Ultracategories seminar
Talk 1: Sheafs on topological spaces

Julie Zangenberg Rasmusen

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Contents

These notes are based on chapter 2 of "*Sheaves in Geometry and Logic*" by Saunders Mac Lane and leke Moerdijk.

1 Introduction to sheaves

Definition 1.1. Let X be a topological space. Then we define the category $\mathcal{O}(X)$ to have all open subsets U of X as objects and defining the morphisms $V \to U$ to be the inclusion $V \subset U$.

Definition 1.2. A *presheaf* is a functor $P : \mathcal{O}(X)^{op} \to Sets$.

This means that every inclusion of open sets $V \subset U$ in X determines a functor

$$P(V \subset U) : P(U) \to P(V)$$

which we will often denote as $t \mapsto t|_V$ for each $t \in P(U)$ (as if they were restriction of ordinary maps)

Definition 1.3. A *sheaf* of sets on a topological space X is a presheaf $F : \mathcal{O}(X)^{op} \to Sets$, such that each open covering $U = \cup_i U_i, i \in I$ of an open set U of X yields an equalizer diagram

$$FU \xrightarrow{e} \prod_i FU_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

where for $t \in FU$

$$e(t) = \{t|_{U_i} | i \in I\}$$

and for a family $t_i \in FU_i$,

$$p\{t_i\} = \{t_i|_{U_i \cap U_j}\}, \;\; q\{t_i\} = \{t_j|_{U_i \cap U_j}\}$$

Definition 1.4. Let X be a topological space. We define a new category $Sh(X)$:

- Objects: All sheaves F of sets on X
- Morphisms $F \to G$: Natural transformations $F \Rightarrow G$.

Note that $Sh(X)$ is a full subcategory of the functor category $\widehat{\mathcal{O}(X)} := Sets^{\mathcal{O}(X)^{op}}$ which is the category of presheaves on X.

Example 1.5. Let X be a topological space and $U \subseteq X$ an open subset. Define

$$CU := \{f | f : U \to \mathbb{R} \text{ is continuous } \}.$$

If $V \subseteq U$ then restricting each $f \mapsto f|_V$ gives us a function $CU \to CV$. This restriction is transitive in the sense that if $W \subseteq V \subseteq U$ are three nested open subsets, then $(f|_V)|_W = f|_W$. This gives us that the assignments

$$U \mapsto CU, \;\; \{C \subseteq U\} \mapsto \{CU \to CV \text{ by } f \mapsto f|_V\}$$

defines a presheaf $C : \mathcal{O}(X)^{op} \to Sets$. We will prove that this is indeed a sheaf. Let $U = U_i$ be an open covering with index set I. Then an I-indexed family of continuous functions $f_i : U_i \to \mathbb{R}$ is an element of the product set $\Pi_i CU_i$. The assignments

$$\{f_i\} \mapsto \{f_i|_{U_i \cap U_j}\}, \text{ and } \{f_i\} \mapsto \{f_j|_{U_i \cap U_j}\}$$

gives two maps p and q from I-indexed set to $I \times I$-indexed set as in

$$\prod_i CU_i \underset{p}{\overset{q}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j).$$

Since being continuous is a local property this gives us an equalizer diagram

$$CU \xrightarrow{e} \prod_i CU_i \underset{p}{\overset{q}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j).$$

where e maps f to the I-indexed set $\{f_i|_{U_i}\}$. This proves that C is a sheaf on X.

Example 1.6. Let X be a locally connected space, then a sheaf F on X is *locally constant* if each point $x \in X$ has a basis of open neighbourhoods $\mathcal{N}_x$ such that whenever $U, V \in \mathcal{N}_x$ with $U \subset V$, the restriction $FV \to FU$ is a bijection.

Example 1.7. Let X be a topological space, $x \in X$ a point and $S \in Set$ a set. We define the *skyscraper sheaf* concentrated at x, denoted by $Sky_x(A) : \mathcal{O}(X)^{op} \to Sets$, by

$$Sky_x(A)(U) = \begin{cases} A & , x \in U \\ 1 & , otherwise \end{cases}$$

where 1 denotes a fixed point and U an open subset of X. We can turn this into a functor $Sky_x : Sets \to Sh(X)$ by saying that for an inclusion $V \hookrightarrow U$ of open sets, the restriction map

$$Sky_x(A)(U) \to Sky_x(A)(V)$$

is the evident maps:

$$A \xrightarrow{id} A \text{ if } x \in V \subseteq U$$
$$A \xrightarrow{unique} 1 \text{ if } x \notin V \subseteq U.$$

Proposition 1.8. *Let X be a topological space, then the category $Sh(X)$ has all small limits and the inclusion of sheaves into presheaves $Sh(X) \to \widehat{\mathcal{O}(X)}$ preserve all small limits.*

2 Sheaf of cross-sections

Definition 2.1. Let X be a topological space. A continuous map $p : Y \to X$ is called a *bundle over X*.

Definition 2.2. Let $p : Y \to X$ be a bundle and U an open subset of X, then $p_U : p^{-1}U \to U$ is a bundle over U. Moreover the square diagram

$$\begin{tikzcd} p^{-1}U \arrow[r, "incl."] \arrow[d, "p_U"'] & Y \arrow[d, "p"] \\ U \arrow[r, "incl."'] \arrow[ur, dotted, "s"] & X \end{tikzcd}$$

is a pullback diagram in Top. A *cross-section* s of the bundle p_U is a continuous map $s : U \to Y$ such that $p \circ s$ is the inclusion $i : U \to X$. Let

$$\Gamma_p U = \{s | s : U \to Y \text{ and } p \circ s = incl.\}$$

denote the set of all such cross-sections over U.

If we have $V \subseteq U$ in the above context, then we have a restriction $\Gamma_p U \to \Gamma_p V$ so we get a functor

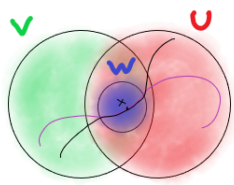
$$\Gamma_p(-) : \mathcal{O}(X)^{op} \to Sets$$

This can be proven to be a sheaf since it's sufficient to check locally if a given function is a cross-section. Hence Γ_p is a sheaf of sets X, called *the sheaf of cross-sections* of the bundle p. So we have a functor

$$\Gamma : Bund(X) \to Sh(X).$$

3 Bundles of germs

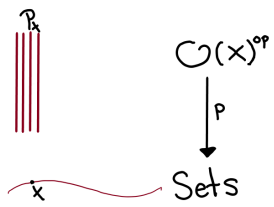
Definition 3.1. Let $P : \mathcal{O}(X)^{op} \to Sets$ be a presheaf, x a point and U, V two open neighbourhoods of x. Further let $u \in PU$ and $v \in PV$. We say that u and v has the *same germ* if there exists an open set $W \subset U \cap V$ such that $u|_W = v|_W \in PW$. This is an equivalence. The equivalence class of any one such u is called the *germ* of u at x. This will be denoted $germ_x u$.



Further let

$$P_x = \{germ_x u \mid u \in PU, x \in U \text{ with } U \text{ open in } X\}$$

be the set of all germs at x. This is called the *stalk* of P at x.



The definition of stalks allow us to consider our sheaf locally.

Example 3.2. Let X be a topological space. A *constant sheaf* F associated to some set A is a sheaf $F \in Sh(X)$ for which it is true that $F_x = A$ for all $x \in X$. A sheaf $F \in Sh(X)$ is *locally constant* if for each $x \in X$ there exists an open neighbourhood $U \subseteq X$ of x such that the restriction $F|_U$ is a constant sheaf on U.

Definition 3.3.

- $\Lambda_P := \coprod_x P_x = \{germ_x s \mid x \in X, s \in PU\}$
- Let $p : \Lambda_p \to X$, $p(germ_x s) = x$
- Let $s \in PU$. This determines a function $\dot{s}$ by

$$\dot{s} : U \to \Lambda_P, \quad \dot{s}x = germ_x s$$

  which is a section of p.

Topologize Λ_P by taking as a base of open sets all the images $\dot{s}(U) \subset \Lambda_P$, so an open set in Λ_P is a union of images of $\dot{s}$. This makes every p and every $\dot{s}$ continuous - in particular is $\dot{s} : U \to \dot{s}(U)$ a homeomorphism. Using this above construction we get a functor

$$\begin{aligned} \Lambda : \widehat{\mathcal{O}(X)} &\to Bund(X) \\ P &\mapsto \Lambda_P \end{aligned}$$

where we by Λ_P mean the bundle $p : \Lambda_P \to X$ as defined above (We will often use this abuse of notation). We note that for a given presheaf P, $\Gamma\Lambda_P$ is a sheaf of sections of bundles, since we have

$$\widehat{\mathcal{O}(X)} \xrightarrow{\Lambda} Bund(X) \xrightarrow{\Gamma} Sh(X) \ .$$

Definition 3.4. Let P be a presheaf over X. Then for each open subset $U \subseteq X$ there is a function

$$\begin{aligned} \eta_U : PU &\to \Gamma\Lambda_P(U) \\ s &\mapsto \dot{s} \end{aligned}$$

Note that $\eta : P \to \Gamma\Lambda_P$ is a natural transformation of functors.
Will only state the following theorem:

Theorem 3.5. *If $F : \mathcal{O}(X)^{op} \to Sets$ is a sheaf, then $\eta : F \to \Gamma\Lambda_F$ is an isomorphism.*

This means that every sheaf is a sheaf of cross-sections.

Theorem 3.6. *Let P be a presheaf. Then η is universal from P to sheaves.*

This means that if F is a sheaf and $\theta : P \to F$ is any map of presheafes, then there exists a unique map $\sigma : \Lambda\Gamma_P \to F$ of sheafs, such that the triangle commutes:

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[dr, "\theta"'] & \Lambda\Gamma_P \arrow[d, dotted, "\exists!\sigma"] \\ & F \end{tikzcd}$$

Proof of Theorem 3.6. By theorem 3.5 we know that $\eta : F \to \Gamma\Lambda_F$ is an isomorphism, so we may define a map $\sigma : \Lambda\Gamma_P \to F$ of sheaves as $\sigma = \eta^{-1}\Gamma\Lambda_\theta$ such that the lower triangle in the following commutes:

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\theta"'] & \Gamma\Lambda_P \arrow[d, "\Gamma\Lambda_\theta"] \arrow[dl, "\sigma"'] \\ F \arrow[r, "\eta"'] & \Gamma\Lambda_F \end{tikzcd}$$

Since η is natural (as noted before it is a natural transformation) we know that the outer square commutes as well. But again, by theorem 3.5, we know that $\eta : F \to \Gamma\Lambda_F$ is an isomorphism, so the upper triangle commutes. It remains to prove that this is indeed unique. □

Corollary 3.7. *For any topological space X the inclusion functor*

$$Sh(X) \hookrightarrow \widehat{\mathcal{O}(X)}$$

admits a left adjoint.

Proof. From theorem 3.6 we get that $\Gamma\Lambda$ is a left adjoint to the inclusion with η as the unit. □

Definition 3.8. A left adjoint functor to the inclusion functor $Sh(X) \hookrightarrow \widehat{\mathcal{O}(X)}$ is called a *sheafification functor*.

so we have that $\Gamma\Lambda$ is a sheafification functor, and in particular a sheafification functor preserves all finite colimits.

4 Étale spaces

Definition 4.1. A bundle $P : E \to X$ is said to be *étale* when P is a local homeomorphism in the following sense: To each $e \in E$ there is an open set V with $e \in V \subset E$ such that PV is open in X and $P|_V$ is a homeomorphism.

Lemma 4.2. *Let $p : E \to X$ be étale and $U \subset X$, then the pullback $E_U \to U$ as in*

$$\begin{tikzcd} E_U \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ U \arrow[r, hook, "i"'] & X \end{tikzcd}$$

is étale over U.

Definition 4.3. A *section* of E, as in the above context, will always mean a continuous map $s : U \to E$ such that $p \circ s = i$.

Theorem 4.4. *Let P be a presheaf over X and Y denote a bundle over X. Then there exists a unit*

$$\eta_P : P \to \Gamma\Lambda P$$

and a counit

$$\epsilon_Y : \Lambda\Gamma Y \to Y$$

such that Λ is a left adjoint to Γ.

To prove this we will need the following techincal proposition:

Theorem 4.5. *For $p : E \to X$ étale, both p and any sections of p are open maps. through every point $d \in E$ there it as least one section $s : U \to E$, and the images sU of all sections form a base for the topology of E. If s and t are two sections, the set $w = \{x | sx = tx\}$ of points where they are both defined and agree, is open in X*

Proof of theorem 4.4. We have already constructed η_p, see definition 3.4. So we wish to construct ε_y. We start by noting that every object in $\Gamma\Lambda U$ has the form $\dot{s}x$ for some $s \in \Gamma\Lambda_Y U$. We define

$$\varepsilon_Y(\dot{s}x) = sx \in Y, \quad x \in U, \quad s \in \Gamma_Y U.$$

We need to check that ε_Y is independent of the choice of s, it is continuous and natural in Y.

- Independent of the choice of s: Assume that $t : U \to Y$ is another section with the same germ as s. Then $\dot{s}x = \dot{t}x$ which means that in a sufficient small neighbourhood of x they agree, so $sx = st$.
- Continuous: It is sufficient to consider ε_Y on an open subset. On each open neighbourhood on each point, ε_Y is a section, which we know from theorem 4.5 is open as well.

- Natural in Y: Let Y and Z be two bundles over X and $p : Y \to Z$ a morphism in $Bund(X)$. Then since Γ and Λ is functors, we first get a morphism $\Gamma Y \to \Gamma Z$ in $\widehat{\mathcal{O}(X)}$ and then a square

$$\begin{tikzcd} \Gamma\Lambda Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \Gamma\Lambda Z \arrow[r] & Z \end{tikzcd}$$

  in $Bund(X)$. For ε_Y to be natural in Y, this square has to commute. This is proven by a simple diagram chase

Now we wish to prove that both the compositions

$$\Gamma \xrightarrow{\eta\Gamma} \Gamma\Lambda\Gamma \xrightarrow{\Gamma\varepsilon} \Gamma$$
$$\Lambda \xrightarrow{\Lambda\eta} \Lambda\Gamma\Lambda \xrightarrow{\varepsilon\Lambda} \Lambda$$

are the identities. Let $Y \to X$ be a bundle and $s \in \Gamma_Y U$ a cross-section. Then the first composite is

$$s \mapsto \dot{s} \mapsto s$$

and the second composite

$$germ_x s \mapsto germ_x \dot{s} \mapsto \dot{s}x = germ_x s$$

for $x \in X$. This means that η and ε are the unit and counit respectively, hence Λ is left adjoint to Γ. □

Proposition 4.6. *If P is a sheaf, then η_P is an isomorphism. If Y is étale, then ε_Y is an isomorphism.*

Proof. We already stated earlier in Theorem 3.5 that if P is a sheaf, then η_P is an isomorphism. Therefore we will now only prove that if Y is étale, then ε_Y is an isomorphism. We wish to construct an inverse θ_Y to ε_Y. So let $p : Y \to X$ be a bundle, $y \in Y$ and $x \in X$ such that $p(y) = x$. Then there exists an open neighbourhood $U \subseteq X$ of x and a section $s : U \to Y$ such that $s(x) = y$, due to theorem 4.5:

$$\begin{tikzcd} p^{-1}U \arrow[r, hook] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, hook] \arrow[ur, dotted, "s"] & X. \end{tikzcd}$$

We define

$$\theta_Y(y) = \dot{s}x \in \Gamma\Lambda_Y.$$

To show that this is the desired inverse, we need to check the following three things:

1. θ_Y independent of the choice of s: We may choose U sufficiently small, so the germs agree.

2. θ_Y is continuous: Due to the topology on Λ_P we know from above that every $\dot{s}$ is continuous.

3. θ_Y is a (2-sided) invers:

$$\begin{aligned} \varepsilon_Y \theta_Y(y) &= \varepsilon_Y(\dot{s}x) = sx = y \\ \theta_Y \varepsilon_Y(\dot{s}x) &= \theta_Y(sx) = \dot{s}x. \end{aligned}$$

□

Let $Etale(X)$ denote the full subcategory of $Bund(X)$ consisting of étale bundles.

Corollary 4.7. *The functor Λ and Γ restricts to an equivalence of categories:*

$$Sh(X) \underset{\Lambda}{\overset{\Gamma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} Etale(X)$$

Recap: So we will now briefly consider what we just done using more abstract categorical methods. So let X be a topological space, and $U \subset X$ an open subset. Then there is a map $A : \mathcal{O}(X) \to Bund(X)$ which sends each U to the inclusion map $U \to X$. Using the yoneda embedding $\mathcal{O}(X) \to \widehat{\mathcal{O}(X)}$ we get that $\widehat{\mathcal{O}(X)}$ is the free cocompletion of $\mathcal{O}(X)$, this can be understood as we are freely adjoining colimits to $\mathcal{O}(X)$. That means we have

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, hook, "\text{yoneda emb.}"] \arrow[dr, "A"'] & \widehat{\mathcal{O}(X)} \\ & Bund(X). \end{tikzcd}$$

Since $Bund(X)$ has all small colimits we have that there exists a unique left Kan extension $\widehat{\mathcal{O}(X)} \to Bund(X)$. Due to uniqueness this has to be Λ as used above. Since both $\widehat{O}(X)$ and $Bund(X)$ are sufficiently nice categories with a sufficiently amount of colimits, we may apply the adjoint functor theorem to Λ, which gives us a the existence of a unique rightadjoint. We will briefly consider this rightadjoint. Let p denote a bundle over X and U an open subset. Then the right adjoint can be expressed by

$$R(p)(U) = Hom_{bund(X)}(A(U), p : Y \to X)$$

which exactly is the commutative triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \\ Y \arrow[ur, "p"'] & \end{tikzcd}$$

which are the cross-sections of p over U. We can then restrict $\widehat{\mathcal{O}(X)}$ to the subcategory where the unit is an isomorphism, this gives us the category $Sh(X)$. Restricting $Bund(X)$ to the case where the counit is an isomorphism gives us $Etale(X)$. So we have the diagram

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, hook, "\text{yoneda emb.}"] \arrow[dr, "A"'] & \widehat{\mathcal{O}(X)} \arrow[d, shift right, "\Gamma"'] & Sh(X) \arrow[l, hook'] \arrow[d, shift right, "\cong"'] \\
& Bund(X) \arrow[u, shift right] & Bund(X). \arrow[l, hook'] \arrow[u, shift right]
\end{tikzcd}$$

We know from Corollary 3.7 that $Sh(X) \hookrightarrow \widehat{\mathcal{O}(X)}$ asmits a left adjoint. Using the adjoint functor theorem on the inclusion functor $Etale(X) \hookrightarrow Bund(X)$ we get that this admits a left adjoints, and we now have the following diagram consisting of four pairs of adjoint functors:

$$\begin{tikzcd}
\widehat{\mathcal{O}(X)} \arrow[r, shift left] \arrow[d, shift left, "\Gamma"] & Sh(X) \arrow[l, shift left] \arrow[d, shift left, "\simeq"'] \\
Bund(X) \arrow[r, shift left] \arrow[u, shift left, "\Lambda"] & Etale(X). \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}$$

where the outer square consists of the left adjoints.

5 Stalks

Recall that the stalk of a presheaf P at x is

$$P_x = \{germ_x u \mid u \in PU, x \in U \text{ with } U \text{ open in } X\}.$$

This gives us the *stalk functor*

$$\begin{aligned} Stalk_x : Sh(X) &\to Sets \\ F &\mapsto F_x \end{aligned}$$

at a given point $x \in X$. We have the following description for morphisms in sheaves using stalks:

Corollary 5.1. *Given sheafs $F, G \in Sh(X)$, a morphism $h : F \to G$ in $Sh(X)$ may be described in any of the three following equivalent ways:*

1. *As a natural transformation $h : F \to G$ of functors,*
2. *as a continuous map $h : \Gamma F \to \Gamma G$ of bundles over X,*
3. *as a family $h_x : F_x \to G_x$ of functions, on the respective fibers over each $x \in X$, such that, for each open set U and each $s \in FU$, the function $x \mapsto h_x(sx)$ is continuous $U \to \Gamma G$.*

Lemma 5.2. *For each points $x \in X$, $Stalk_x$ is left adjoint to Sky_x.*

Proof. Let A be a set and F a sheaf. We need to describe a bijective correspondence between set functions $\phi : F_x \to A$ and sheaf maps $h : F \to Sky_x(A)$.

- Given a set function $\phi : F_x \to A$ we define a sheaf map $h : F \to Sky_x(A)$ component wise
$$h_U : F(U) \to Sky_x(A)(U).$$
If $x \notin U$, then there is only one such function
$$F(U) \to Sky_x(U) = 1.$$
If $x \in U$, then for any $s \in FU$ we define
$$\begin{aligned} F(U) &\xrightarrow{h_U} Sky_x(A)(U) \\ s &\mapsto \phi(germ_x(s)) \end{aligned}$$
where $\phi(germ_x(s)) \in A = Sky_x(A)(U)$.

- Given a sheaf map $h : F \to Sky_x(A)$ we wish to construct a set function $\phi : F_x \to A$. Each element in F_x are of the form $germ_x(s)$ for some $s \in FU$ for some open neighbourhood U of x, so we define
$$\begin{aligned} F_x &\xrightarrow{\phi} A = Sky_x(A)(U) \\ germ_x(s) &\mapsto h_U(s). \end{aligned}$$

It's necessary to check that this definition of ϕ and h is welldefined, natural in F and A and that ϕ and h is mutually inverse. □

Proposition 5.3. *A map $h : F \to G$ of sheaves on a space X is an epimorphism (resp. monomorphism) in the category $Sh(X)$ if and only if for each $x \in X$ the map of stalks $h_x : F_x \to G_x$ is a surjection (reps. injection) of sets.*

Proof. We will only prove the case with epimorphism and surjection.

$\Leftarrow$: Let $h : F \to G$ be a map of sheaves such that each stalk i a surjection. We wish to prove that h is an epimorphism. Consider sheaf maps $k, l : G \to G$ such that $h = lh$. for each $x \in X$ we have
$$k_x \circ h_x = (k \circ h)_x = (l \circ h)_x = l_x \circ h_x$$
so since h_x is a surjcetion by assumption we get that $k_x = l_x$. By corollary 5.1 we know that k and l are determined by their effect on stalks, hence $k = l$.

$\Rightarrow$: By Lemma 5.2 we know that the stalk functor has a right adjoint, hence preserves colimits. In particular it preserves epimorphisms, since h is an epimorphisms if and only if

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "h"'] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

is a pushout.

$\square$

6 Direct and inverse image sheaves

A continuous map $f : X \to Y$ between topological spaces will induce functors in both directions between the associated categories of sheaves. In this section we will consider these, so let a continuous map $f : X \to Y$ be given.

Definition 6.1. Let $F \in Sh(X)$ be any sheaf. We can then define a new sheaf f_*F called the *direct image* of F under f, by

$$f_*F(V) := F(f^{-1}V)$$

for V open in Y.

That means f_*F is the composite functor

$$\mathcal{O}(Y)^{op} \xrightarrow{f^{-1}} \mathcal{O}(X)^{op} \xrightarrow{F} Sets.$$

This clearly gives us a functor

$$f_* : Sh(X) \to Sh(Y).$$

To understand how to get the induced functor in the other direction we have to consider a construction called inverse image sheaves, which is a bit more complicated.

Definition 6.2. Let $E \to Y$ be a bundle. Then the pullback of the bundle along f yields a bundle $f^*E \to X$ over X:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

This yields a functor

$$f^* : Bund(Y) \to Bund(X).$$

We have to go through étale bundles to associate this to sheaves.

Lemma 6.3. *If $p : E \to Y$ is étale over Y, then $f^*E \to X$ is étale over X.*

Proof. In the pullback f^*E consider any point $\langle x, e\rangle$ consisting of $x \in X$ and $e \in E$ with $f(x) = p(e)$. Since p is assumed étale, there exists an open neighbourhood U of e in E mapped homeomorphically by p onto an open set pU in Y. Then $f^{-1}(pU) \times U$ is an open neigbourhodd of $\langle x, e\rangle$ in $X \times E$. Since f^*E is a subspace of $X \times E$ we have $f^{-1}(pU) \times U \cap f^*E$ is an open neighbourhood of $\langle x, e\rangle$ in f^*E. This is mapped homeomorphically onto $f^{-1}(pU)$ in X, hence f^*E is étale. □

Using corollary 4.7 we know that $Sh(X) \simeq Etale(X)$ by Γ and Λ, so we get the composition

$$Sh(Y) \xrightarrow{\Gamma} Etale(Y) \xrightarrow{f^*} Etale(X) \xrightarrow{\Lambda} Sh(X).$$

Definition 6.4. For a continuous map $f : X \to Y$ we have a functor

$$f^* : Sh(Y) \to Sh(X).$$

For a sheaf G on Y, the value $f^*(G)$ is called the *inverse image* of G, under f.

So for any continuous map $f : X \to Y$ we have functors

$$Sh(X) \underset{f_*}{\overset{f^*}{\leftrightarrows}} Sh(Y).$$

The most important relation between these two functors is the following:

Theorem 6.5. *If $f : X \to Y$ is a continuous map, then the inverse image functor f^* is left adjoint to the direct image functor f_*.*

This is actually a generalization of the the stalk functor and the skyscraber sheaf. Consider the example where x is a fixed object of a topological space X. Then we have the inclusion map $\{x\} \hookrightarrow X$. The theorem above then gives us that i^* is left adjoint to i_*

$$Sh(X) \rightleftarrows Sh(\{x\}) \simeq Sets\ ,$$

where $i^* = Stalk_x$ and $i_* = Sky_x$.