \text{Bund}(X)$$

$$P \mapsto \Lambda_P$$

where we by $\Lambda_P$ mean the bundle $p : \Lambda_P \to X$ as defined above (We will often use this abuse of notation). We note that for a given presheaf $P$, $\Gamma \Lambda P$ is a sheaf of sections of bundles, since we have

$$\widehat{\mathcal{O}(X)} \xrightarrow{\Lambda} \text{Bund}(X) \xrightarrow{P} Sh(X).$$

Definition 3.4. Let $P$ be a presheaf over $X$. Then for each open subset $U \subseteq X$ there is a function

$$\eta_U : PU \to \Gamma \Lambda_P(U)$$

$$s \mapsto \hat{s}$$

Note that $\eta : P \to \Gamma \Lambda_P$ is a natural transformation of functors.

Will only state the following theorem:

Theorem 3.5. If $F : \mathcal{O}(X)^{op} \to \text{Sets}$ is a sheaf, then $\eta : F \to \Gamma \Lambda_F$ is an isomorphism.

This means that every sheaf is a sheaf of cross-sections.

Theorem 3.6. Let $P$ be a presheaf. Then $\eta$ is universal from $P$ to sheaves.
This means that if $F$ is a sheaf and $\theta : P \rightarrow F$ is any map of presheaves, then there exists a unique map $\sigma : \Lambda \Gamma P \rightarrow F$ of sheaves, such that the triangle commutes:

$$
P \xrightarrow{\eta} \Lambda \Gamma P \\
\downarrow \theta \quad \downarrow \exists \sigma \\
F$$

**Proof of Theorem 3.6.** By theorem 3.5 we know that $\eta : F \rightarrow \Gamma \Lambda F$ is an isomorphism, so we may define a map $\sigma : \Lambda \Gamma P \rightarrow F$ of sheaves as $\sigma = \eta^{-1} \Gamma \Lambda \theta$ such that the lower triangle in the following commutes:

$$
P \xrightarrow{\eta} \Gamma \Lambda P \\
\downarrow \theta \quad \downarrow \sigma \\
\Gamma \Lambda F \xrightarrow{\eta} F$$

Since $\eta$ is natural (as noted before it is a natural transformation) we know that the outer square commutes as well. But again, by theorem 3.5, we know that $\eta : F \rightarrow \Gamma \Lambda F$ is an isomorphism, so the upper triangle commutes. It remains to prove that this is indeed unique.

**Corollary 3.7.** For any topological space $X$ the inclusion functor

$$Sh(X) \hookrightarrow \overline{O}(X)$$

admits a left adjoint.

**Proof.** From theorem 3.6 we get that $\Gamma \Lambda$ is a left adjoint to the inclusion with $\eta$ as the unit.

**Definition 3.8.** A left adjoint functor to the inclusion functor $Sh(X) \hookrightarrow \overline{O}(X)$ is called a sheafification functor.

so we have that $\Gamma \Lambda$ is a sheafification functor, and in particular a sheafification functor preserves all finite colimits.

### 4 Étale spaces

**Definition 4.1.** A bundle $P : E \rightarrow X$ is said to be étale when $P$ is a local homeomorphism in the following sense: To each $e \in E$ there is an open set $V$ with $e \in V \subset E$ such that $PV$ is open in $X$ and $P|_{V}$ is a homeomorphism.
Lemma 4.2. Let $p : E \to X$ be étale and $U \subset X$, then the pullback $E_U \to U$ as in

$$
\begin{array}{ccc}
E_U & \longrightarrow & E \\
\downarrow & & \downarrow p \\
U & \longrightarrow & X
\end{array}
$$

is étale over $U$.

Definition 4.3. A section of $E$, as in the above context, will always mean a continuous map $s : U \to E$ such that $p \circ s = i$.

Theorem 4.4. Let $P$ be a presheaf over $X$ and $Y$ denote a bundle over $X$. Then there exists a unit

$$
\eta_P : P \to \Gamma \Lambda P
$$

and a counit

$$
\epsilon_Y : \Lambda \Gamma Y \to Y
$$

such that $\Lambda$ is a left adjoint to $\Gamma$.

To prove this we will need the following technical proposition:

Theorem 4.5. For $p : E \to X$ étale, both $p$ and any sections of $p$ are open maps, through every point $d \in E$ there it at least one section $s : U \to E$, and the images $sU$ of all sections form a base for the topology of $E$. If $s$ and $t$ are two sections, the set $w = \{x \mid sx = tx\}$ of points where they are both defined and agree, is open in $X$.

Proof of Theorem 4.4. We have already constructed $\eta_p$, see Definition 3.4. So we wish to construct $\epsilon_Y$. We start by noting that every object in $\Gamma \Lambda U$ has the form $\hat{s}x$ for some $s \in \Gamma \Lambda Y U$. We define

$$
\epsilon_Y(\hat{s}x) = sx \in Y, \quad x \in U, \quad s \in \Gamma \Lambda Y U.
$$

We need to check that $\epsilon_Y$ is independent of the choice of $s$, it is continuous and natural in $Y$.

- Independent of the choice of $s$: Assume that $t : U \to Y$ is another section with the same germ as $s$. Then $\hat{t}x = \hat{t}x$ which means that in a sufficient small neighbourhood of $x$ they agree, so $sx = st$.

- Continuous: It is sufficient to consider $\epsilon_Y$ on an open subset. On each open neighbourhood on each point, $\epsilon_Y$ is a section, which we know from theorem 4.5 is open as well.
Natural in $Y$: Let $Y$ and $Z$ be two bundles over $X$ and $p : Y \to Z$ a morphism in $\textbf{Bund}(X)$. Then since $\Gamma$ and $\Lambda$ are functors, we first get a morphism $\Gamma_Y \to \Gamma_Z$ in $\mathcal{O}(X)$ and then a square

$$\begin{array}{ccc}
\Gamma \Lambda Y & \longrightarrow & Y \\
\downarrow & & \downarrow p \\
\Gamma \Lambda Z & \longrightarrow & Z
\end{array}$$

in $\textbf{Bund}(X)$. For $\varepsilon_Y$ to be natural in $Y$, this square has to commute. This is proven by a simple diagram chase.

Now we wish to prove that both the compositions

$$\begin{align*}
\Gamma & \xrightarrow{\eta} \Gamma \Lambda \Gamma \\
\Lambda & \xrightarrow{\Lambda \eta} \Lambda \Gamma \Lambda
\end{align*}$$

are the identities. Let $Y \to X$ be a bundle and $s \in \Gamma_Y U$ a cross-section. Then the first composite is

$$s \mapsto \dot{s} \mapsto s$$

and the second composite

$$\text{germ}_x s \mapsto \text{germ}_x \dot{s} \mapsto \dot{s}x = \text{germ}_x s$$

for $x \in X$. This means that $\eta$ and $\varepsilon$ are the unit and counit respectively, hence $\Lambda$ is left adjoint to $\Gamma$. $\square$

**Proposition 4.6.** If $P$ is a sheaf, then $\eta_P$ is an isomorphism. If $Y$ is étale, then $\varepsilon_Y$ is an isomorphism.

**Proof.** We already stated earlier in Theorem 3.5 that if $P$ is a sheaf, then $\eta_P$ is an isomorphism. Therefore we will now only prove that if $Y$ is étale, then $\varepsilon_Y$ is an isomorphism. We wish to construct an inverse $\theta_Y$ to $\varepsilon_Y$. So let $p : Y \to X$ be a bundle, $y \in Y$ and $x \in X$ such that $p(y) = x$. Then there exists an open neighbourhood $U \subseteq X$ of $x$ and a section $s : U \to Y$ such that $s(x) = y$, due to theorem 4.5:

$$\begin{array}{ccc}
p^{-1}U & \xrightarrow{s} & Y \\
\downarrow & & \downarrow p \\
U & \xrightarrow{x} & X
\end{array}$$

We define

$$\theta_Y(y) = \dot{s}x \in \Gamma \Lambda Y.$$

To show that this is the desired inverse, we need to check the following three things:

1. $\theta_Y$ independent of the choice of $s$: We may choose $U$ sufficiently small, so the germs agree.
2. \(\theta_Y\) is continuous: Due to the topology on \(\Lambda p\) we know from above that every \(s\) is continuous.

3. \(\theta_Y\) is a (2-sided) invers:

\[
\varepsilon_Y \theta_Y(y) = \varepsilon_Y(sx) = sx = y
\]
\[
\theta_Y \varepsilon_Y(sx) = \theta_Y(sx) = sx.
\]

\[\square\]

Let \(\text{Etale}(X)\) denote the full subcategory of \(\text{Bund}(X)\) consisting of étale bundles.

**Corollary 4.7.** The functor \(\Lambda\) and \(\Gamma\) restricts to an equivalence of categories:

\[
\text{Sh}(X) \xrightarrow{\Gamma} \text{Etale}(X) \xleftarrow{\Lambda} \text{Bund}(X)
\]

**Recap:** So we will now briefly consider what we just done using more abstract categorical methods. So let \(X\) be a topological space, and \(U \subset X\) an open subset. Then there is a map \(A : \mathcal{O}(X) \rightarrow \text{Bund}(X)\) which sends each \(U\) to the inclusion map \(U \rightarrow X\). Using the yoneda embedding \(\mathcal{O}(X) \rightarrow \hat{\mathcal{O}(X)}\) we get that \(\hat{\mathcal{O}(X)}\) is the free cocompletion of \(\mathcal{O}(X)\), this can be understood as we are freely adjoining colimits to \(\mathcal{O}(X)\). That means we have

\[
\mathcal{O}(X) \xrightarrow{\text{yoneda emb.}} \hat{\mathcal{O}(X)} \xrightarrow{A} \text{Bund}(X).
\]

Since \(\text{Bund}(X)\) has all small colimits we have that there exists a unique left Kan extension \(\hat{\mathcal{O}(X)} \rightarrow \text{Bund}(X)\). Due to uniqueness this has to be \(\Lambda\) as used above. Since both \(\hat{\mathcal{O}(X)}\) and \(\text{Bund}(X)\) are sufficiently nice categories with a sufficiently amount of colimits, we may apply the adjoint functor theorem to \(\Lambda\), which gives us a the existence of a unique right adjoint. We will briefly consider this rightadjoint. Let \(p\) denote a bundle over \(X\) and \(U\) an open subset. Then the right adjoint can be expressed by

\[
R(p)(U) = \text{Hom}_{\text{Bund}(X)}(A(U), p : Y \rightarrow X)
\]

which exactly is the commutative triangles

\[
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \nearrow & \\
\end{array}
\]
which are the cross-sections of $P$ over $U$. We can then restrict $\widehat{\mathcal{O}(X)}$ to the subcategory where the unit is an isomorphism, this gives us the category $Sh(X)$. Restricting $Bund(X)$ to the case where the counit is an isomorphism gives us $Etale(X)$. So we have the diagram

\[
\begin{array}{c}
\mathcal{O}(X) \xrightarrow{\text{yoneda emb.}} \widehat{\mathcal{O}(X)} \xleftarrow{\Gamma} Sh(X) \\
\downarrow A \quad \downarrow \Rightarrow \quad \downarrow \cong \\
Bund(X) \xleftarrow{\text{yoneda emb.}} \widehat{Bund(X)}. \\
\end{array}
\]

We know from Corollary 3.7 that $Sh(X) \hookrightarrow \widehat{\mathcal{O}(X)}$ admits a left adjoint. Using the adjoint functor theorem on the inclusion functor $Etale(X) \hookrightarrow Bund(X)$ we get that this admits a left adjoints, and we now have the following diagram consisting of four pairs of adjoint functors:

\[
\begin{array}{c}
\mathcal{O}(X) \xrightarrow{\text{yoneda emb.}} \widehat{\mathcal{O}(X)} \xleftarrow{\Gamma} Sh(X) \\
\downarrow A \quad \downarrow \Rightarrow \quad \downarrow \cong \\
Bund(X) \xleftarrow{\text{yoneda emb.}} \widehat{Bund(X)}. \\
\end{array}
\]

where the outer square consists of the left adjoints.

5 Stalks

Recall that the stalk of a presheaf $P$ at $x$ is

\[ P_x = \{ \text{germ}_x u \mid u \in PU, x \in U \text{ with } U \text{ open in } X \}. \]

This gives us the stalk functor

\[ \text{Stalk}_x : Sh(X) \to Sets \]

\[ F \mapsto F_x \]

at a given point $x \in X$. We have the following description for morphisms in sheaves using stalks:

**Corollary 5.1.** Given sheaves $F, G \in Sh(X)$, a morphism $h : F \to G$ in $Sh(X)$ may be described in any of the three following equivalent ways:

1. As a natural transformation $h : F \to G$ of functors,
2. as a continuous map $h : \Gamma F \to \Gamma G$ of bundles over $X$,
3. as a family $h_x : F_x \to G_x$ of functions, on the respective fibers over each $x \in X$, such that, for each open set $U$ and each $s \in FU$, the function $x \mapsto h_x(sx)$ is continuous $U \to \Gamma G$.
Lemma 5.2. For each points $x \in X$, $\text{Stalk}_x$ is left adjoint to $\text{Sky}_x$.

Proof. Let $A$ be a set and $F$ a sheaf. We need to describe a bijective correspondence between set functions $\phi : F_x \to A$ and sheaf maps $h : F \to \text{Sky}_x(A)$.

- Given a set function $\phi : F_x \to A$ we define a sheaf map $h : F \to \text{Sky}_x(A)$ component wise

  \[ h_U : F(U) \to \text{Sky}_x(A)(U). \]

  If $x \notin U$, then there is only one such function

  \[ F(U) \to \text{Sky}_x(U) = 1. \]

  If $x \in U$, then for any $s \in FU$ we define

  \[ F(U) \xrightarrow{h_U} \text{Sky}_x(A)(U) \]

  \[ s \mapsto \phi(\text{germ}_x(s)) \]

  where $\phi(\text{germ}_x(s)) \in A = \text{Sky}_x(A)(U)$.

- Given a sheaf map $h : F \to \text{Sky}_x(A)$ we wish to construct a set function $\phi : F_x \to A$.

  Each element in $F_x$ are of the form $\text{germ}_x(s)$ for some $s \in FU$ for some open neighbourhood $U$ of $x$, so we define

  \[ F_x \xrightarrow{\phi} A = \text{Sky}_x(A)(U) \]

  \[ \text{germ}_x(s) \mapsto h_U(s). \]

It’s necessary to check that this definition of $\phi$ and $h$ is well-defined, natural in $F$ and $A$ and that $\phi$ and $h$ is mutually inverse. \qed

Proposition 5.3. A map $h : F \to G$ of sheaves on a space $X$ is an epimorphism (resp. monomorphism) in the category $\text{Sh}(X)$ if and only if for each $x \in X$ the map of stalks $h_x : F_x \to G_x$ is a surjection (reps. injection) of sets.

Proof. We will only prove the case with epimorphism and surjection.

$\Leftarrow$: Let $h : F \to G$ be a map of sheaves such that each stalk is a surjection. We wish to prove that $h$ is an epimorphism. Consider sheaf maps $k, l : G \to G$ such that $h = lh$. for each $x \in X$ we have

\[ k_x \circ h_x = (k \circ h)_x = (l \circ h)_x = l_x \circ h_x. \]

so since $h_x$ is a surjection by assumption we get that $k_x = l_x$. By corollary 5.1 we know that $k$ and $l$ are determined by their effect on stalks, hence $k = l$.  

\[ \]
6 Direct and inverse image sheaves

A continuous map \( f : X \to Y \) between topological spaces will induce functors in both directions between the associated categories of sheaves. In this section we will consider these, so let a continuous map \( f : X \to Y \) be given.

**Definition 6.1.** Let \( F \in \text{Sh}(X) \) be any sheaf. We can then define a new sheaf \( f_*F \) called the direct image of \( F \) under \( f \), by

\[
f_*F(V) := F(f^{-1}V)
\]

for \( V \) open in \( Y \).

That means \( f_*F \) is the composite functor

\[
O(Y)^{\text{op}} \xrightarrow{f^{-1}} O(X)^{\text{op}} \xrightarrow{F} \text{Sets}.
\]

This clearly gives us a functor

\[
f_* : \text{Sh}(X) \to \text{Sh}(Y).
\]

To understand how to get the induced functor in the other direction we have to consider a construction called inverse image sheaves, which is a bit more complicated.

**Definition 6.2.** Let \( E \to Y \) be a bundle. Then the pullback of the bundle along \( f \) yields a bundle \( f^*E \to X \) over \( X \):

\[
\begin{array}{ccc}
  f^*E & \to & E \\
  \downarrow & & \downarrow \\
  X & \xrightarrow{f} & Y.
\end{array}
\]

This yields a functor

\[
f^* : \text{Bund}(Y) \to \text{Bund}(X).
\]

We have to go through \( \text{étale} \) bundles to associate this to sheaves.
Lemma 6.3. If $p : E \to Y$ is étale over $Y$, then $f^*E \to X$ is étale over $X$.

Proof. In the pullback $f^*E$ consider any point $\langle x, e \rangle$ consisting of $x \in X$ and $e \in E$ with $f(x) = p(e)$. Since $p$ is assumed étale, there exists an open neighbourhood $U$ of $e$ in $E$ mapped homeomorphically by $p$ onto an open set $pU$ in $Y$. Then $f^{-1}(pU) \times U$ is an open neighbourhood of $\langle x, e \rangle$ in $X \times E$. Since $f^*E$ is a subspace of $X \times E$ we have $f^{-1}(pU) \times U \cap f^*E$ is an open neighbourhood of $\langle x, e \rangle$ in $f^*E$. This is mapped homeomorphically onto $f^{-1}(pU)$ in $X$, hence $f^*E$ is étale.

Using corollary 4.7 we know that $Sh(X) \simeq Etale(X)$ by $\Gamma$ and $\Lambda$, so we get the composition

$$Sh(Y) \xrightarrow{\Gamma} Etale(Y) \xrightarrow{f^*} Etale(X) \xrightarrow{\Lambda} Sh(X).$$

Definition 6.4. For a continuous map $f : X \to Y$ we have a functor

$$f^* : Sh(Y) \to Sh(X).$$

For a sheaf $G$ on $Y$, the value $f^*(G)$ is called the inverse image of $G$ under $f$.

So for any continuous map $f : X \to Y$ we have functors

$$Sh(X) \xrightarrow{f^*} Sh(Y).$$

The most important relation between these two functors is the following:

Theorem 6.5. If $f : X \to Y$ is a continuous map, then the inverse image functor $f^*$ is left adjoint to the direct image functor $f_*$. 

This is actually a generalization of the the stalk functor and the skyscraper sheaf. Consider the example where $x$ is a fixed object of a topological space $X$. Then we have the inclusion map $\{x\} \hookrightarrow X$. The theorem above then gives us that $i^*$ is left adjoint to $i_*$

$$Sh(X) \xrightarrow{i_*} Sh(\{x\}) \simeq Sets,$$

where $i^* = Stalk_x$ and $i_* = Sky_x$. 

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