Universal Property of L-theory spectra on Poincaré ∞-categories

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Abstract

We lay down the fundamental theory necessary to define and understand hermitian and Poincaré ∞-categories as done in [1] and [2]. The goal of this project is to construct the Groethendieck-Witt spectrum $\mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ as well as the L-theory spectrum $\mathrm{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ on such Poincaré ∞-categories. We further explain how these two notions corresponds by in the end proving the universal property of L-theory, which tells us that L is the bordification of GW, and furthermore that L is the initial bordism invariant additive functor which has a natural transformation from the functor of Poincaré objects $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ to the infinite loop space $\Omega^{\infty} L$.

Contents

Introduction

This project contains a rather concise exposition of L-theory spectra and Groethendieck-Witt spectra on Poincaré ∞-categories following [1] and [2], and contains no original results. All results in this project which are from any of these two papers will therefore be labelled with a reference, where we will reference to [1] and [2] with I and II respectively.

We will start by considering the necessary foundations to construct and understand hermitian ∞-categories, which is a pair $(\mathcal{C}, \kappa)$ consisting of a small stable ∞-category $\mathcal{C}$ together with a so called quadratic functor $\kappa : \mathcal{C}^{op} \to \mathrm{Sp}$, which means that it is reduced and 2-excisive. Using the Groethendieck construction, these ∞-categories can be formed into an ∞-category $\mathrm{Cat}^{\mathrm{h}}_{\infty}$. The quadratic functor in such a hermitian ∞-category $(\mathcal{C}, \kappa)$ induces a duality functor $D_\kappa : \mathcal{C}^{op} \to \mathcal{C}$, and a Poincaré ∞-category is a hermitian ∞-category with some extra assumptions on the quadratic functor, such that in particular this duality functor is an equivalence. These ∞-categories then form a non-full subcategory $\mathrm{Cat}^{\mathrm{p}}_{\infty} \subseteq \mathrm{Cat}^{\mathrm{h}}_{\infty}$, which has as morphisms those functors which preserve this duality.

This is the main type of ∞-categories which we will focus on in this project, and the goal is to construct L-theory spectra and Groethendieck-Witt spectra on such ∞-categories and understand the correspondence between these two constructions, which is captured in the following main theorem, which we will unravel the meaning of in a moment.

Theorem 0.1 (theorem 3.16)**.** *The natural transformation bord* $: GW \Rightarrow L$*, between the Groethendieck-Witt spectrum* $GW : Cat^p_\infty \to Sp$ *and the L-theory spectrum* $L : Cat^p_\infty \to Sp$*, exhibits L as the bordification of GW.*

In particular,

$$L : Cat^p_\infty \to Sp$$

is the initial bordism invariant, additive functor equipped with a natural transformation

$$Pn \Rightarrow \Omega^\infty L \simeq \mathcal{L}$$

from the functor of Poincare objects $Pn : Cat^p_\infty \to \mathscr{S}$ *to the L-theory space* $\mathcal{L} : Cat^p_\infty \to \mathscr{S}$*.*

That one of the main properties of these two functors L and GW is that they are bordism invariant is a hint at the fact that the notion of cobordism in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ is an

important tool when constructing and understanding these. Another important notion to understand, is the one of Poincaré objects in a Poincaré ∞-category, namely the functor $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ defined by

$$\mathrm{Pn}(\mathcal{C}, \kappa) = \{(x, q) | x \in \mathcal{C},\ q \in \Omega^{\infty}\kappa(x),\ \text{s.t. } x \simeq D_{\kappa}(x)\}.$$

After introducing Poincaré ∞-categories and the Poincaré objects, we will then turn to defining cobordism in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ and for $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ an additive functor the F-based n-extended cobordism category $\mathrm{Cob}^{F}_{n}(\mathcal{C}, \kappa)$, which is an n-fold Segal space. A cobordism is a specific kind of span between Poincaré objects and we can consider morphisms in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$, so called Poincaré functors, as Poincaré objects in the functor category which is how we define cobordism between Poincaré functors.

In the case where $F = \mathrm{Pn}$ we have that the n-fold Segal space $\mathrm{Cob}^{\mathrm{Pn}}_{n}(\mathcal{C}, \kappa)$ informally models the (∞, n)-category which has as objects the Poincaré objects in $(\mathcal{C}, \Omega^{n}\kappa)$, as morphisms cobordisms between these Poincaré objects, the 2-morphisms are cobordisms between cobordisms and so forth up to degree n.

We can then extend this definition to be a spectrum valued functor in the case where $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ is space-valued:

$$\begin{aligned} \mathbb{C}\mathrm{ob}^{F} &: \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{PSp} \\ &(\mathcal{C}, \kappa) \mapsto [\mathrm{Cob}^{F}_{0}(\mathcal{C}, \kappa), |\mathrm{Cob}^{F}_{1}(\mathcal{C}, \kappa)|, |\mathrm{Cob}^{F}_{2}(\mathcal{C}, \kappa)|, ...]. \end{aligned}$$

This construction is what we use to extend both the L-theory space and the Groethendieck-Witt space functors

$$\mathcal{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S},\ \ \mathcal{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$$

to the spectrum-valued functors

$$\mathrm{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp},\ \ \mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}.$$

The Groethendieck-Witt space functor $\mathcal{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ is the group-completion of the before mentioned Poincaré-object functor $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ inside the category of additive functors from $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ to $\mathscr{S}$. We then obtain the Groethendieck-Witt spectrum $\mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$, which satisfies $\Omega^{\infty}\mathrm{GW} \simeq \mathcal{GW}$, by taking $\mathbb{C}\mathrm{ob}$ of $\mathcal{GW}$, and we will show that this has the following universal property:

Corollary 0.2 (corollary 3.4)**.** *The Groethendieck-Witt spectrum functor $GW : Cat^p_\infty \to Sp$ is additive, and is the initial such functor that is equipped with a natural transformation*

$$Pn \Rightarrow \Omega^\infty GW \simeq \mathcal{GW}$$

of functors $Cat^p_\infty \to \mathscr{S}$.

The L-theory space functor is a bit more involved to define, but it is the geometric realization of the Poincaré objects of the so called ρ-construction, which is a way of turning a Poincaré ∞-category into a simplicial object in $\mathrm{Cat}^{\mathrm{p}}_\infty$, i.e. we define it as $\mathcal{L}(\mathcal{C}, \kappa) := |\mathrm{Pn}\rho(\mathcal{C}, \kappa)|$. An important property of this functor is that it is bordism invariant, which means that it is additive and it takes bordism equivalences in $\mathrm{Cat}^{\mathrm{p}}_\infty$ to equivalences in $\mathscr{S}$. An important consequence of being bordism invariant is that it gives us another way to lift it to spectra, since for any such $F : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathscr{S}$ we will show that

$$\Omega F(\mathcal{C}, \kappa) \simeq F(\mathcal{C}, \Omega\kappa),$$

hence we can consider the spectrum

$$[F(\mathcal{C}, \kappa), F(\mathcal{C}, \Omega\kappa), F(\mathcal{C}, \Omega^2\kappa), ...].$$

We will show that this spectrum structure is equivalent to the one from when applying $\mathbb{C}\mathrm{ob}^F$. This will be very useful for us when we define the L-theory spectrum as $\mathbb{C}\mathrm{ob}^{\mathcal{L}}$, since we then get this easier description of the spectrum. This functor is again additive, bordism invariant and it satisfies $\Omega^\infty \mathrm{L} \simeq \mathcal{L}$. We get a natural transformation

$$\mathrm{Pn} \Rightarrow \Omega^\infty \mathrm{L} \simeq \mathcal{L},$$

so using that the $\mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Sp}$ is the initial additive functor with a natural transformation $\mathrm{Pn} \Rightarrow \Omega^\infty \mathrm{GW}$, we get that this natural transformation extends to a natural transformation

$$\mathrm{bord} : \mathrm{GW} \Rightarrow \mathrm{L}.$$

We will then finally be able to prove the main theorem stated above.

Notations

Throughout this project we will work in the setting of ∞-categories as developed by Joyal and Lurie, with [4] being the standard reference. The two main examples of ∞-categories which will be used throughout is the ∞-category of spectra Sp and the ∞-category of spaces $\mathscr{S}$. General ∞-categories will be denoted by $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, and objects by x and y. The ∞-category of ∞-categories will be denoted by Cat_∞.

In the case where $\mathcal{C} \in \mathrm{Cat}_\infty$ and $x, y \in \mathcal{C}$ we have the *mapping space* which will be denoted by $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathscr{S}$. If we further assume that $\mathcal{C}$ is stable, then this mapping space has a canonical structure of an infinite loop space and can be extended to a *mapping spectrum* $\mathrm{map}_{\mathcal{C}}(x, y) \in \mathrm{Sp}$, such that $\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \Omega^\infty \mathrm{map}_{\mathcal{C}}(x, y)$.

1 Poincaré ∞-categories

The goal of this section is to lay down the foundations necessary to define and understand Poincaré ∞-categories, but we start of by recalling some basic definitions and notations. First recall that an ∞-category is called *pointed* if it admits a zero object, i.e. an object which is both terminal and initial. We will always fix such a zero object and denote it by 0. A functor between two pointed ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is called *reduced* if it preserves the zero object, and we write $\mathrm{Fun}_*(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the full subcategory spanned by the reduced functors.

A pointed ∞-category $\mathcal{C}$ is called *stable* if it admits pushout and pullbacks such that a square is a pushout if and only if it is a pullback. Such squares are called *exact*, and a functor between two stable ∞-categories is said to be *exact* if it preserves zero objects as well as exact squares. If $\mathcal{C}$ and $\mathcal{D}$ are two stable ∞-categories, we write $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the full subcategory spanned by exact fnctors. We further write $\mathrm{Cat}_\infty^{\mathrm{ex}} \subseteq \mathrm{Cat}_\infty$ for the non-full subcategory of stable ∞-categories and exact functors between these. The main example for us is the ∞-category of spectra Sp.

Note that it can be shown that any stable ∞-category admits all finite limits and colimits, and a functor between stable ∞-categories is exact if and only if it preserves finite colimits which it does if and only if it preserves finite limits.

Finally recall that a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories, where $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ finite limits, is called 2-*excisive* if it maps strongly cocartesian 3-cubes to cartesian 3-cubes. In a stable ∞-category we note that a 3-cube is cocartesian exactly when it is cartesian.

We are now ready to begin introducing the notions and constructions necessary to define Poincaré ∞-categories.

Definition 1.1. Let $\mathcal{C}$ be a small stable ∞-category. A reduced functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ is called *bireduced* if $B(x,y) \simeq 0$ whenever either x or y is a zero object.

We write $\mathrm{BiFun}(\mathcal{C}) \subset \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$ for the full subcategory spanned by bireduced functors.

We want to have a canonical way of extending a reduced functor to a bireduced functor, so let $\mathcal{C}$ be a stable ∞-category and $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ a reduced functor. Then the canonical maps $0 \to x \to 0$ and $0 \to y \to 0$ for any $x, y \in \mathcal{C}$ induces a canonical retract diagram

$$B(x,0) \oplus B(0,y) \xrightarrow{i} B(x,y) \xrightarrow{r} B(x,0) \oplus B(0,y),$$

with composition being the identity, since $B(0,0) \simeq 0$. This means that we get a canonical splitting

$$B(x,y) \simeq B^{\mathrm{red}}(x,y) \oplus B(x,0) \oplus B(0,y),$$

where $B^{\mathrm{red}}(x,y)$ can be identified with either $\mathrm{fib}(r)$ or $\mathrm{cofib}(i)$. This gives us a bireduced functor

$$B^{\mathrm{red}}(-,-) : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp},$$

which we call the *universal bireduced replacement*. That this is indeed the correct construction follows from the following lemma.

Lemma 1.2 (I.1.1.3)**.** *Let $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to Sp$ be a reduced functor with $\mathcal{C}$ stable. Then the split inclusion $B^{red}(-,-) \Rightarrow B(-,-)$ is universal among natural transformations from a bireduced functor into B, and the projection $B(-,-) \Rightarrow B^{red}(-,-)$ is the universal natural transformation from B into a bireduced functor.*

This in particular gives us that $B \mapsto B^{red}$ is both left and right adjoint to the inclusion $BiFun(\mathcal{C}) \subseteq Fun_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp)$.

Proof. We first note that $\mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$ is stable, since Sp is such, and we further see that since $\mathrm{BiFun}(\mathcal{C})$ is closed under limits in $\mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$, $\mathrm{BiFun}(\mathcal{C})$ is a stable full subcategory. So to prove both claims, it is sufficient to prove that for any $B \in \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$, the associated functors

$$(x,y) \mapsto B(x,0) \text{ and } (x,y) \mapsto B(0,y),$$

considered as objects in $\mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$, have trivial mapping spectra into and from any bireduced functor $F \in \mathrm{BiFun}(\mathcal{C})$, since in this case we get that

$$\begin{aligned}
\mathrm{map}(B^{\mathrm{red}}(x,y), F(x,y)) &\simeq \mathrm{fib}[\mathrm{map}(B(x,y), F(x,y)) \to \mathrm{map}(B(x,0) \oplus B(0,y), F(x,y))] \\
&\simeq \mathrm{fib}[\mathrm{map}(B(x,y), F(x,y)) \to 0] \\
&\simeq \mathrm{map}(B(x,y), F(x,y)),
\end{aligned}$$

and similarly

$$\mathrm{map}(F(x,y), B^{\mathrm{red}}(x,y)) \simeq \mathrm{map}(F(x,y), B(x,y)).$$

To see that these mapping spectra is indeed trivial we first note that since $0 \in \mathcal{C}^{op}$ is the zero object, the inclusion $\mathcal{C}^{op} \times \{0\} \subseteq \mathcal{C}^{op} \times \mathcal{C}^{op}$ is both left and right adjoint to the projection

$$\mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{C}^{op} \times \{0\}.$$

So restricting along the inclusion is left and right adjoint to restricting along the projection. The same result holds when we consider inclusion in the other factor $\{0\} \times \mathcal{C}^{op} \subseteq \mathcal{C}^{op} \times \mathcal{C}^{op}$.

We will only spell out how this implies that one of the mapping spectra are trivial. To do so, we first note that the above argument in particular gives us the following adjunction

$$\begin{tikzcd} \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp}) \arrow[r, bend right, "\pi_!"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_*(\mathcal{C}^{op}, \mathrm{Sp}) \arrow[l, bend right, "\pi^*"'] \end{tikzcd}$$

where

$$\pi_!(B) = B(-, 0), \quad \pi^*(F) = ((x, y) \mapsto F(x)).$$

Assuming that F is bireduced, this adjunction then gives us

$$\begin{aligned} \mathrm{map}((x, y) \mapsto B(x, 0), F) &\simeq \mathrm{map}(\pi^*\pi_! B, F) \\ &\simeq \mathrm{map}(\pi_! B, \pi_! F) \\ &\simeq \mathrm{map}(\pi_! B, F(-, 0)) \\ &\simeq \mathrm{map}(\pi_! B, 0) \\ &\simeq 0. \end{aligned}$$

The other case follows by using the other adjunction. □

Definition 1.3. Let $\mathcal{C}$ be a stable ∞-category and $\kappa : \mathcal{C}^{op} \to \mathrm{Sp}$ a reduced functor. Then the association $(x, y) \mapsto \kappa(x \oplus y)$ is again reduced, and we denote by $B_\kappa \in \mathrm{BiFun}(\mathcal{C})$ the bireduced replacement of this functor, i.e.

$$B_\kappa(-, -) := \kappa((-) \oplus (-))^{\mathrm{red}} : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}.$$

We call this functor the *cross-effect* of κ, and this yields a functor

$$\begin{aligned} B_{(-)} : \mathrm{Fun}_*(\mathcal{C}^{op}, \mathrm{Sp}) &\to \mathrm{BiFun}(\mathcal{C}) \\ \kappa &\mapsto B_\kappa. \end{aligned}$$

This functor admits both a left and a right adjoint, but to understand these we will need the following constructions.

Definition 1.4. Let $\mathcal{C}$ be a stable ∞-category. Then the diagonal functor $\Delta : \mathcal{C}^{op} \to \mathcal{C}^{op} \times \mathcal{C}^{op}$ induces a pullback functor

$$\Delta^* : \mathrm{BiFun}(\mathcal{C}) \to \mathrm{Fun}_*(\mathcal{C}^{op}, \mathrm{Sp}).$$

For any $B \in \mathrm{BiFun}(\mathcal{C})$ we write $B^\Delta := \Delta^* B$ for the restriction of B along the diagonal.

Now, let $\kappa : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with $\mathcal{C}$ stable. Then for any $x \in \mathcal{C}$ we have that the diagonal $\Delta_x : x \to x \oplus x$ and the collapse map $\nabla_x : x \oplus x \to x$ induces reduced maps

$$\kappa(x \oplus x) \to \kappa(x) \to \kappa(x \oplus x).$$

Hence we get natural maps

$$B_\kappa(x, x) \to \kappa(x) \to B_\kappa(x, x),$$

which can be considered as natural transformations

$$B_\kappa^\Delta \Rightarrow \kappa \Rightarrow B_\kappa^\Delta. \tag{1}$$

The cross-effect functor then have the following universal property.

Lemma 1.5 (I.1.1.7)**.** *Let $\kappa : \mathcal{C}^{op} \to Sp$ be reduced and $\mathcal{C}$ stable. Then the two natural transformations $B_\kappa^\Delta \Rightarrow \kappa \Rightarrow B_\kappa^\Delta$ in (1) acts as unit and counit exhibiting the cross-effect functor $B_{(-)}$ as left and right adjoint to the restriction functor*

$$\Delta^* : BiFun(\mathcal{C}) \to Fun_*(\mathcal{C}^{op}, Sp).$$

Proof. Using that $\mathcal{C}$ is stable, hence $\mathcal{C}^{op}$ is stable, we have that the direct sum functor

$$\oplus : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{C}^{op}$$

induces both the product and coproduct, so it is both left and right adjoint to the diagonal $\Delta : \mathcal{C}^{op} \to \mathcal{C}^{op} \times \mathcal{C}^{op}$, with unit and counit the diagonal and collapse map respectively, of the object in $\mathcal{C}$. This gives us that restricting along the direct sum is both left and right adjoint to restricting along the diagonal, with unit and counit induced by the diagonal and the collapse maps. The result then follows by lemma 1.2, by recalling that the two natural transformations

$$B_\kappa^\Delta \Rightarrow \kappa \Rightarrow B_\kappa^\Delta$$

are induced by Δ_x and ∇_x. □

Next we wish to take a C_2-action into account. Recall that we say that something is *symmetric* if it is a homotopy fixed point with respect to a C_2-action.

Lemma 1.6 (I.1.1.9)**.** *Let $\kappa : \mathcal{C}^{op} \to Sp$ be a reduced functor with $\mathcal{C}$ stable and assume that it is symmetric. Then the cross-effect B_κ is again symmetric, i.e. it canonically refines to an element of $BiFun(\mathcal{C})^{hC_2}$.*

Proof. By [6, 6.1.4.3 and 6.1.4.4] we have that

$$(-)^{\mathrm{red}} : \mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp}) \to \mathrm{BiFun}(\mathcal{C})$$

refines to a compatible functor

$$\mathrm{Fun}_*(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})^{hC_2} \to \mathrm{BiFun}(\mathcal{C})^{hC_2}$$

on C_2-equivariant objects.

Recalling that

$$B_\kappa(-,-) := \kappa((-) \oplus (-))^{\mathrm{red}} \in \mathrm{BiFun}(\mathcal{C})$$

we see that it is therefore sufficient for us to show that

$$(x, y) \mapsto \kappa(x \oplus y)$$

refines to a C_2-equivariant object. For this, it is further sufficient to see that

$$\oplus : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{C}^{op}$$

is equipped with a C_2-equivariant structure with respect to the flip action on $\mathcal{C}^{op} \times \mathcal{C}^{op}$ and the trivial action on $\mathcal{C}^{op}$.

We have this due to the symmetric monoidal structure on the direct sum, which are induced by the fact that this is the coproduct in $\mathcal{C}^{op}$. □

Remark 1.7. We can use this (or rather, the proof of this lemma as well as the proof of the one above) to canonically equip $B_\kappa(x, x)$ with a C_2-action. Assume that $\mathcal{C}$ is a stable ∞-category and note that $\mathrm{BiFun}(\mathcal{C})$ admits a C_2-action by simply inverting the elements in the source. We then further note that the diagonal map $\Delta : \mathcal{C}^{op} \to \mathcal{C}^{op} \times \mathcal{C}^{op}$ is

invariant under the flip C_2-action on the right hand side. This implies that the pullback functor

$$\Delta^* : \mathrm{BiFun}(\mathcal{C}) \to \mathrm{Fun}_*(\mathcal{C}^{op}, \mathrm{Sp})$$

is equivariant under the trivial C_2-action on the target. Since $B_\kappa \in \mathrm{BiFun}(\mathcal{C})^{hC_2}$ is symmetric we get that B_κ^Δ is a C_2-object of $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp})$. In particular this means that for any $x \in \mathcal{C}$ we have that $B_\kappa(x,x)$ is a spectrum with a C_2-action induced by the canonical C_2-action on $x \oplus x$ which swaps the components.

Lemma 1.8 (I.1.1.10)**.** *The natural transformations in (1) both refines naturally to C_2-equvariant maps with respect to the above C_2-action on B_κ^Δ. In particular they induce natural transformations*

$$(B_\kappa^\Delta)_{hC_2} \Rightarrow \kappa \Rightarrow (B_\kappa^\Delta)^{hC_2}.$$

Proof. By inspecting the construction of the natural transformations in (1) we see that it is sufficient to put a C_2-equivariant structure on the diagonal and collapse natural transformations

$$\Delta : id \Rightarrow id \oplus id, \text{ and } \nabla : id \oplus id \Rightarrow id,$$

of functors $\mathcal{C} \to \mathcal{C}$. Since the direct sum monoidal structure is both cartesian and cocartesian, we get the desired structure by using that [6, 2.4.3.8] tells us that every object canonically is a commutative algebra object with respect to coproducts. □

Definition 1.9. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be stable ∞-categories. We say that a functor $B : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is *bilinear* if it is exact in each variable. We write $\mathrm{Fun}^b(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$ for the full subcategory of bilinear functors.

We note that $\mathrm{Fun}^b(\mathcal{C})$ is closed under the C_2-action on $\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$ which flips the entries, hence it inherits it. Using this action we then write $\mathrm{Fun}^s(\mathcal{C}) := (\mathrm{Fun}^b(\mathcal{C}))^{hC_2}$, and we call such functors *symmetric bilinear functors* on $\mathcal{C}$.

Using this notion we can now state the following proposition.

Proposition 1.10 (I.1.1.13)**.** *Let $\mathcal{C}$ be a stable ∞-category and $\kappa : \mathcal{C}^{op} \to Sp$ a functor. Then the following are equivalent:*

i) κ is reduced and 2-excisive,

ii) B_κ is bilinear and fib$(\kappa(x) \to B_\kappa(x,x)^{hC_2})$ is exact in x,

iii) B_κ is bilinear and cofib$(B_\kappa(x,x)_{hC_2} \to \kappa(x))$ is exact in x,

where in ii) and iii) the maps comes from lemma 1.8

Definition 1.11. A functor $\kappa : \mathcal{C}^{op} \to \mathrm{Sp}$ with $\mathcal{C}$ stable is called *quadratic* if it satisfies the above equivalent conditions. We write $\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp})$ for the full subcategory spanned by such functors.

Using lemma 1.6 we get that the cross-effect refines to a functor

$$B_{(-)} : \mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^s(\mathcal{C}).$$

Given a quadratic functor $\kappa \in \mathrm{Fun}^q(\mathcal{C})$ we refer to $B_\kappa \in \mathrm{Fun}^s(\mathcal{C})$ as the *symmetric bilinear part of* κ and the underlying bilinear functor of B_κ as the *bilinear part* of κ.

Example 1.12. Any exact functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ is quadratic, since it per definition is reduced and it can be shown that exact functors are n-excisive for any n, so in particular 2-excisive. This gives us an exact full inclusion of stable ∞-categories $\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp})$.

Definition 1.13. Let $\kappa \in \mathrm{Fun}^q(\mathcal{C})$. Then we denote by $L_\kappa : \mathcal{C}^{op} \to \mathrm{Sp}$ the cofiber $\mathrm{cofib}((B_\kappa^\Delta)_{hC_2} \Rightarrow \kappa)$, which is exact by proposition 1.10, and we call this the *linear part* of κ. By construction this fits into an exact sequence

$$B_\kappa(x, x)_{hC_2} \to \kappa(x) \to L_\kappa(x),$$

and we can then organize L_κ into a functor

$$L_{(-)} : \mathrm{Fun}^q(\mathcal{C}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp}).$$

Note that if we postcompose $L_{(-)}$ with the inclusion $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp}) \subseteq \mathrm{Fun}^q(\mathcal{C})$ it carries a natural transformation from the identity $\kappa \Rightarrow L_\kappa$, which is exactly the second map in the exact sequence above. This in fact gives us the unit of the following adjunction.

Lemma 1.14 (I.1.1.24)**.** *For any quadratic functor κ, the natural transformation $\kappa \Rightarrow L_\kappa$ is a unit exhibiting the functor $L_{(-)}$ as the left adjoint to the inclusion $Fun^{ex}(\mathcal{C}^{op}, Sp) \subseteq Fun^q(\mathcal{C})$.*

The above construction of the linear part of a quadratic functor will be very relevant for us later, but we will now turn our focus to Poincaré ∞-categories which will be the main player in this project. Before we can define these, we need to understand the weaker notion of hermitian ∞-categories.

Definition 1.15. A *hermitian ∞-category* is a pair $(\mathcal{C}, \kappa)$ consisting of a small stable ∞-category $\mathcal{C}$ and a quadratic functor $\kappa : \mathcal{C}^{op} \to \mathrm{Sp}$ which we call the *hermitian structure* on $\mathcal{C}$. We then define the potentially large ∞-category $\mathrm{Cat}^{\mathrm{h}}_{\infty}$ of hermitian ∞-categories by applying the cartesian Groethendieck construction to the functor

$$\begin{aligned} \mathrm{Cat}^{\mathrm{ex}}_{\infty} &\to \mathrm{CAT}_{\infty} \\ \mathcal{C} &\mapsto \mathrm{Fun}^{q}(\mathcal{C}), \end{aligned}$$

where CAT_{∞} denotes the ∞-category of potentially large ∞-categories. The morphisms in this $\mathrm{Cat}^{\mathrm{h}}_{\infty}$ is called the *hermitian functors*.

By unravelling the definition we can obtain a more explicit description of hermitian functors, namely that a hermitian functor is a pair

$$(f, \eta) : (\mathcal{C}, \kappa) \to (\mathcal{D}, \varphi)$$

consisting of an exact functor $f : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\eta : \kappa \Rightarrow f^{*}\varphi$ of functors $\mathcal{C}^{op} \to \mathrm{Sp}$, where we write $f^{*}\varphi = \varphi \circ f^{op}$.

Using [2, 1.1.6] it can be shown that given such a hermitian functor (f, η) there exists a natural transformation $(f \times f)^{*}B_{\varphi} \simeq B_{f^{*}\varphi}$, hence η induces a natural transformation

$$\beta_{\eta} : B_{\kappa} \Rightarrow (f \times f)^{*}B_{\kappa}. \tag{2}$$

Given a hermitian ∞-category we want to extend its hermitian structure, to the ∞-category of functors which maps into the underlying stable ∞-category, such that we obtain a new hermitian ∞-category.

Definition 1.16. Let $\mathcal{D}$ be a small ∞-category, $(\mathcal{C}, \kappa)$ a hermitian ∞-category and write $\mathcal{C}^{\mathcal{D}} := \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ for the ∞-category of functors $\mathcal{D} \to \mathcal{C}$, which we know is stable by [6, 1.1.3.1] . Further write $\mathrm{ev} : \mathcal{D} \times \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ for the evaluation functor which corresponds to the functor which associates to each $d \in \mathcal{D}$ the functor $\mathrm{ev}_d : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$, which evaluates at d, under the exponential equivalence

$$\begin{aligned} \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathcal{D}}, \mathcal{C})) &\xrightarrow{\simeq} \mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathcal{D}}, \mathcal{C}) \\ (d \mapsto \mathrm{ev}_d) &\mapsto \mathrm{ev}. \end{aligned}$$

We then define a functor $\kappa^{\mathcal{D}} : (\mathcal{C}^{\mathcal{D}})^{op} \to \mathrm{Sp}$ as the limit $\lim_{d \in \mathcal{D}} \mathrm{ev}_d^{*}\kappa$, which on an object

$\varphi \in \mathcal{C}^{\mathcal{D}}$ is given by

$$\kappa^{\mathcal{D}}(\varphi) = \lim_{d \in \mathcal{D}^{op}} \kappa(\varphi(d)).$$

Using [1, 6.3.2] we get that $\kappa^{\mathcal{D}}$ is quadratic, hence we get a hermitian ∞-category $(\mathcal{C}^{\mathcal{D}}, \kappa^{\mathcal{D}})$ which we call the *cotensor of* $(\mathcal{C}, \kappa)$ *by* $\mathcal{D}$.

We will now turn our focus to further define the main kind of ∞-categories which we will work with in this project, namely Poincaré ∞-categories. A Poincaré ∞-category is a hermitian ∞-category $(\mathcal{C}, \kappa)$ where the hermitian structure κ satiesfies two non-degeneracy conditions, which only depends on its symmetric bilinear part. To understand these conditions we first need to note that the exponential equivalence

$$\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}))$$

restricts to an equivalence

$$\mathrm{Fun}^b(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp})),$$

so we can consider any bilinear functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ as an exact functor $\mathcal{C}^{op} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp})$.

We will also need to recall the stable Yoneda embedding:

Remark 1.17. Let $\mathcal{C}$ be a stable ∞-category, then by [4, 5.1.3.1] we have that the Yoneda embedding

$$\begin{aligned} j : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{op}, \mathscr{S}) \\ x &\mapsto \mathrm{Map}_{\mathcal{C}}(-, x) \end{aligned}$$

is fully faithful. Using that by [4, 5.5.2.2] we have that for any $x \in \mathcal{C}$, the functor $\mathrm{Map}_{\mathcal{C}}(-, x) : \mathcal{C}^{op} \to \mathscr{S}$ preserves all limits and therefore is an object in the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{op}, \mathscr{S}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathscr{S})$ consisting of the left exact functors, it follows by [6, 1.4.2.23] that for a stable ∞-category $\mathcal{C}$ we can formulate the Yoneda embedding as

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{op}, \mathscr{S}) & \xrightarrow{\simeq} & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp}) \\ x & \longmapsto & \mathrm{Map}_{\mathcal{C}}(-, x) & \longmapsto & \mathrm{map}_{\mathcal{C}}(-, x), \end{array}$$

where $\mathrm{map}_{\mathcal{C}}(-, -)$ denotes the mapping spectrum.

Definition 1.18. Let $\mathcal{C}$ be a stable ∞-category and $F : \mathcal{C}^{op} \to \mathrm{Sp}$ an exact functor. Then we say F is *representable* if it is in the essential image of the stable Yoneda embedding,

i.e. if it is equivalent to $\mathrm{map}_{\mathcal{C}}(-,x)$ for some $x \in \mathcal{C}$. In this case we say F is *represented by* x.

Definition 1.19. A bilinear functor $B \in \mathrm{Fun}^b(\mathcal{C})$ is *right non-degenerate* if the associated functor $B(-,y) \in \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp})$ is representable for all $y \in \mathcal{C}$. In this case we can factor the functor

$$\begin{aligned} \mathcal{C}^{op} &\to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp}) \\ y &\mapsto B(-,y) \end{aligned}$$

essentially uniquely as a duality functor

$$D_B : \mathcal{C}^{op} \to \mathcal{C}$$

followed by the inclusion $\mathcal{C} \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{op}, \mathrm{Sp})$ such that we obtain an equivalence

$$B(x,y) \simeq \mathrm{map}_{\mathcal{C}}(x, Dy).$$

In the same way we say that $B \in \mathrm{Fun}^b(\mathcal{C})$ is *left non-degenerate* if $B(x,-)$ is representable for all $x \in \mathcal{C}$. If $B \in \mathrm{Fun}^b(\mathcal{C})$ is both right and left non-degenerate we call it *non-degenerate*, in which case the two dualities are adjoints.

We will further say that a symmetric functor is non-degenerate if the underlying bilinear functor is such, and in this case it suffices to show that it is right non-degenerate. In this case the two dualities are equivalent and we call the representing functor D_B the *duality* functor associated to the non-degenerate symmetric functor B.

Given a hermitian ∞-category $(\mathcal{C}, \kappa)$ we say that the hermitian structure κ is *non-degenerate* if the symmetric bilinear part B_κ is, in which case we call $(\mathcal{C}, \kappa)$ a *non-degenerate* hermitian ∞-category. In this case we denote the duality D_{B_κ} by D_κ.

Remark 1.20. Let $B \in \mathrm{Fun}^s(\mathcal{C})$ be non-degenerate and write $D := D_B$ for the duality. We then get an equivalence

$$\begin{aligned} \mathrm{map}_{\mathcal{C}}(x, Dy) &\simeq B(x,y) \\ &\simeq B(y,x) \\ &\simeq \mathrm{map}_{\mathcal{C}}(y, Dx) \\ &\simeq \mathrm{map}_{\mathcal{C}^{op}}(D^{op}x, y), \end{aligned}$$

hence we have an adjunction

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\perp"] & \mathcal{C}. \arrow[l, bend right, "D^{op}"'] \end{tikzcd}$$

We write ev : $id \Rightarrow DD^{op}$ for the unit of this adjunction and we call it the *evaluation map of D*.

Lemma 1.21 (I.1.2.4)**.** *Let $(\mathcal{C}, \kappa)$ and $(\mathcal{D}, \varphi)$ be non-degenerate hermitian ∞-categories and let D_κ, D_φ denote the associated dualities. Further let $f, g : \mathcal{C} \to \mathcal{D}$ be two exact functors. Then there is a natural equivalence*

$$nat(B_\kappa, (f \times g)^* B_\varphi) \simeq nat(f D_\kappa, D_\kappa g^{op}),$$

between the spectra of natural transformations between functors $\mathcal{C}^{op} \times \mathcal{C}^{op} \to Sp$ and $\mathcal{C}^{op} \to \mathcal{D}$ respectively.

Before we can prove this result, we need the following technical lemma regarding left Kan extenstions.

Lemma 1.22. *The left Kan extension of a representable functor $\mathcal{C}^{op} \to Sp$ along a functor $\mathcal{C}^{op} \to \mathcal{D}^{op}$ is again representable.*

Proof. Given the following diagram

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "\mathrm{map}_{\mathcal{C}}(-,x)"] \arrow[d, "f"] & \mathrm{Sp} \\ \mathcal{D}^{op} & \end{tikzcd}$$

we have that the left Kan extension assembles into the following functor

$$f_! : \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{D}^{op}, \mathrm{Sp}),$$

which is left adjoint to composition with f, i.e. we have an equivalence between the spectra of natural transformations

$$\mathrm{nat}(f_! g, h) \simeq \mathrm{nat}(f, h \circ f).$$

This gives us the following equivalence

$$\begin{aligned}\mathrm{nat}(f_!\mathrm{map}_{\mathcal{C}}(-,x),h) &\simeq \mathrm{nat}(\mathrm{map}_{\mathcal{C}}(-,x),h\circ f)\\ &\simeq h(f(x))\\ &\simeq \mathrm{nat}(\mathrm{map}_{\mathcal{C}}(-,f(x)),h).\end{aligned}$$

So we get

$$f_!(\mathrm{map}_{\mathcal{C}}(-,x)) \simeq \mathrm{map}_{\mathcal{D}}(-,f(x))$$

as desired. □

Proof of lemma 1.21. We first note that we have a left Kan extension

$$\begin{tikzcd} \mathcal{C}^{op}\times\mathcal{C}^{op} \arrow[r, "\kappa\times\kappa"] \arrow[d, "(f\times id)"'] & \mathrm{Sp} \\ \mathcal{D}^{op}\times\mathcal{C}^{op} \arrow[ur, dotted] & \end{tikzcd}$$

and by the functoriality of this, we get a functor

$$(f\times id)_! : \mathrm{Fun}(\mathcal{C}^{op}\times\mathcal{C}^{op},\mathrm{Sp}) \to \mathrm{Fun}(\mathcal{D}^{op}\times\mathcal{C}^{op},\mathrm{Sp}).$$

This functor is then left adjoint to the functor

$$\mathrm{Fun}(\mathcal{D}^{op}\times\mathcal{C}^{op},\mathrm{Sp}) \to \mathrm{Fun}(\mathcal{C}^{op}\times\mathcal{C}^{op},\mathrm{Sp}),$$

which is induced by restricting along this Left Kan extension. Under this adjunction we get that a natural transformation

$$B_\kappa \Rightarrow (f\times g)^* B_\varphi \simeq (f\times id)^*(id\times g)^* B_\varphi$$

of functors $\mathcal{C}^{op}\times\mathcal{C}^{op}\to\mathrm{Sp}$ corresponds to a natural transformation

$$(f\times id)_! B_\kappa \Rightarrow (id\times g)^* B_\kappa$$

of functors $\mathcal{D}^{op}\times\mathcal{C}^{op}\to\mathrm{Sp}$. Using the pointwise description of Left Kan extension as e.g. given in [7, 2.11], we get that

$$((f\times id)_! B_\kappa)|_{\mathcal{D}\times\{y\}} \simeq (f\times\{y\})_!((B_\kappa)|_{\mathcal{C}\times\{y\}}).$$

Since $B_\kappa(-,y)$ is represented by $D_\kappa(y)$, and left Kan extension preserves representable functors by lemma 1.22, we get for $(x,y) \in \mathcal{D}^{op} \times \mathcal{C}^{op}$ that

$$((f \times id)_! B_\kappa)(x,y) \simeq \mathrm{map}_{\mathcal{D}^{op}}(x, f(D_\kappa(y))).$$

On the other hand by a similar argument we have

$$((id \times g)^* B_\varphi)(x,y) \simeq \mathrm{map}_{\mathcal{D}^{op}}(x, D_\varphi(g(y))),$$

so an object in $\mathrm{nat}(B_\kappa, (f \times g)^* B_\varphi)$ can equivalently be described as a natural transformation

$$\mathrm{map}_{\mathcal{D}^{op}}(x, f(D_\kappa(y))) \Rightarrow \mathrm{map}_{\mathcal{D}^{op}}(x, D_\varphi(g(y))).$$

So by the fully faithfulness of the Yoneda embedding we get the desired natural equivalence

$$\mathrm{nat}(B_\kappa, (f \times g)^* B_\varphi) \simeq \mathrm{nat}(f D_\kappa, D_\kappa g^{op}).$$

□

Remark 1.23. Given a hermitian functor $(f,\eta) : (\mathcal{C},\kappa) \to (\mathcal{D},\varphi)$ we know from 2 that we have a natural transformation

$$\beta_\eta \in \mathrm{nat}(B_\kappa, (f \times f)^* B_\varphi).$$

Under lemma 1.21 we get that β_η corresponds to a natural transformation

$$\tau_\eta : f D_\kappa \Rightarrow D_\varphi f^{op}.$$

We further have that we can also construct β_η from τ_η as the following composition:

$$\begin{aligned} B_\kappa(x,y) &\simeq \mathrm{map}_{\mathcal{C}}(x, D_\kappa(y)) \\ &\to \mathrm{map}_{\mathcal{D}}(f(x), f D_\kappa(y)) \\ &\to \mathrm{map}_{\mathcal{D}}(f(x), D_\varphi f(y)) \\ &\simeq B_\varphi(f(x), f(y)), \end{aligned}$$

where the equivalences comes from the assumption that $B_\kappa(-,y)$ and $B_\varphi(-,y)$ are represented by D_κ and D_φ respectively, the first arrow is induced by f's action on mapping spectra and the second map is induced by post composing with τ_n. That this compo-

sition corresponds to τ_n under the equivalence of lemma 1.21 follows by the triangle identities of the adjunction $(f \times id)_! \dashv (f \times id)^*$.

Definition 1.24. A hermitian functor $(f, \eta) : (\mathcal{C}, \kappa) \to (\mathcal{D}, \varphi)$ between non-degenerate hermitian ∞-categories are called *duality preserving* if the natural transformation

$$\tau_\eta : f D_\kappa \Rightarrow D_\varphi f^{op}$$

is an equivalence.

Definition 1.25. A symmetric bilinear functor $B \in \mathrm{Fun}^s(\mathcal{C})$ is said to be *perfect* if the evaluation map $\mathrm{ev} : \mathrm{id}_{\mathcal{C}} \Rightarrow D_B D_B^{op}$ is an equivlaence, and we say that a hermitian structure κ is *Poincaré* if its symmetric bilinear part $B_\kappa \in \mathrm{Fun}^s(\mathcal{C})$ is perfect. So a hermitian ∞-category $(\mathcal{C}, \kappa)$ is called *Poincaré* if the hermitian structure κ is Poincaré.

We write $\mathrm{Cat}_\infty^{\mathrm{p}} \subseteq \mathrm{Cat}_\infty^{\mathrm{h}}$ for the non-full subcategory spanned by the Poincaré ∞-categories and the duality preserving hermitian functors between such. We call such functors for *Poincaré functors*.

Definition 1.26. Let $\mathcal{C}$ be a stable ∞-category, then we write $\mathrm{Fun}^p(\mathcal{C}) \subseteq \mathrm{Fun}^q(\mathcal{C})$ for the subcategory spanned by those hermitian structures κ on $\mathcal{C}$ which are Poincaré, together with the duality preserving natural transformations between such functors, i.e. those $\eta : \kappa \Rightarrow \varphi$ for which

$$(id, \eta) : (\mathcal{C}, \kappa) \to (\mathcal{C}, \varphi)$$

is Poincaré.

Example 1.27. Let $(\mathcal{C}, \kappa)$ be a Poincaré ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a full stable subcategory such that the duality D_κ maps $\mathcal{D}$ to itself. Then $(\mathcal{D}, \kappa|_{\mathcal{D}})$ is again Poincaré with duality $D_{(\kappa|_{\mathcal{D}})} = (D_\kappa)|_{\mathcal{D}}$.

Example 1.28. Let $\mathcal{C}$ be a stable ∞-category and $\kappa \in \mathrm{Fun}^q(\mathcal{C})$. For $n \in \mathbb{Z}$ we then write $\kappa^{[n]} : \mathcal{C}^{op} \to \mathrm{Sp}$ for the *n-fold suspension* of κ given by

$$\kappa^{[n]}(x) = \Sigma^n \kappa(x).$$

Since $\mathrm{Fun}^q(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp})$ is a stable subcategory and both $L_{(-)}$ and $B_{(-)}$ are exact, we get that $\kappa^{[n]}$ is again in $\mathrm{Fun}^q(\mathcal{C})$ with bilinear part $B_{\Sigma^n \kappa} \simeq \Sigma^n B_\kappa$ and linear part $L_{\Sigma^n \kappa} \simeq \Sigma^n L_\kappa$.

We note that $(\mathcal{C}, \kappa^{[n]})$ is Poincaré if and only if $(\mathcal{C}, \kappa)$ is.

An important example of a Poincaré ∞-category which will be useful later, is the following:

Definition 1.29. Let $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$. We define the associated *metabolic category* $\mathrm{Met}(\mathcal{C}, \kappa)$ to be the hermitian ∞-category with underlying stable ∞-category $\mathrm{Ar}(\mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C})$ and hermitian structure the composite

$$\kappa_{\mathrm{met}} : \mathrm{Ar}(\mathcal{C})^{op} \simeq \mathrm{Ar}(\mathcal{C}^{op}) \xrightarrow{\mathrm{Ar}(\kappa)} \mathrm{Ar}(\mathrm{Sp}) \xrightarrow{\mathrm{fib}} \mathrm{Sp},$$

which on objects is given by

$$\kappa_{\mathrm{met}}([w \to x]) = \mathrm{fib}(\kappa(x) \to \kappa(w)).$$

To see that this does indeed form a hermitian ∞-category, we first note that $\mathrm{Fun}(\Delta^1, \mathcal{C})$ is stable since $\mathcal{C}$ is so by assumption and that κ_{met} clearly is reduced. To see that κ_{met} is 2-excisive, we first note that $\mathrm{Ar}(\kappa)$ is 2-excisive since limits and colimits in arrow categories are pointwise. Since $\mathrm{fib} : \mathrm{Ar}(\mathrm{Sp}) \to \mathrm{Sp}$ preserves limits, we therefore get that the composite κ_{met} is indeed 2-excisive.

If we unravel the definition we see that the underlying symmetric bilinear functor of κ_{met} is

$$B_{\mathrm{met}}[(w \to x), (w' \to x')] = \mathrm{fib}[B_\kappa(x, x') \to B_\kappa(w, w')].$$

So we see that if B_κ is perfect with duality D, then B_{met} is perfect with duality

$$D_{\mathrm{met}}(w \to x) = (\mathrm{fib}[D(x) \to D(w)] \to D(x)),$$

hence $\mathrm{Met}(\mathcal{C}, \kappa)$ is a Poincaré ∞-category.

Lemma 1.30 (I.2.3.7)**.** *Let $(\mathcal{C}, \kappa) \in Cat^p_\infty$. Then the maps*

$$(\mathcal{C}, \kappa^{[-1]}) \xrightarrow{i} Met(\mathcal{C}, \kappa) \xrightarrow{met} (\mathcal{C}, \kappa),$$

given respectively by $i(x) = (x \to 0)$ and $met(w \to x) = x$, extends to morphisms in Cat^p_∞.

Proof. We first spell out the definitions of the maps which we are interested in. The map

$$(r, \varepsilon) = i : (\mathcal{C}, \kappa^{[-1]}) \to \mathrm{Met}(\mathcal{C}, \kappa)$$

consists of the functor

$$\begin{aligned} r : \mathcal{C} &\to \mathrm{Ar}(\mathcal{C}) \\ x &\mapsto (x \to 0) \end{aligned}$$

together with the natural transformation

$$\begin{aligned} \varepsilon : \Omega\kappa = \kappa^{[-1]} &\Rightarrow r^*\kappa_{\mathrm{met}} \\ \varepsilon_x : \Omega\kappa(x) &\xrightarrow{\simeq} \mathrm{fib}(\kappa(0) \to \kappa(x)). \end{aligned}$$

We further have that the map

$$(t, \eta) = \mathrm{met} : \mathrm{Met}(\mathcal{C}, \kappa) \to (\mathcal{C}, \kappa)$$

consists of the target functor

$$\begin{aligned} t : \mathrm{Ar}(\mathcal{C}) &\to \mathcal{C} \\ (f : w \to c) &\mapsto c \end{aligned}$$

together with the natural transformation

$$\begin{aligned} \eta : \kappa_{\mathrm{met}} &\Rightarrow t^*\kappa \\ \eta_f : \mathrm{fib}(\kappa(c) \xrightarrow{f^*} \kappa(w)) &\to \kappa(c), \end{aligned}$$

which objectwise is given by the canonical map.

Now, to see that i extends to a morphism in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ we note that it is fully faithfull, that $r^*\kappa_{\mathrm{met}} \simeq \Omega\kappa$ and that the image of i is closed under the duality D_{met} since

$$\begin{aligned} D_{\mathrm{met}}(x \to 0) &= [\mathrm{fib}(D(0) \to D(x)) \to D(0)] \\ &= [D(x) \to 0] \\ &= i(D(x)). \end{aligned}$$

The desired then follows from example 1.27, since this gives us that

$$(i(\mathrm{Ar}(\mathcal{C})), \kappa_{\mathrm{met}}|_{i(\mathrm{Ar}(\mathcal{C}))})$$

is Poincaré.

That the functor met is Poincaré follows by using the explicit description of the duality D_{met} we gave above, since this gives us

$$\begin{aligned} t(D_{\mathrm{met}}(w \to x)) &= (\mathrm{fib}[D_\kappa(x) \to D_\kappa(w)]) \\ &= D_\kappa(x) \\ &= D_\kappa t^{op}(w \to x). \end{aligned}$$

□

1.1 Poincaré objects

In this section we will introduce the notion of Poincaré objects, which given an Poincaré ∞-category $(\mathcal{C}, \kappa)$ is an object $x \in \mathcal{C}$ together with a point in the infinite loop space $\Omega^\infty \kappa(x)$ which induces a specific map $x \to D_\kappa(x)$ that we assume to be an equivalence. To make this construction precise, we first make the definition of hermitian object which is a weaker notion than that of Poincaré objects.

Definition 1.31. Let $(\mathcal{C}, \kappa)$ be a hermitian ∞-category and $x \in \mathcal{C}$. Then a *hermitian form* on x is a point q in the space $\Omega^\infty \kappa(x)$. We call such a pair (x, q) a *hermitian object* in $\mathcal{C}$. We define the *∞-category of hermitian objects* $\mathrm{He}(\mathcal{C}, \kappa)$ to be the source of the right fibration we obtain by unstraightening the functor $\Omega^\infty \kappa : \mathcal{C}^{op} \to \mathscr{S}$.

We further let $\mathrm{Fm}(\mathcal{C}, \kappa) \subseteq \mathrm{He}(\mathcal{C}, \kappa)$ denote the maximal subgroupoid and we refer to this as the *space of hermitian objects.*

It can be shown that this construction He extends to a functor on hermitian ∞-categories.

Lemma 1.32 (I.2.1.2)**.** *Let $(\mathcal{C}, \kappa) \in Cat_\infty^h$, then the assignment $(\mathcal{C}, \kappa) \mapsto He(\mathcal{C}, \kappa)$ canonically extends to a functor*

$$He : Cat_\infty^h \to Cat_\infty,$$

together with a natural transformation from He to the forgetful functor which simply forgets the hermitian structure.

We are mainly interested in hermitian objects (x, q) on Poincaré ∞-categories $(\mathcal{C}, \kappa)$ which satiesfies a unimodular condition. To specify this, we first note that such a hermitian object canonically determines a map $q_\# : x \to D_\kappa(x)$ as the image of q under

$$\Omega^\infty \kappa(x) \to \Omega^\infty B_\kappa(x, x) \simeq \mathrm{Map}_{\mathcal{C}}(x, D_\kappa(x)),$$

where the equivalence comes from the assumption that $(\mathcal{C}, \kappa)$ is non-degenerate.

Definition 1.33. A hermitian form q on $x \in \mathcal{C}$ is a *Poincaré form* if the associated map $q_{\#} : x \to D_\kappa(x)$ is an equivalence, and in this case we call the pair (x, q) a *Poincaré object* in $\mathcal{C}$. We write $\mathrm{Pn}(\mathcal{C}, \kappa) \subseteq \mathrm{He}(\mathcal{C}, \kappa)$ for the maximal subgroupoid spanned by the Poincaré objects, and we call this the *space of Poincaré objects* in $\mathcal{C}$.

This construction assembles into a functor by the following result.

Lemma 1.34 (I.2.1.5)**.** *Let $(f, \eta) : (\mathcal{C}, \kappa) \to (\mathcal{D}, \varphi)$ be a duality preserving hermitian functor between non-degenerate hermitian ∞-categories. Then the induced functor*

$$f_* : Fm(\mathcal{C}, \kappa) \to Fm(\mathcal{D}, \varphi)$$

preserves Poincaré objects, i.e. it sends $Pn(\mathcal{C}, \kappa) \subset Fm(\mathcal{C}, \kappa)$ to $Pn(\mathcal{D}, \varphi) \subset Fm(\mathcal{D}, \varphi)$. In particular we get that for $(\mathcal{C}, \kappa) \in Cat^p_\infty$ the assignment $(\mathcal{C}, \kappa) \mapsto Pn(\mathcal{C}, \kappa)$ extends to a functor

$$Pn : Cat^p_\infty \to \mathscr{S}.$$

We will later need to assign to an object of $\mathcal{C}$ a Poincaré object in a canonical way, which we do by the following construction.

Definition 1.35. Let $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_\infty$ with duality D. Then given $x \in \mathcal{C}$ we define the *hyperbolic Poincaré object on x* as the Poincaré object with underlying object $x \oplus Dx$ and Poincaré form the image of the identiy under the composition

$$\mathrm{Map}_{\mathcal{C}}(x, x) \xrightarrow{(ev_x)_*} \mathrm{Map}_{\mathcal{C}}(x, DD(x)) \simeq \Omega^\infty B_\kappa(x, D(x)) \longrightarrow \Omega^\infty \kappa(x \oplus Dx).$$

We denote this by $\mathrm{hyp}(x) \in \mathrm{Pn}(\mathcal{C}, \kappa)$.

To understand these better, we define a new Poincaré ∞-category, such that the Poincaré objects are exactly the hyperbolic Poincaré objects.

Definition 1.36. Let $\mathcal{C}$ be a stable ∞-category. We then define its *hyperbolic ∞-category*, denoted by $\mathrm{Hyp}(\mathcal{C}) \in \mathrm{Cat}^{\mathrm{h}}_\infty$, as the hermitian ∞-category with $\mathcal{C} \oplus \mathcal{C}^{op}$ as the underlying stable ∞-category, and hermitian structure given by $\kappa_{\mathrm{hyp}}(x, y) = \mathrm{map}_{\mathcal{C}}(x, y)$.

By unwinding the definitions we see that the symmetric bilinear part of κ_{hyp} is given by

$$B_{\mathrm{hyp}}((x, y), (x', y')) = \mathrm{map}_{\mathcal{C}}(x, y') \oplus \mathrm{map}_{\mathcal{C}}(x', y)$$

with trivial linear part. This gives us that B_{hyp} is perfect with duality $D_{\mathrm{hyp}}(x,y) = (y,x)$, so in particular we get that $\mathrm{Hyp}(\mathcal{C})$ is always a Poincaré ∞-category.

For a Poincaré ∞-category $(\mathcal{C},\kappa)$ with duality D, the hyperbolic category $\mathrm{Hyp}(\mathcal{C})$ relates to $(\mathcal{C},\kappa)$ through the following Poincaré functors

$$\mathrm{Hyp}(\mathcal{C}) \xrightarrow{\mathrm{hyp}} (\mathcal{C},\kappa) \xrightarrow{fgt} \mathrm{Hyp}(\mathcal{C}),$$

where the first functor, hyp, is given by the exact functor $(x,y) \mapsto x \oplus Dy$, together with the natural transformation

$$\mathrm{map}_{\mathcal{C}}(x,y) \xrightarrow{(ev_y)_*} \mathrm{map}_{\mathcal{C}}(x,DDy) \simeq B_\kappa(x,Dy) \longrightarrow \kappa(x \oplus Dy).$$

The other map $fgt : (\mathcal{C},\kappa) \to \mathrm{Hyp}(\mathcal{C})$ is given by the exact functor $(x,y) \mapsto x \oplus Dy$ together with the natural transformation

$$\kappa(x) \to B_\kappa(x,x) \to \mathrm{map}_{\mathcal{C}}(x,Dx) = \kappa_{\mathrm{hyp}}(x,Dx).$$

1.2 Additive functors

We will now take a technical detour to understand what it means for a functor to be additive, in particular what it means when considering Poincaré functors, which will be an important property of functors in this project.

Definition 1.37. A sequence

$$(\mathcal{C}',\kappa') \xrightarrow{(f',\eta')} (\mathcal{C},\kappa) \xrightarrow{(f,\eta)} (\mathcal{C}'',\kappa'')$$

in $\mathrm{Cat}^{\mathrm{p}}_\infty$ with vanishing composition is called a *Poincaré-Verdier sequence* if it is both a fiber and a cofiber sequence in $\mathrm{Cat}^{\mathrm{p}}_\infty$. In this case we call (f',η') the *Poincaré-Verdier inclusion* and (f,η) the *Poincaré-Verdier projection.*

We further say such a sequence is *split* if the underlying functor

$$f' : \mathcal{C}' \to \mathcal{C},$$

or equivalently the underlying functor $f : \mathcal{C} \to \mathcal{C}''$, admits both adjoints when considered in $\mathrm{Cat}^{\mathrm{ex}}_\infty$.

Definition 1.38. A *(split) Poincaré-Verdier sqare* is a commutative square

$$\begin{tikzcd} (\mathcal{C},\kappa) \arrow[r] \arrow[d] & (\mathcal{D},\varphi) \arrow[d] \\ (\mathcal{C}',\kappa') \arrow[r] & (\mathcal{D}',\varphi') \end{tikzcd}$$

in $\mathrm{Cat}_\infty^{\mathrm{p}}$ which is cartesian, and whose vertical maps are (split) Poincaré-Verdier projections.

Example 1.39 (II.1.2.5)**.** The most important example for us will be the *metabolic fibre sequence* from lemma 1.30

$$(\mathcal{C},\kappa^{[-1]}) \xrightarrow{i} \mathrm{Met}(\mathcal{C},\kappa) \xrightarrow{\mathrm{met}} (\mathcal{C},\kappa)$$

which we claim is a split Poincaré-Verdier sequence.

We wish to give a sketch of why this holds, but the full proof relies on a deeper analysis and understanding of Poincaré-Verdier sequences which we won't go into in this project.

We start of by noting that the sequence have vanishing composition

$$\begin{aligned} (\mathcal{C},\kappa^{[-1]}) \xrightarrow{i} &\mathrm{Met}(\mathcal{C},\kappa) \xrightarrow{\mathrm{met}} (\mathcal{C},\kappa) \\ x \mapsto &(x \to 0) \mapsto 0. \end{aligned}$$

As above we write $i = (r,\varepsilon)$ and $\mathrm{met} = (t,\eta)$ when we wish to be precise in which part of the functors we are referring to. Before we show that the sequence is a Poincaré-Verdier sequence we see that by [2, A.2.11] the underlying sequence

$$\mathcal{C} \xrightarrow{r} \mathrm{Ar}(\mathcal{C}) \xrightarrow{t} \mathcal{C}$$

in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is both a fiber and a cofiber sequence. By the same result we also see that both r and t admits both adjoints, namely

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] & \mathrm{Ar}(\mathcal{C}) \arrow[l, bend right, "s"'] \arrow[l, bend left, "\mathrm{fib}"] \arrow[r, "t"] & \mathcal{C}, \arrow[l, bend right, "g"'] \arrow[l, bend left, "\delta"] \end{tikzcd}$$

where

$$\begin{aligned} s(x \to y) &= x \\ g(y) &= (0 \to y) \\ \delta(y) &= id_y, \end{aligned}$$

so if we show that it is a Poincaré-Verdier sequence it follows from this that it is also split.

That the sequence is a fibre sequence in $\mathrm{Cat}_\infty^{\mathrm{p}}$ follows from [2, 1.1.4 (i)], since the underlying sequence in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is a fiber sequence by the above, and the fact that $\varepsilon : \kappa^{[-1]} = \Omega\kappa \to r^*\kappa_{\mathrm{met}} \simeq \Omega\kappa$ is an equivalence.

We can then finally use [2, 1.2.2 (i)] to conclude the desired, since we by the above know that t admits a fully faithful left adjoint g and it can be shown by unraveling the definitions that the transformation

$$g^*\kappa_{\mathrm{met}} \xRightarrow{g^*\eta} g^*t^*\kappa \xRightarrow{u^*} \kappa$$

is an eqiuvalence, where $u : id_{\mathcal{C}} \Rightarrow tg$ denotes an adjunction unit.

Definition 1.40. Let $\mathcal{E}$ be an ∞-category with finite limits and $F : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E}$ some functor. We say that F is *reduced* if $F(0)$ is the terminal object in $\mathcal{E}$. We further say such a reduced functor F is *additive* if it takes split Poincaré-Verdier squares in $\mathrm{Cat}_\infty^{\mathrm{p}}$ to cartesian squares in $\mathcal{E}$.

We denote the full subcategory spanned by such functors by

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathcal{E}).$$

2 The cobordism category

The goal of this section is to define the ∞-category of cobordisms in a Poincaré ∞-category, for which it is necessary to construct a hermitian version of the Q-construction. We start out with considering cobordisms between Poincaré objects.

Definition 2.1. Let $(\mathcal{C}, \kappa) \in \mathrm{Cat}_\infty^{\mathrm{p}}$ and $(x, q), (x', q')$ two Poincaré objects in $\mathcal{C}$. A *cobordism* from (x, q) to (x', q') is a span in $\mathcal{C}$ of the form

$$x \xleftarrow{\alpha} w \xrightarrow{\beta} x',$$

together with a path $\eta : \alpha^*q \to \beta^*q'$ in the space $\Omega^\infty\kappa(w)$, such that w satiesfies the so called *Poincaré-Lefschetz condtition* with respect to x and x', which demands that

$$\mathrm{fib}(w \to x) \simeq \mathrm{fib}(x' \to x \cup_w x') \to \mathrm{fib}(x' \to D_\kappa w) \simeq \Omega D_\kappa(\mathrm{fib}(w \to x'))$$

is an equivalence, where the middle map are induced by the map $w \to D_\kappa x \times_{D_\kappa w} D_\kappa x'$ which comes from η.

Remark 2.2. We can intepret a cobordism in a Poincaré ∞-category $(\mathcal{C}, \kappa)$ as a Poincaré object in the diagram category $(\mathrm{Fun}(\mathbb{P}, \mathcal{C}), \kappa^{\mathbb{P}})$, where

$$\mathbb{P} = \cdot \leftarrow \cdot \rightarrow \cdot$$

and $\kappa^{\mathbb{P}}$ is the Poincaré structure on the diagram category given by the limit of the values of κ on the diagram.

Due to this, it makes sense that we will now construct a hermitian version of the Q-construction to better understand such cobordisms.

Definition 2.3. Let K be some poset and $(\mathcal{C}, \kappa)$ a hermitian ∞-category. We then let $Q_K(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{TwAr}(K), \mathcal{C})$ denote the full subcategory which is spanned by those functors F which satisfies that for for every $i \leq j \leq k \leq l \in K$, the square

$$\begin{tikzcd} F(i \leq j) \arrow[r] \arrow[d] & F(j \leq l) \arrow[d] \\ F(i \leq k) \arrow[r] & F(j \leq k), \end{tikzcd}$$

is bicartestian. We can then equip this with a hermitian structure

$$\kappa^{\mathrm{TwAr}(K)}(F) = \lim_{\mathrm{TwAr}(K)^{op}} \kappa \circ F^{op},$$

and write $Q_K(\mathcal{C}, \kappa)$ for the resulting hermitian ∞-category. In the case where $K = \Delta^n$ we denote this hermitian ∞-category by $Q_n(\mathcal{C}, \kappa)$, and we write κ_n for the hermitian structure.

As mentioned when we defined the cotensor on a hermitian ∞-category it follows from [1, 6.3.2] that this indeed gives a hermitian ∞-category.

Example 2.4 (II.2.2.3)**.** Let $(\mathcal{C}, \kappa)$ be a hermitan ∞-category

- Consider the case $K = \Delta^1$. Since $\mathrm{TwAr}(\Delta^1)$ can be thought of as the category $\mathbb{P} = \cdot \leftarrow \cdot \rightarrow \cdot$ we have that $\mathrm{Fun}(\mathrm{TwAr}(\Delta^1), \mathcal{C})$ consists of the spans in $\mathcal{C}$, so the associated hermitian ∞-category $Q_1(\mathcal{C}, \kappa)$ can be interpreted as the category of cobordisms in $(\mathcal{C}, \kappa)$
- Next, consider the case where $K = \Delta^2$. Then $Q_2(\mathcal{C}, \kappa)$ consists of diagrams of the form

$$\begin{tikzcd}
 & & F(0 \leq 2) \arrow[dl] \arrow[dr] & & \\
 & F(0 \leq 1) \arrow[dl] \arrow[dr] & & F(1 \leq 2) \arrow[dl] \arrow[dr] & \\
F(0 \leq 0) & & F(1 \leq 1) & & F(2 \leq 2),
\end{tikzcd}$$

where the square is bicartesian. Hence we can think of $Q_2(\mathcal{C}, \kappa)$ as the category of two composable cobordisms equipped with a chosen composite.

- We can extend this understanding to a general $K = \Delta^n$. To do so, we first write $J_n \subseteq \mathrm{TwAr}(\Delta^n)$ for the full subcategory spanned by the pairs (i, j) with $j \leq i+1$. In the case $n = 2$, J_2 is the lower zig-zag part of the diagram above, i.e. of the form

$$\begin{tikzcd}
 & (0 \leq 1) \arrow[dl] \arrow[dr] & & (1 \leq 2) \arrow[dl] \arrow[dr] & \\
(0 \leq 0) & & (1 \leq 1) & & (2 \leq 2).
\end{tikzcd}$$

It is shown in [2, 2.1.3.(iv)] that the restriction functor

$$Q_n(\mathcal{C}, \kappa) \to (\mathcal{C}^{J_n}, \kappa^{J_n})$$

is an equivalence of hermitian ∞-categories. So we can consider $Q_n(\mathcal{C}, \kappa)$ as the ∞-category of n composable cobordisms in $(\mathcal{C}, \kappa)$.

This construction is functorial in the sense that it extends to a functor

$$\begin{aligned} \mathrm{Posets}^{op} \times \mathrm{Cat}^{\mathrm{h}}_{\infty} &\to \mathrm{Cat}^{\mathrm{h}}_{\infty} \\ (K, (\mathcal{C}, \kappa)) &\mapsto Q_K(\mathcal{C}, \kappa), \end{aligned}$$

where Posets denotes the category of posets. So by restricting along the inclusion $\Delta \subset \mathrm{Posets}$ and adjoining, we obtain a simplicial object $Q(\mathcal{C}, \kappa) \in s\mathrm{Cat}^{\mathrm{h}}_{\infty}$. For details regarding this construction see [2, p.45].

Definition 2.5. We call the above functor

$$Q : \mathrm{Cat}^{\mathrm{h}}_{\infty} \to s\mathrm{Cat}^{\mathrm{h}}_{\infty}$$

the *hermitian Q-construction.*

Lemma 2.6 (II.2.2.7)**.** *The functor $Q : Cat^h_\infty \to sCat^h_\infty$ restricts to a functor $Q : Cat^p_\infty \to sCat^p_\infty$, and $Q(\mathcal{C}, \kappa)$ is a complete Segal object in Cat^p_∞ for any $(\mathcal{C}, \kappa) \in Cat^p_\infty$.*

This Q-construction will be used to define the cobordism category, but for that we will need some basic properties, which we will not prove here.

Lemma 2.7 (II.2.3.1)**.** *Let $(\mathcal{C}, \kappa) \in Cat^p_\infty$ and $F : Cat^p_\infty \to \mathscr{S}$ an additive functor. Then $FQ(\mathcal{C}, \kappa)$ is a Segal space, and if we further assume that F preserves arbitrary pullbacks, then $FQ(\mathcal{C}, \kappa)$ is complete.*

Recalling that complete Segal spaces gives rise to ∞-categories in a canonical way, see [3], we can now make the following definition.

Definition 2.8. Let $F : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathscr{S}$ be an additive functor which preserves arbitrary pullbacks, $(\mathcal{C}, \kappa)$ a Poincaré ∞-category and recall that then $(\mathcal{C}, \kappa^{[1]})$ is again a Poincaré ∞-category. We then denote the ∞-category associated to the complete Segal space $FQ(\mathcal{C}, \kappa^{[1]})$ with $\mathrm{Cob}^F(\mathcal{C}, \kappa)$ and call it the *F-based cobordism category.* In the case where $F = \mathrm{Pn}$ we simply write $\mathrm{Cob}(\mathcal{C}, \kappa)$ and call it the *cobordism category.*

We note that this definition does indeed make sense, since Pn is both additive and preserves arbitrary pullbacks by [2, 1.5.10].

By [2, p.51] we get that the construction of these ∞-categories assemble into a functor

$$\begin{aligned} \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathscr{S}) \times \mathrm{Cat}^{\mathrm{p}}_\infty &\to \mathrm{Cat}_\infty \\ (F, (\mathcal{C}, \kappa)) &\mapsto \mathrm{Cob}^F(\mathcal{C}, \kappa). \end{aligned}$$

Given a Poincaré ∞-category $(\mathcal{C}, \kappa)$ we want to iterate this construction to obtain an n-fold Segal object in $\mathrm{Cat}^{\mathrm{p}}_\infty$. To do so, we first note that we have an n-fold simplicial object in $\mathrm{Cat}^{\mathrm{p}}_\infty$ given by

$$\begin{aligned} Q^{(n)}(\mathcal{C}, \kappa) : (\Delta^{op})^n &\to \mathrm{Cat}^{\mathrm{p}}_\infty \\ ([m_1], [m_2], ..., [m_n]) &\mapsto Q_{m_1} Q_{m_2} ... Q_{m_n}(\mathcal{C}, \kappa), \end{aligned}$$

which we call the *n-fold iterated hermitian Q-construction.* It follows by lemma 2.6 and [2, 2.2.5] that $Q^{(n)}(\mathcal{C}, \kappa)$ is an n-fold Segal object of $\mathrm{Cat}^{\mathrm{p}}_\infty$. We can therefore make the following definition.

Definition 2.9. Let $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ be an additive functor which preserves arbitrary pullbacks and $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$. We then call the n-fold Segal space $FQ^{(n)}(\mathcal{C}, \kappa^{[n]})$ the *F-based n-extended cobordism category* of $(\mathcal{C}, \kappa)$ and we denote it by $\mathrm{Cob}^F_n(\mathcal{C}, \kappa)$. Again, in the case $F = \mathrm{Pn}$ we denote this by $\mathrm{Cob}_n(\mathcal{C}, \kappa)$.

Remark 2.10. In the above setting we see that $\mathrm{Cob}^F_0(\mathcal{C}, \kappa) \simeq F(\mathcal{C}, \kappa)$ and $\mathrm{Cob}^F_1(\mathcal{C}, \kappa) = FQ(\mathcal{C}, \kappa^{[1]})$, which is the Segal space that gives rise to $\mathrm{Cob}^F(\mathcal{C}, \kappa)$. In general, we have natural equivalences

$$|\mathrm{Cob}_i^{|\mathrm{Cob}^F_j|}(\mathcal{C}, \kappa)| \simeq |\mathrm{Cob}^F_{i+j}(\mathcal{C}, \kappa)|.$$

Remark 2.11. Let $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ be an additive functor, which preserves arbitrary pullbacks. Then by [2, 2.5.1] we get that $|\mathrm{Cob}^F|$ is additive, so by induction it follows by remark 2.10 that $|\mathrm{Cob}^F_i|$ is again additive.

Example 2.12. Consider the additive functor $F = \mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ and let $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$. In this case the n-fold Segal space $\mathrm{Cob}_n(\mathcal{C}, \kappa)$ informally models the (∞, n)-category which has as objects the Poincaré objects of $(\mathcal{C}, \kappa^{[n]})$, as morphisms the cobordisms between these Poincaré objects, and as 2-morphisms the cobordisms between the cobordisms and so on up to degree n.

We quickly just note the following property of $\pi_0|\mathrm{Cob}^F(\mathcal{C}, \kappa)|$ which will be relevant later.

Proposition 2.13 (II.2.2.6)**.** *Let $(\mathcal{C}, \kappa)$ be a Poincaré ∞-category and $F : Cat^{p}_{\infty} \to \mathscr{S}$ an additive functor. Then the Poincaré functor $(id_{\mathcal{C}}, -id_{\kappa}) : (\mathcal{C}, \kappa) \to (\mathcal{C}, \kappa)$ induces an inversion map on $\pi_0|Cob^F(\mathcal{C}, \kappa)|$.*

The next goal is to extend this construction to a functor, which goes into the ∞-category PSp of prespectra. To do so, we first note that for any additive functor $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ it follows by [2, 3.3] the commutative square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & |\mathrm{Cob}^F_{0/}| \arrow[d] \\ 0 \arrow[r] & |\mathrm{Cob}^F| \end{tikzcd}$$

exhibits $|\mathrm{Cob}^F|$ as the suspension of F in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S})$. By iterating this construction we get a model for the suspension prespectrum of any such additive functor F.

Applying this to the functor $|\mathrm{Cob}_i^F|$, which we by remark 2.11 know is additive, we obtain the structure maps necessary to turn it into a prespectrum. Hence we can now make the following definition:

Definition 2.14. Let $F : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathscr{S}$ be an additive functor which preserves arbitrary pullbacks. We then define

$$\begin{aligned} \mathbb{C}\mathrm{ob}^F : \mathrm{Cat}_\infty^{\mathrm{p}} &\to \mathrm{PSp} \\ (\mathcal{C}, \kappa) &\mapsto [\mathrm{Cob}_0^F(\mathcal{C}, \kappa), |\mathrm{Cob}_1^F(\mathcal{C}, \kappa)|, |\mathrm{Cob}_2^F(\mathcal{C}, \kappa)|, ...], \end{aligned}$$

using the structure maps determined above

Remark 2.15. By construction we have that the 0th term in $\mathbb{C}\mathrm{ob}^F(\mathcal{C}, \kappa)$ is $\mathrm{Cob}_0^F(\mathcal{C}, \kappa) \simeq F(\mathcal{C}, \kappa)$, so we have a natural map

$$F(\mathcal{C}, \kappa) \to \Omega^\infty \mathbb{C}\mathrm{ob}^F(\mathcal{C}, \kappa).$$

Remark 2.16. As mentioned above, we have that $|\mathrm{Cob}^F|$ is the suspension of F in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathscr{S})$ so using that $|\mathrm{Cob}_n^F| \simeq |\mathrm{Cob}_{n-1}^{|\mathrm{Cob}^F|}|$ from remark 2.10 we get that $|\mathrm{Cob}_n^F|$ is a model for the n-fold suspension. We can therefore consider $\mathbb{C}\mathrm{ob}^F$ as the suspension pre-spectrum of F in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathscr{S})$.

Before we prove the main properties of this functor $\mathbb{C}\mathrm{ob}^F$ we need the following result and definition.

Lemma 2.17 (II.1.5.7)**.** *The forgetful functor*

$$\mathit{Fun}^{add}(\mathit{Cat}_\infty^p, \mathit{Mon}_{E_\infty}(\mathcal{E})) \to \mathit{Fun}^{add}(\mathit{Cat}_\infty^p, \mathcal{E})$$

is an equivalence.

Definition 2.18. An additive functor $F : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E}$ is *group-like* if the canonical lift of F to $\mathrm{Mon}_{E_\infty}(\mathcal{E})$, by the equivalence above, takes values in the full subcategory $\mathrm{Grp}_{E_\infty}(\mathcal{E})$ of $\mathrm{Mon}_{E_\infty}(\mathcal{E})$.

Remark 2.19. We note that if $\mathcal{E}$ is additive, then any additive functor $F : \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E}$ is naturally group-like, since in this case both forgetful functors

$$\mathrm{Mon}_{E_\infty}(\mathcal{E}) \to \mathcal{E}, \;\; \mathrm{Grp}_{E_\infty}(\mathcal{E}) \to \mathcal{E}$$

are equivalences.

Example 2.20. By proposition 2.13 we get that for any additive functor $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$, the functor $|\mathrm{Cob}^F_n|$ is group-like for all $n \geq 1$.

We get from [2, 3.3.6] that the inclusion

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Grp}_{E_\infty}(\mathscr{S})) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S})$$

admits a left adjoint which we denote by $(-)^{\mathrm{grp}}$ and call the *group-completion functor*. This is the canonical way of turning an additive functor $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ into a group-like functor.

We can now prove the following main properties of $\mathbb{C}\mathrm{ob}^F$.

Proposition 2.21 (II.3.4.5)**.** *Let $F \in Fun^{add}(Cat^p_\infty, \mathscr{S})$ and $(\mathcal{C}, \kappa) \in Cat^p_\infty$. Then the following properties holds:*

1. *$\mathbb{C}ob^F : Cat^p_\infty \to PSp$ is additive and takes values in positive Ω-spectra.*
2. *If F is group-like, then $\mathbb{C}ob^F(\mathcal{C}, \kappa)$ is an Ω-spectrum so we can consider $\mathbb{C}ob^F$ as a functor $\mathbb{C}ob^F : Cat^p_\infty \to Sp$, and $\mathbb{C}ob^F$ is additive when considered as such a functor.*
3. *The natural map*
$$\mathbb{C}ob^F(\mathcal{C}, \kappa) \to \mathbb{C}ob^{F^{grp}}(\mathcal{C}, \kappa)$$
exhibits the right hand side as the spectrification of the left hand side.

In particular we get that (3) *and* (1) *respectively implies the following equivalences for $n \geq 1$:*

$$F^{grp}(\mathcal{C}, \kappa) \simeq \Omega^\infty \mathbb{C}ob^F(\mathcal{C}, \kappa)$$
$$|Cob^F_n(\mathcal{C}, \kappa)| \simeq \Omega^{\infty - n} \mathbb{C}ob^F(\mathcal{C}, \kappa).$$

Proof. By remark 2.10 we know that

$$|\mathrm{Cob}^F_n| \simeq |\mathrm{Cob}^{|\mathrm{Cob}^F|}_{n-1}|,$$

and by example 2.20 we have that this is group-like for $n \geq 1$. So in this case it follows by [2, 3.3.4.(ii)] that we have the following equivalence in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S})$:

$$|\mathrm{Cob}^F_n| \xrightarrow{\simeq} \Omega|\mathrm{Cob}^F_{n+1}|.$$

If F is group-like this clearly also holds in the case $n = 0$, so we get that $\mathbb{C}\mathrm{ob}^F$ takes values in positive Ω-specra, and in spectra if F is further group-like.

To show the additivity part of statement 1. and 2., we recall that remark 2.11 gives us that $|\mathrm{Cob}^F|$ is additive if F is. Using that fibre sequences in both spectra and pre-spectra are detected degree-wise we then get that $\mathbb{C}\mathrm{ob}^F$ is also additive.

To prove the third statement, we note that by the second statement which we have just shown, we get that the spectrification of $\mathbb{C}\mathrm{ob}^F$ is given by

$$[\Omega|\mathrm{Cob}_1^F(\mathcal{C},\kappa)|, |\mathrm{Cob}_1^F(\mathcal{C},\kappa)|, |\mathrm{Cob}_2^F(\mathcal{C},\kappa)|, ...]. \tag{3}$$

Using that $|\mathrm{Cob}_{n+1}^F| \simeq |\mathrm{Cob}_n^{|\mathrm{Cob}^F|}|$, it follows by [2, 3.3.6], which tells us that $\Omega|\mathrm{Cob}^F(-)| \simeq F^{grp}$, together with [2, 3.3.7], which tells us that $|\mathrm{Cob}^F(\mathcal{C},\kappa)| \simeq \mathrm{Cob}^{F^{grp}}(\mathcal{C},\kappa)$, that (3) agrees with

$$[F^{grp}(\mathcal{C},\kappa), |\mathrm{Cob}_1^{F^{grp}}(\mathcal{C},\kappa)|, |\mathrm{Cob}_2^{F^{grp}}(\mathcal{C},\kappa)|, ...].$$

□

The universal property of the iterated hermitian Q-construction can then be shown:

Corollary 2.22 (II.3.4.6)**.** *Let $F \in Fun^{add}(Cat_\infty^p, \mathscr{S})$ be group-like. Then the functor*

$$\mathbb{C}ob^F : Cat_\infty^p \to Sp$$

is the initial additive functor under $\mathbb{S}[F]$, the pointwise suspension spectrum of F, i.e. we have an adjunction

$$Fun^{add}(Cat_\infty^p, Grp_{E_\infty}) \underset{Forget}{\overset{\mathbb{C}ob}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Fun^{add}(Cat_\infty^p, Sp).$$

We further have an adjunction

$$Fun^{add}(Cat_\infty^p, \mathscr{S}) \underset{Forget}{\overset{\mathbb{C}ob\circ(-)^{grp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Fun^{add}(Cat_\infty^p, Sp).$$

Proof. By adjunction we have an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{p}},\mathrm{Sp})}(\mathbb{S}[F], G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{p}},\mathscr{S})}(F, \Omega^\infty G),$$

and using that we by lemma 2.17 have the equivalence

$$\mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathscr{S}) \simeq \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathrm{Mon}_{E_\infty}(\mathscr{S})),$$

we further get that this is equivalent to

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty},\mathscr{S})}(F,\Omega^{\infty}G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty},\mathrm{Mon}_{E_{\infty}})}(F,\Omega^{\infty}G).$$

On the other hand, we have that the space of natural transformations $\mathbb{C}\mathrm{ob}^F \Rightarrow G$, of functors $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$, is given by

$$\lim_{n\in\mathbb{N}} \mathrm{Nat}(\Omega^{\infty-n}\mathbb{C}\mathrm{ob}^F, \Omega^{\infty-n}G) \simeq \lim_{n\in N} \mathrm{Nat}(|\mathrm{Cob}^F_n|, \Omega^{\infty-n}G),$$

where the equivalence follows from proposition 2.21, and Nat denotes the space of natural transformations. Using that remark 2.16 tells us that $|\mathrm{Cob}^F_n|$ is the n-fold suspension of F, we get that the colimit system is constant with value $\mathrm{Nat}(F,\Omega^{\infty}G)$, which gives the desired adjunction. □

2.1 Cobordism and Bordism Invariance

To understand cobordisms between Poincaré functors, we will need to define a hermitian ∞-category $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\kappa),(\mathcal{D},\varphi))$. Let $(\mathcal{C},\kappa)$ and $(\mathcal{D},\varphi)$ be two hermitian ∞-categories. Then we set

$$\begin{aligned} \mathrm{nat}^{\varphi}_{\kappa} : \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D})^{op} &\to \mathrm{Sp} \\ f &\mapsto \mathrm{nat}(\kappa, f^*\varphi), \end{aligned}$$

where nat denotes the spectrum of natural transformations between two spectrum valued functors. By [1, 6.2.2] this functor is quadratic, and we denote the resulting hermitian ∞-category by

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\kappa),(\mathcal{D},\varphi)) := (\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D}), \mathrm{nat}^{\varphi}_{\kappa}),$$

and call it the *internal functor category* from $(\mathcal{C},\kappa)$ to $(\mathcal{D},\varphi)$. If both $(\mathcal{C},\kappa)$ and $(\mathcal{D},\varphi)$ are Poincaré ∞-categories then so is $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\kappa),(\mathcal{D},\phi))$ by [1, 6.2.4].

Definition 2.23. Let $(\mathcal{C},\kappa)$ and $(\mathcal{D},\varphi)$ be Poincaré ∞-categories and $f,g : (\mathcal{C},\kappa) \to (\mathcal{D},\varphi)$ two Poincaré functors. We then define a *cobordism* from f to g to be a cobordism in the Poincaré ∞-category $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\kappa),(\mathcal{D},\varphi))$ between the Poincaré objects corresponding to f and g. If there exists a cobordism between f and g we say that they are *cobordant*.

Remark 2.24. We see that a cobordism from f to g is a span

$$f \xleftarrow{\alpha} h \xrightarrow{\beta} g$$

in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$, with α, β natural transformations. Recalling the construction of the Poincaré ∞-category $Q_1(\mathcal{C}, \kappa)$ which has spans of $\mathcal{C}$ as the underlying stable ∞-category, we note that the data of a cobordism from f to g equivalently can be encoded in a Poincaré functor

$$\psi : (\mathcal{C}, \kappa) \to Q_1(\mathcal{D}, \varphi)$$

which satisfies

$$d_0\psi \simeq f, \;\; d_1\psi \simeq g.$$

Definition 2.25. A Poincaré functor $f : (\mathcal{C}, \kappa) \to (\mathcal{D}, \varphi)$ is called a *bordism equivalence* if there exists a Poincaré functor $g : (\mathcal{D}, \varphi) \to (\mathcal{C}, \kappa)$ such that the composites $f \circ g$ and $g \circ f$ are cobordant to the respective identities.

Definition 2.26. Let $\mathcal{E}$ be an ∞-category with finite products. We say that an additive functor $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is *bordism invariant* if it sends bordism equivalences in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ to equivalences in $\mathcal{E}$. We write $\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$ for the full subcategory spanned by these bordism invariant functors.

Remark 2.27. We wish to show that bordism invariant functors vanishes on all metabolic Poincaré ∞-categories. To do so, let $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ be a bordism invariant functor with $\mathcal{E}$ an ∞-category with finite products. We then want to show that for any $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$

$$F(\mathrm{Met}(\mathcal{C}, \kappa)) \simeq 0.$$

Since F in particular is reduced we have that $F((0, \kappa)) = 0$, where $(0, \kappa)$ is the zero-object in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$, so we wish to show that we have a bordism equivalence

$$\mathrm{Met}(\mathcal{C}, \kappa) \rightleftarrows (0, \kappa).$$

First we note that it is clear, that the composite

$$(0, \kappa) \to \mathrm{Met}(\mathcal{C}, \kappa) \to (0, \kappa)$$

is cobordant to the identity, so we only need to consider the other direction, namely the composite

$$\mathrm{Met}(\mathcal{C}, \kappa) \to (0, \kappa) \to \mathrm{Met}(\mathcal{C}, \kappa).$$

We see that this composite is the 0-map, so we want a bordism between

$$\mathrm{Met}(\mathcal{C},\kappa) \underset{0}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Met}(\mathcal{C},\kappa).$$

This cobordism is given by the following diagram which describes a span in $\mathrm{Met}(\mathcal{C},\kappa)$:

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "f"] & c \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ d & \mathrm{fib}(f^*q_\#) \arrow[l, "\sim"'] \arrow[r] & 0 \end{tikzcd}$$

with $q \in \Omega^\infty\kappa(d)$.

It turns out that in the case where the functor $F : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{E}$ is also group-like the converse holds as well.

Lemma 2.28 (II.3.5.4)**.** *Let $F : Cat^p_\infty \to \mathcal{E}$ be a group-like additive functor. Then F is bordism invariant if and only if F vanishes on all metabolic Poincaré ∞-categories.*

We can now use this to show the following very important property of bordism invariant functors.

Proposition 2.29 (II.3.5.8)**.** *Let $\mathcal{E}$ be an ∞-category with finite limits and $F : Cat^p_\infty \to \mathcal{E}$ a bordism invariant functor. Then the natural map*

$$\Omega F(\mathcal{C},\kappa) \to F(\mathcal{C},\kappa^{[-1]})$$

arising from the metabolic Poincaré-Verdier sequence is an equivalence. In particular, it follows that F is automatically group-like.

Proof. Recall the metabolic fibre sequence

$$(\mathcal{C},\kappa^{[-1]}) \xrightarrow{i} \mathrm{Met}(\mathcal{C},\kappa) \xrightarrow{\mathrm{met}} (\mathcal{C},\kappa)$$

from lemma 1.30, which we showed is a Poincaré-Verdier sequence in example 1.39. Since F in particular is additive, we get a fibre sequence

$$F(\mathcal{C},\kappa^{[-1]}) \to F(\mathrm{Met}(\mathcal{C},\kappa)) \to F(\mathcal{C},\kappa).$$

By remark 2.27 we know that since F is bordism invariant, it vanishes on metabolics, hence we get a fibre sequence

$$F(\mathcal{C},\kappa^{[-1]}) \to * \to F(\mathcal{C},\kappa),$$

which exactly gives us the desired equivalence. □

This means that we can deloop bordism invariant functors by simply shifting the Poincaré structure, so for any bordism invariant functor $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ and $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$ we get

$$[F(\mathcal{C}, \kappa), F(\mathcal{C}, \kappa^{[1]}), F(\mathcal{C}, \kappa^{[2]}), ...]$$

which when F is space-valued, gives a spectrum with structure maps those arising from proposition 2.29.

It can be shown that the converse also holds whenever $\mathcal{E}$ is stable.

Proposition 2.30 (II.3.5.8)**.** *Let $F : Cat^p_\infty \to \mathcal{E}$ be an additive functor with $\mathcal{E}$ stable. Then F is bordism invariant if and only if the natural map*

$$\Omega F(\mathcal{C}, \kappa) \to F(\mathcal{C}, \kappa^{[1]})$$

arising from the metabolic Poincaré-Verdier sequence is an equivalence.

Using this we can show that the spectrum structure arising on $\mathbb{C}\mathrm{ob}^F$ through this, agrees with the one described when defining $\mathbb{C}\mathrm{ob}^F$, when $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ is bordism invariant.

Theorem 2.31 (II.3.5.9)**.** *The forgetful functor, given by postcomposing with Ω^∞,*

$$Fun^{bord}(Cat^p_\infty, Sp) \to Fun^{bord}(Cat^p_\infty, \mathscr{S}),$$

is an equivalence with inverse

$$F \mapsto \mathbb{C}ob^F.$$

In particular this means that any additive bordism invariant functor $F : Cat^p_\infty \to \mathscr{S}$ admits an essentially unique lift to another such functor $Cat^p_\infty \to Sp$.

Proof. We first note that it is clear that the essential image of Ω^∞ is contained in bordism invariant functors, hence the functor does indeed exists.

Now, assume that $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ is bordism invariant, then F is group-like by proposition 2.29, so we further get by proposition 2.21 that $\mathbb{C}\mathrm{ob}^F$ is additive and takes values in spectra. We wish to show that this $\mathbb{C}\mathrm{ob}^F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ is bordism invariant. Due to the equivalences

$$\Omega^{\infty-n}\mathbb{C}\mathrm{ob}^F \simeq |\mathrm{Cob}^F_n| \simeq |\mathrm{Cob}^{|\mathrm{Cob}^F|}_{n-1}|,$$

which follows by proposition 2.21 and remark 2.10 respectively, it follows by induction that it is sufficient to show that $|\mathrm{Cob}^F|$ is bordism invariant. To see this, we note that a cobordism of Poincaré functors $(\mathcal{C}, \kappa) \to Q_1(\mathcal{D}, \Phi)$ induces a cobordism

$$Q_n(\mathcal{C}, \kappa) \to Q_n(Q_1(\mathcal{D}, \varphi)) \simeq Q_1(Q_n(\mathcal{D}, \varphi)).$$

Hence a bordism equivalence

$$(\mathcal{C}, \kappa) \to (\mathcal{D}, \varphi)$$

induces an equivalence of simplicial objects

$$FQ(\mathcal{C}, \kappa) \to FQ(\mathcal{D}, \varphi),$$

since F is assumed to be bordism invariant. So we also obtain an equivalence on geometric realization, hence $|\mathrm{Cob}^F|$ is bordism invariant.

This means that the adjunction between Ω^∞ and $\mathbb{C}\mathrm{ob}$ from corollary 2.22 restricts as claimed, and that Ω^∞ is essentially surjective.

We therefore next wish to show that Ω^∞ is fully faithfull, since then we get that it is an equivalence and the inverse is exactly its adjoint. To show this, we show that the counit

$$c : \mathbb{C}\mathrm{ob}^{\Omega^\infty F}(\mathcal{C}, \kappa) \Rightarrow F(\mathcal{C}, \kappa)$$

from the above adjunction, is an equivalence for any $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_\infty$ and any $F \in \mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Sp})$. By proposition 2.21 we know that $F^{\mathrm{grp}}(\mathcal{C}, \kappa) \simeq \Omega^\infty \mathbb{C}\mathrm{ob}^F(\mathcal{C}, \kappa)$, so using that F is group-like by proposition 2.29 we get that $\Omega^\infty c$ is an equivalence. But since both target and domain of c is bordism invariant it follows by proposition 2.30 that $\Omega^{\infty - n} c$ is an equivalence for all $n \geq 0$, and the desired follows. □

There is a canonical way to obtain a bordism invariant functor from an additive one, with target either Sp or $\mathscr{S}$, using the following theorem:

Theorem 2.32 (II. 3.6.1)**.** *The inclusions*

$$\mathit{Fun}^{\mathit{bord}}(\mathit{Cat}^p_\infty, \mathit{Sp}) \subseteq \mathit{Fun}^{\mathit{add}}(\mathit{Cat}^p_\infty, \mathit{Sp})$$
$$\mathit{Fun}^{\mathit{bord}}(\mathit{Cat}^p_\infty, \mathscr{S}) \subseteq \mathit{Fun}^{\mathit{add}}(\mathit{Cat}^p_\infty, \mathscr{S})$$

both admits left and right adjoints. We refer to the left adjoint as the bordification and denote it by $(-)^{\mathit{bord}}$.

Recalling that $(-)^{\mathrm{grp}}$ is the left adjoint to the inclusion

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Grp}_{E_\infty}(\mathscr{S})) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S}),$$

we get that these bordification functors fits into the following commutative square consisting of forgetful functors and their left adjoints, which are denoted by dotted arrows:

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Sp}) \arrow[r, shift right, "(-)^{\mathrm{bord}}"'] \arrow[d, shift right, "\mathbb{C}\mathrm{ob}"'] \arrow[d, phantom, "\sim"] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Sp}) \arrow[l, shift right] \arrow[d, shift right, "\mathbb{C}\mathrm{ob}\circ(-)^{\mathrm{grp}}"'] \\
\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S}) \arrow[r, shift right, "(-)^{\mathrm{bord}}"'] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S}), \arrow[l, shift right]
\end{tikzcd}$$

Due to this diagram, we will often consider bordification of a space valued functor F and the spectrum valued functor $\mathbb{C}\mathrm{ob}^{F^{\mathrm{grp}}}$ as the same case, since we can reconstruct one from the other.

There are several different models for bordification, but the one which is interesting for us is the so called "ad"-construction. For this, we first need to define the ρ-construction. Fix $[n] \in \Delta$ and write $\tau_n := P_0([n])^{op}$ for the opposite ∞-category, of the ∞-category of non-empty posets with maps the inclusions. Given a Poincaré ∞-category $(\mathcal{C}, \kappa)$ we further write

$$\rho_n(\mathcal{C}, \kappa) := (\mathrm{Fun}(\tau_n, \mathcal{C}), \kappa^{\tau_n})$$

for the cotensor of $(\mathcal{C}, \kappa)$ by τ_n. Using that τ_n is functorial in $[n]$ we get that this gives a cosimplicial category which we denote by $\rho(\mathcal{C}, \kappa)$. By [1, 6.6.1] it follows that each $\rho_n(\mathcal{C}, \kappa)$ is Poincaré, and by [1, 6.6.2] it further follows that the hermitian functor $\sigma^* : \rho_n(\mathcal{C}, \kappa) \to \rho_m(\mathcal{C}, \kappa)$ induced by every $\sigma : [m] \to [n]$ in Δ, is Poincaré. Hence we can consider $\rho(\mathcal{C}, \kappa)$ as the following simplicial object in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$

$$\rho(\mathcal{C}, \kappa) : \Delta^{op} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}.$$

Definition 2.33. Let $\mathcal{E}$ be an ∞-category with sifted colimits and $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ some functor. Then we denote by $\mathrm{ad}F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ the functor given by

$$\mathrm{ad}F(\mathcal{C}, \kappa) = |F\rho(\mathcal{C}, \kappa)|.$$

By functoriality of the cotensor construction, we can extend this to a functor

$$\begin{aligned} \mathrm{ad} : \mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Sp}) &\to \mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E}) \\ F &\mapsto \mathrm{ad}F. \end{aligned}$$

By inclusion of vertices we then further obtain a natural transformation $b_F : F \Rightarrow \mathrm{ad}F$.

We want to restrict this ad-functor to just the additive functors into Sp. Using that for any stable ∞-category $\mathcal{E}$, the collection of additive functors $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is closed under colimits when considered in $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{E})$, and the fact that by [2, 1.4.14] we get that $\rho_n : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$ preserves split Poincaré-Verdier sequences, it follows that $\mathrm{ad}F$ is additive whenever $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ is. Hence we can consider ad as a functor

$$\mathrm{ad} : \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Sp}).$$

We want to show that ad is a model for the bordification. To do so, we need the following lemma.

Lemma 2.34 (II.3.6.9)**.** *Let $B : Fun^{add}(Cat^{p}_{\infty}, Sp) \to Fun^{add}(Cat^{p}_{\infty}, Sp)$ be a functor equipped with a natural transformation $\beta : id \Rightarrow B$, and assume the following holds:*

1. *B commutes with colimits,*
2. *If $F \in Fun^{bord}(Cat^{p}_{\infty}, Sp)$, then $\beta_F : F \Rightarrow BF$ is an equivalence,*
3. *$B(F^{hyp}) \simeq 0$ for all $F \in Fun^{add}(Cat^{p}_{\infty}, Sp)$.*

Then β exhibits B as a bordification functor.

Proof. Let $F : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ be an additive functor. From [2, 3.6.7] we get a fibre sequence

$$F^{\mathrm{hyp}}_{hC_2} \to F \to F^{\mathrm{bord}},$$

so by applying B we get a commutative rectangle

$$\begin{tikzcd} F^{\mathrm{hyp}}_{hC_2} \arrow[r] \arrow[d, "\beta_{F^{\mathrm{hyp}}_{hC_2}}"] & F \arrow[r] \arrow[d, "\beta_F"] & F^{\mathrm{bord}} \arrow[d, "\beta_{F^{\mathrm{bord}}}"] \\ B(F^{\mathrm{hyp}}_{hC_2}) \arrow[r] & B(F) \arrow[r] & B(F^{\mathrm{bord}}), \end{tikzcd}$$

where both rows are bifibre sequences. Using that F^{bord} is bordism invariant, we get by assumption (2) that $\beta_{F^{\mathrm{bord}}}$ is an equivalence. This implies that the left square is

bicartesian. By assumption (1) and (3) we further get that $B(F^{\mathrm{hyp}}_{hC_2}) \simeq 0$, so the map

$$B(F) \to B(F^{\mathrm{bord}})$$

is an equivalence as well. So we get

$$B(F) \xrightarrow{\simeq} B(F^{\mathrm{bord}}) \xleftarrow{\simeq} F^{\mathrm{bord}},$$

hence B is equivalent to the bordification under the identity of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{Sp})$. □

We now finish this section by arguing why ad is indeed a model for bordification.

Proposition 2.35 (II.3.6.13)**.** *The natural transformation b exhibits $ad : Fun^{add}(Cat^{p}_{\infty}, Sp) \to Fun^{add}(Cat^{p}_{\infty}, Sp)$ as a bordification functor.*

Proof. It is sufficient for us to show that b and ad satiesfies the three assumptions of lemma 2.34. The first one is clear and the last two follows from [2, 3.6.12]. □

3 Groethendieck-Witt and L-theory Spectra

We have now finished the preliminary technical notions which is necessary for us to start considering the main objects of interest in this project, namely the Groethendieck-Witt spectrum and the L-theory spectrum on Poincaré ∞-categories. Firt we wish to define the Groethendieck-Witt spectrum, for which we will need to define the Groethendieck-Witt space of a Poincaré ∞-category.

Definition 3.1. We define the *Groethendieck-Witt space* functor $\mathcal{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Grp}_{E_\infty}$ to be the group-complection of the additive functor $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ inside $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathscr{S})$

$$\mathcal{GW}(\mathcal{C}, \kappa) := \mathrm{Pn}^{\mathrm{grp}}(\mathcal{C}, \kappa).$$

We further define the *Grothendieck-Witt groups* of a Poincaré ∞-category $(\mathcal{C}, \kappa)$ as

$$\mathrm{GW}_i(\mathcal{C}, \kappa) := \pi_i \mathcal{GW}(\mathcal{C}, \kappa).$$

Note that $\mathcal{GW}$ is not the levelwise group-completion, i.e. $\mathcal{GW}(\mathcal{C}, \kappa)$ is not in general the group-completion of $\mathrm{Pn}(\mathcal{C}, \kappa)$.

Remark 3.2. Using the universal property of group-completion it follows directly that $\mathcal{GW} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ is additive and group-like, and furthermore it is the initial such functor under $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$.

Using that corollary 2.22 gives us that the inclusion

$$\Omega^\infty : \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{E_\infty})$$

admits $\mathbb{C}\mathrm{ob}$ as left adjoint, together with the fact that $\mathcal{GW} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Sp}$ is additive, we get that we can extend the above definition to spectra.

Definition 3.3. We define the *Groethendieck-Witt spectrum* functor $\mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Sp}$ by assigning

$$\mathrm{GW}(\mathcal{C}, \kappa) := \mathbb{C}\mathrm{ob}^{\mathcal{GW}}(\mathcal{C}, \kappa),$$

and we write $\mathrm{GW}_i(\mathcal{C}, \kappa) := \pi_i \mathrm{GW}(\mathcal{C}, \kappa)$ for the homotopy groups.

This means that $\mathrm{GW}(\mathcal{C}, \kappa)$ can be thought of as the spectrum

$$[\mathcal{GW}(\mathcal{C}, \kappa), |\mathrm{Cob}^{\mathcal{GW}}_1(\mathcal{C}, \kappa)|, |\mathrm{Cob}^{\mathcal{GW}}_2(\mathcal{C}, \kappa)|, ...].$$

We will in a moment show in corollary 3.5 that the definition of GW_i as the Groethendieck-witt groups in definition 3.1 and as the homotopy groups of $\mathrm{GW}(\mathcal{C}, \kappa)$ in definition 3.3 agrees, but we will first note some consequences from this definition.

Corollary 3.4 (II.4.2.2)**.** *The Groethendieck-Witt spectrum functor $GW : Cat^p_\infty \to Sp$ is additive, and is the initial such functor that is equipped with a natural transformation*

$$Pn \Rightarrow \Omega^\infty GW$$

of functors $Cat^p_\infty \to \mathscr{S}$.

Proof. We first note that by the universal property of the iterated hermitian Q-construction given in corollary 2.22, we know that GW is the initial additive spectra valued functor which is equipped with a natural transformation $\mathcal{GW} \Rightarrow \Omega^\infty \mathrm{GW}$ of E_∞-groups. The desired result then follows by combining this with the universal property of $\mathcal{GW}$ established in remark 3.2. □

Corollary 3.5 (II.4.2.3)**.** *Let $(\mathcal{C}, \kappa) \in Cat^p_\infty$. We then have the following canonical equivalences for any $i \geq 1$, natural in $(\mathcal{C}, \kappa)$:*

$$\mathcal{GW}(\mathcal{C}, \kappa) \simeq \Omega^\infty GW(\mathcal{C}, \kappa) \ \ and \ \ |Cob_i(\mathcal{C}, \kappa)| \simeq \Omega^{\infty - i} GW(\mathcal{C}, \kappa).$$

In particular we obtain isomorphisms

$$\pi_i GW(\mathcal{C}, \kappa) \cong GW_i(\mathcal{C}, \kappa).$$

Proof. Follows immediately from proposition 2.21. □

We are now ready to construct the L-theory space as well as the L-theory spectrum in a way similar to how we defined the Groethendieck-Witt space and spectrum above. We then finally wish to prove the main theorem of this project, namely that the L-theory spectrum is the bordification of GW.

Definition 3.6. We define the *L-theory space* functor $\mathcal{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ as the geometric realization of the ρ-construction applied to Pn, i.e.

$$\mathcal{L}(\mathcal{C}, \kappa) = |\mathrm{Pn}\rho(\mathcal{C}, \kappa)|.$$

Before we move on and extend this definition to spectra, we wish to consider some of the important properties it which it admits. To do so we first state the following property, which is necessary to prove an important corollary.

Proposition 3.7 (II.4.4.2)**.** *Given a Poincaré-Verdier sequence*

$$(\mathcal{C}', \kappa') \to (\mathcal{C}, \kappa) \to (\mathcal{C}'', \kappa''),$$

the induced functor

$$Pn(\rho(\mathcal{C}, \kappa)) \to Pn(\rho(\mathcal{C}'', \kappa''))$$

is a Kan fibration of simplicial spaces with fibre $Pn(\rho(\mathcal{C}', \kappa'))$.

Corollary 3.8. *The L-theory space functor* $\mathcal{L} : Cat^{p}_{\infty} \to \mathscr{S}$ *is additive.*

Proof. We first need to show that $\mathcal{L}$ is reduced, which follows by the observation that ρ_n of $(\mathcal{C} = *, \kappa : \mathcal{C}^{op} \to \mathrm{Sp}$ s.t. $\kappa(*) = 0)$, which is the zero-object in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$, will always just be the trivial Poincaré ∞-category for any n. To further see that $\mathcal{L}$ indeed is additive we use proposition 3.7 together with the fact that geometric realization takes Kan fibrations with fibre to cartesian squares. □

Remark 3.9. We wish to describe $\pi_0\mathcal{L}(\mathcal{C}, \kappa)$ more explicit, and to do so we first note that when we consider $\mathcal{L}(\mathcal{C}, \kappa)$ as a simplicial space, i.e. before applying the geometric realization, it is Kan for all $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$ by proposition 3.7. Using that $\rho_0(\mathcal{C}, \kappa) \simeq (\mathcal{C}, \kappa)$, so $\mathcal{L}(\mathcal{C}, \kappa)_0 \simeq \mathrm{Pn}(\mathcal{C}, \kappa)$, we get that

$$\begin{aligned}\pi_0\mathcal{L}(\mathcal{C},\kappa) &= \mathrm{coeq}(d_0, d_1 : \pi_0\mathcal{L}(\mathcal{C},\kappa)_1 \to \pi_0\mathcal{L}(\mathcal{C},\kappa)_0)\\ &\simeq \pi_0\mathcal{L}(\mathcal{C},\kappa)_0/\sim\\ &\simeq \pi_0\mathrm{Pn}(\mathcal{C},\kappa)/\sim,\end{aligned}$$

where two Poincaré objects $(c,q),(c',q') \in \mathrm{Pn}(\mathcal{C},\kappa)$ are equivalent when there exists an $x \in \pi_0(\mathrm{Pn}(\mathcal{C},\kappa))$ such that

$$d_0 x = (c,q), \quad d_1 x = (c',q'),$$

so exactly when they are cobordant. Hence we can consider

$$\pi_0\mathcal{L}(\mathcal{C},\kappa) = \text{ forms/cobordism }.$$

Proposition 3.10 (II.3.5.5)**.** *The L-theory space functor $\mathcal{L} : Cat^p_\infty \to \mathscr{S}$ is bordism invariant.*

To prove this proposition we will need the following lemma, which in itself is a rather technical result, and the proof will therefore be omitted.

Lemma 3.11 ([5] Lecture 7, prop. 9)**.** *For any $(\mathcal{C},\kappa) \in Cat^p_\infty$ we have*

$$\pi_n\mathcal{L}(\mathcal{C},\kappa) \cong \pi_0\mathcal{L}(\mathcal{C},\kappa^{[-n]}).$$

Proof of proposition 3.10. We first claim that cobordant functors between Poincaré ∞-categories induce equivalent maps on the homotopy groups of the L-theory spaces. To see this, we first consider the maps

$$d_0, d_1 : Q_1(\mathcal{C},\kappa) \to (\mathcal{C},\kappa),$$

which sends a span $x \leftarrow y \to z$ to x or z respectively. We see that when applying $\pi_n\mathcal{L}(-)$ to these, the class of the above span in $\pi_n\mathcal{L}(Q_1(\mathcal{C},\kappa))$ is mapped to classes which are cobordant, hence equivalent by lemma 3.11 and remark 3.9. So we get that d_0 and d_1 induce the same map on the homotopy groups of the L-theory spaces.

Now, let $f,g : (\mathcal{C},\kappa) \to (\mathcal{D},\varphi)$ be cobordant Poincaré functors. By remark 2.24 we can equivalently describe this as a Poincaré functor

$$H : (\mathcal{C},\kappa) \to Q_1(\mathcal{D},\varphi)$$

which satisfies that

$$d_0 H \simeq f, \;\; d_1 H \simeq g.$$

Combining this with the above observation, we get that

$$\begin{aligned} \pi_n \mathcal{L}(f) &\simeq \pi_n \mathcal{L}(d_0 H) \\ &\simeq \pi_n \mathcal{L}(d_1 H) \\ &\simeq \pi_n \mathcal{L}(g), \end{aligned}$$

hence any cobordant functors between Poincaré ∞-categories induces equivalent maps on the homotopy groups of the L-theory spaces.

Now, assume that $f : (\mathcal{C}, \kappa) \to (\mathcal{D}, \phi)$ is a bordism equivalence between Poincaré ∞-categories, with inverse g. Per definition this mean that $f \circ g$ and $g \circ f$ are corbordant to the respective identities. Hence by the above we get that both composites induces identities on $\pi_n \mathcal{L}$, so f induces an isomorphism with inverse induced by g. From this we conclude that $\pi_n \mathcal{L}(f)$ is an isomorphism for all n, so $\mathcal{L}(f)$ is an equivalence as desired. □

Using that we now know that the L-theory space functor $\mathcal{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathscr{S}$ is additive and bordism invariant, it follows from theorem 2.31 that it admits an essentially unique lift to an additive bordism invariant functor $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$. This is the reasoning behind the following definition.

Definition 3.12. We define the *L-theory spectrum* functor $\mathrm{L} : \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Sp}$ by

$$\mathrm{L}(\mathcal{C}, \kappa) := \mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C}, \kappa),$$

for $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$. We further write $\mathrm{L}_i(\mathcal{C}, \kappa) := \pi_i \mathrm{L}(\mathcal{C}, \kappa)$ for the homotopy groups.

We note that for any $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_{\infty}$ we can more explicitly write the L-theory spectrum as

$$[\mathcal{L}(\mathcal{C}, \kappa), |\mathrm{Cob}^{\mathcal{L}}_1(\mathcal{C}, \kappa)|, |\mathrm{Cob}^{\mathcal{L}}_2(\mathcal{C}, \kappa)|, ...].$$

Using that $\mathcal{L}$ is bordism invariant by proposition 3.10 it follows by theorem 2.31 that we equivalently can consider this as the spectrum

$$[\mathcal{L}(\mathcal{C}, \kappa), \mathcal{L}(\mathcal{C}, \kappa^{[1]}), \mathcal{L}(\mathcal{C}, \kappa^{[2]}), ...],$$

using the structure maps arising in proposition 2.29.

We now see the following basic properties of the L-theory spectrum functor.

Corollary 3.13 (II.4.4.5)**.** *For any $(\mathcal{C}, \kappa) \in Cat^p_\infty$ there are canonical equivalences*

$$\Omega^{\infty - i} L(\mathcal{C}, \kappa) \simeq \mathcal{L}(\mathcal{C}, \kappa^{[i]})$$

for all $i \in \mathbb{Z}$. In particular we get isomorphisms

$$L_i(\mathcal{C}, \kappa) \cong L_0(\mathcal{C}, \kappa^{[-i]}),$$

also for negative i.

Proof. Follows directly by proposition 2.21 and proposition 2.29. □

As an almost immediate consequence of this, we get the following.

Corollary 3.14 (II.4.4.6)**.** *The L-theory spectrum functor $L : Cat^p_\infty \to Sp$ is additive and bordism invariant.*

Proof. We first note that $\mathcal{L} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathscr{S}$ is additive by corollary 3.8 which implies that $L = \mathbb{C}\mathrm{ob}^{\mathcal{L}}$ is additive by proposition 2.21. To see that L is bordism invariant, we note that by corollary 3.13 we have that for any $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_\infty$,

$$\begin{aligned} \pi_i \mathrm{L}(\mathcal{C}, \kappa) &\simeq \pi_0 \Omega^{\infty + i} \mathrm{L}(\mathcal{C}, \kappa) \\ &\simeq \pi_0 \mathcal{L}(\mathcal{C}, \kappa^{[-i]}), \end{aligned}$$

so the desired follows by proposition 3.10. □

Remark 3.15. Using the definition of $\mathcal{L}(\mathcal{C}, \kappa)$ together with the fact that $\rho_0(\mathcal{C}, \kappa) \simeq (\mathcal{C}, \kappa)$ and geometric realization is just a specific kind of colimit, we get a natural transformation $\mathrm{Pn} \Rightarrow \mathcal{L}$. Noting that corollary 3.13 gives us an equivalence $\Omega^\infty \mathrm{L} \simeq \mathcal{L}$, we therefore have a natural transformation

$$\mathrm{Pn} \Rightarrow \Omega^\infty \mathrm{L} \simeq \mathcal{L}. \tag{4}$$

Since we have showed in corollary 3.14 that $L : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Sp}$ is additive it then follows from corollary 3.4, which tells us that $\mathrm{GW} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Sp}$ is the initial additive functor with a natural transformation $\mathrm{Pn} \Rightarrow \Omega^\infty \mathrm{GW}$, that (4) extends to a natural transformation

$$\mathrm{bord} : \mathrm{GW} \Rightarrow \mathrm{L}$$

of functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Sp}$.

We are finally ready to turn out focus to the main theorem of this project.

Theorem 3.16 (II.4.4.12)**.** *The natural transformation bord : $GW \Rightarrow L$ exhibits L as the bordification of GW.*

In particular,

$$L : Cat^p_\infty \to Sp$$

is the initial bordism invariant, additive functor equipped with a natural transformation

$$Pn \Rightarrow \Omega^\infty L \simeq \mathcal{L}$$

of functors $Cat^p_\infty \to \mathscr{S}$.

Proof. Throughout let $(\mathcal{C}, \kappa) \in \mathrm{Cat}^{\mathrm{p}}_\infty$. We first note that $\rho_n Q_m(\mathcal{C}, \kappa) \simeq Q_m \rho_n(\mathcal{C}, \kappa)$, since

$$\mathrm{Fun}(\tau_n, \mathrm{Fun}(\mathrm{TwAr}(\Delta^m), \mathcal{C})) \simeq \mathrm{Fun}(\mathrm{TwAr}(\Delta^m), \mathrm{Fun}(\tau_n, \mathcal{C})),$$

so we get that

$$\rho Q^{(m)}(\mathcal{C}, \kappa) \simeq Q^{(m)} \rho(\mathcal{C}, \kappa).$$

This will be used to show that $\mathrm{L}(\mathcal{C}, \kappa)$ can be identified with $|\mathrm{GW}\rho(\mathcal{C}, \kappa)|$. First we see that

$$\begin{aligned} \mathrm{L}(\mathcal{C}, \kappa) :&= \mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C}, \kappa) \\ &= [\mathrm{Cob}^{\mathcal{L}}_0(\mathcal{C}, \kappa), |\mathrm{Cob}^{\mathcal{L}}_1(\mathcal{C}, \kappa)|, |\mathrm{Cob}^{\mathcal{L}}_2(\mathcal{C}, \kappa)|, ...] \end{aligned}$$

where each term is given by

$$\begin{aligned} \mathrm{Cob}^{\mathcal{L}}_n(\mathcal{C}, \kappa) :&= \mathcal{L}(Q^{(n)}(\mathcal{C}, \kappa^{[n]})) \\ &= |\mathrm{Pn}\rho(Q^{(n)}(\mathcal{C}, \kappa^{[n]}))|. \end{aligned}$$

Using that $\mathcal{L}$ is bordism-invariant by proposition 3.10, hence group-like by proposition 2.29, it follows by proposition 2.21 that $\mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C}, \kappa)$ is equivalent to its spectrification, which is given by $\mathbb{C}\mathrm{ob}^{\mathcal{L}^{\mathrm{grp}}}(\mathcal{C}, \kappa)$.

Using proposition 2.21 again, we have that $\mathrm{GW} = \mathbb{C}\mathrm{ob}^{\mathcal{GW}}(\mathcal{C}, \kappa)$ is the spectrification of $\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\mathcal{C}, \kappa)$, using that $\mathcal{GW} = \mathrm{Pn}^{\mathrm{grp}}$. Therefore it is sufficient for us to show that $\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\rho(\mathcal{C}, \kappa))$ is equivalent to $|\mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C}, \kappa)|$ considered as prespectra. By unravelling the definition of $\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\rho(\mathcal{C}, \kappa))$ we see that

$$\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\rho(\mathcal{C},\kappa)) = [\mathrm{Cob}_0^{\mathrm{Pn}}(\rho(\mathcal{C},\kappa)), |\mathrm{Cob}_1^{\mathrm{Pn}}(\rho(\mathcal{C},\kappa))|, |\mathrm{Cob}_2^{\mathrm{Pn}}(\rho(\mathcal{C},\kappa))|, ...],$$

where each term is given by

$$\begin{aligned}\mathrm{Cob}_n^{\mathrm{Pn}}(\rho(\mathcal{C},\kappa)) &= \mathrm{Pn}(Q^{(n)}(\rho(\mathcal{C},\kappa^{[n]}))) \\ &= \mathrm{Pn}(\rho(Q^{(n)}(\mathcal{C},\kappa^{[n]}))),\end{aligned}$$

so we get that $\mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C},\kappa)$ and $|\mathbb{C}\mathrm{ob}^{\mathrm{Pn}}(\rho(\mathcal{C},\kappa))|$ are equivalent as prespectra as desired, hence

$$\mathrm{L}(\mathcal{C},\kappa) \simeq |\mathrm{GW}(\rho(\mathcal{C},\kappa))|$$

as spectra. So we get the identification

$$\mathrm{L} = |\mathrm{GW}\rho| = \mathrm{ad}(\mathrm{GW}).$$

By proposition 2.35 we know that ad is a bordification functor, hence the first part of the theorem holds.

The second part follows directly from corollary 3.4. □

Corollary 3.17. *The L-theory space functor $\mathcal{L} : Cat^p_\infty \to \mathscr{S}$ is the initial bordism invariant additive functor under either Pn or $\mathcal{GW}$.*

Proof. Follows by theorem 3.16 and remark 3.2. □

Bibliography

[1] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. Hermitian k-theory for stable ∞-categories i: Foundations. arXiv:2009.07223, 2020.

[2] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. Hermitian k-theory for stable ∞-categories ii: Cobordism categories and additivity. arXiv:2009.07224, 2020.

[3] A. Joyal and M. Tierney. Quasi-categories vs segal spaces. arXiv:math/0607820, 2006.

[4] J. Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. ISBN 9780691140490.

[5] J. Lurie. Algebraic l-theory and surgery (287x). Lecture notes from course `https://www.math.ias.edu/~lurie/287x.html`, 2011.

[6] J. Lurie. Higher algebra. 2017.

[7] J. D. Quigley and J. Shah. On the parametrized tate construction and two theories of real p-cyclotomic spectra, 2019.