Goodwillie Calculus - Talk notes

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The goal of this talk is to prove the following proposition from [Kuh03][Proposition 1.9]:

Theorem 0.1. *Let $F : Sp \to Sp$ be any homotopy functor. For all $n \geq 1$ there is a homotopy pullback diagram*

$$\begin{tikzcd} P_nF(X) \arrow[r] \arrow[d] & (\Delta_n F(X))_{h\Sigma_n} \arrow[d] \\ P_{n-1}F(X) \arrow[r] & (\Delta_n F(X))^{t\Sigma_n}. \end{tikzcd}$$

This diagram is natural in both X and F.

Before we try to prove this theorem, we will first consider some of the constructions in this theorem and explain why the statement even makes sense.

Throughout this, let $F : Sp \to Sp$ be any homotopy functor where Sp denotes the stable infinity category of spectra. Let $n \geq 1$ and $\bar{n} = \{1, ..., n\}$. Then we can first of all define the *n'th cross effect* cr_nF of F by

$$\begin{aligned} cr_nF &: Sp^n \to Sp \\ (cr_nF)(X_1, ..., X_n) &= Tfib_{T \subset \bar{n}} F(\vee_{i \in \bar{n} \setminus T} X_i). \end{aligned}$$

Another construction which is needed is that

$$\begin{aligned} \Delta_nF &: Sp \to Sp^{\Sigma_n} \\ \Delta_nF(X) &= P_{(1,...,1)}(cr_nF)(X, ..., X). \end{aligned}$$

The definition of the n'th cross-effect let's us state two theorems from Goodwillies paper which will be used a couple of times:

Theorem 0.2 (Theorem 3.5)**.** *Let $L : Sp^n \to Sp$ be a multilinear functor, then we can define*

$$(\rho_n L)(X) = L(X, ..., X)_{h\Sigma_n},$$

which are mutual inverse to the n'th cross effect

$$\mathcal{H}_n(Sp, Sp) \underset{\rho_n}{\overset{cr_n}{\rightleftarrows}} \mathcal{L}_n(Sp, Sp).$$

Theorem 0.3 (Theorem 6.1)**.** *For any homotopy functor $F : Sp \to Sp$ we have*

$$D^{(n)}F = cr_n D_n F \simeq P_{(1,...,1)}(cr_n F).$$

Using Theorem 6.1 in [Goo03] we get that

$$\begin{aligned} \Delta_n F(X)_{h\Sigma_n} &= (P_{(1,...,1)}(cr_n F)(X, ..., X))_{h\Sigma_n} \\ &\simeq (D^{(n)}F)(X, ..., X)_{h\Sigma_n} \\ &= (cr_n D_n F)(X, ..., X)_{h\Sigma_n}. \end{aligned}$$

Since $\Delta_n F \to \Delta_n P_n F$ always is an equivalence, we may assume that F is n-excisive, then $D_n F$ is n-homogeneous so Theorem 3.5 in [Goo03] gives us that

$$(cr_n D_n F)(X, ..., X)_{h\Sigma_n} = (\rho_n cr_n D_n F)(X) \simeq D_n F(X).$$

We will also need to consider the dual constructions of the above constructions. First of all we have the *dual n'th cross effect* defined by

$$\begin{aligned} cr^n F &: Sp^n \to Sp \\ (cr^n F)(X_1, ..., X_n) &= Tcofib_{T \subset \bar{n}} F(\prod_{i \in T} X_i), \end{aligned}$$

which is then again used to define

$$\begin{aligned} \Delta^n F &: Sp \to Sp^{\Sigma_n} \\ \Delta^n F(X) &= P_{(1,...,1)}(cr^n F)(X, ..., X). \end{aligned}$$

We have a natural transformation between the two cross effects $cr_n F \Rightarrow cr^n F$ given by the composition

$$cr_n F(X_1, ..., X_n) \to F(X_1 \vee \cdots \vee X_n) \xrightarrow{\simeq} F(X_1 \times \cdots X_n) \to cr^n F(X_1, .., X_n),$$

where the first morphism is the unique morphism from the total fiber, the second morphism is an equivalence since both Sp and Sp^{Σ_n} are stable, and the last morphism is the unique morphism to the total fiber.

Lemma 0.4. *The natural transformation $cr_n F \Rightarrow cr^n F$ described above is a weak equivalence.*

Proof. We will only sketch the case $n = 2$. Let $f : F \Rightarrow G$ be a natural transformation between functors $F, G : Sp \to Sp$. Then we have a diagram

$$\begin{tikzcd} cr_2 F(X_1, X_2) \arrow[r] \arrow[d, "cr_2(f)_{X_1,X_2}"] & F(X_1 \vee X_2) \arrow[r, "\simeq"] \arrow[d, "f_{X_1 \vee X_2}"] & F(X_1 \times X_2) \arrow[r] \arrow[d, "f_{X_1 \times X_2}"] & cr^2 F(X_1, X_2) \arrow[d, "cr^2(f)_{X_1,X_2}"] \\ cr_2 G(X_1, X_2) \arrow[r] & G(X_1 \vee X_2) \arrow[r, "\simeq"] & G(X_1 \times X_2) \arrow[r] & cr^2 G(X_1, X_2), \end{tikzcd}$$

where the middle square commutes. This shows that it is sufficient to prove that both

$$cr_2F \xrightarrow{a} F(- \vee -) \text{ and } F(- \times -) \xrightarrow{b} cr^2F$$

are natural transformations. We will only sketch the proof for a, the proof for b is dual. We need to show that

$$\begin{tikzcd}
cr_2F(X_1,X_2) \arrow[r, "cr_2((X_1,X_2)\to(Y_1,Y_2))"] \arrow[d, "a_{X_1,X_2}"] & cr_2F(Y_1,Y_2) \arrow[d, "a_{Y_1,Y_2}"] \\
F(X_1 \vee X_2) \arrow[r, "F((X_1\to Y_1)\vee(X_2\to Y_2))"] & F(Y_1 \vee Y_2)
\end{tikzcd}$$

commutes. This follows by considering the following diagram, where we have used the universal property of pullback diagrams, and that we are in a stable category:

$$\begin{tikzcd}
 & cr_2F(X_1,X_2) \arrow[dl, dotted] \arrow[dr] & & & & & \\
cr_2F(Y_1,Y_2) \arrow[dr] & & F(X_1 \vee X_2) \arrow[rrrr] \arrow[dl, dotted] \arrow[dddd] \arrow[ddrr] & & & & F(X_1) \arrow[ddll, bend right] \arrow[dl, dotted] \arrow[dddd] \\
 & F(Y_1 \vee Y_2) \arrow[rrrr] \arrow[dddd] \arrow[ddrr] & & & & F(Y_1) \arrow[ddll, bend left] \arrow[dddd] & \\
 & & & & F(X_1) \times F(X_2) \arrow[dl, dotted] & & \\
 & & & F(Y_1) \times F(Y_2) & & & \\
 & & F(X_2) \arrow[uurr, bend left] \arrow[rrrr] \arrow[dl, dotted] & & & & * \arrow[dl, dotted] \\
 & F(Y_2) \arrow[uurr, bend right] \arrow[rrrr] & & & & * &
\end{tikzcd}$$

$\square$

Using this result, we get that the natural transformation

$$\Delta_n F \Rightarrow \Delta^n F$$

is also a weak equivalence. Noting that if we assume that F is n-excisive, then $\Delta^n F(X)$ may be identified with $(cr^n F)(X, ..., X)$, so by the definition of cr^n (the totalcofiber), we have a unique morphism $F(X^n) \to (cr^n F)(X, ..., X)$. We use this to define a weak natural transformation

$$\alpha^n(X) : F(X) \to (\Delta_n F)(X)^{h\Sigma_n}$$

as the composition

$$F(X) \xrightarrow{F(X \to X^n)} F(X^n)^{h\Sigma_n} \to (\Delta^n F)(X)^{h\Sigma_n} \overset{\simeq}{\leftarrow} (\Delta_n F)(X)^{h\Sigma_n}.$$

We will use the following lemma, which will be stated without proof (Lemma 5.1 in [Kuh03]):

Lemma 0.5. *Let $F : Sp \to Sp$ be n-excisive, then*

1. *$cr^n(\alpha^n)$, and thus $D_n(\alpha^n)$, is an equivalence,*
2. *There is a natural equivariant weak equivalence*

$$cr^n(\Delta^n F) \simeq Map_{Sp}(\Sigma_{n+}, cr^n F).$$

Here we need to assume that F is n-excisive, to use α^n. This lemma will be used to prove the following lemma (Lemma 5.2 in [Kuh03]), which is the main tool for proving our main theorem.

Lemma 0.6. *Let F be n-excisive. Then $(\Delta_n F)^{t\Sigma_n}$ is $(n-1)$-excisive, and the cofibration sequence*

$$D_n((\Delta_n F)^{h\Sigma_n}) \to P_n((\Delta_n F)^{h\Sigma_n}) \to P_{n-1}((\Delta_n F)^{h\Sigma_n})$$

identifies with the norm fibration sequence

$$(\Delta_n F)_{h\Sigma_n} \to (\Delta_n F)^{\Sigma_n} \to (\Delta_n F)^{t\Sigma_n}.$$

A main tool for proving this lemma is the following lemma:

Lemma 0.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two stable ∞-categories. Then the category $Exc_n(\mathcal{C}, \mathcal{D})$ is closed under finite limits and finite colimits.*

Abstract proof. The inclusion functor

$$Exc^n(\mathcal{C}, \mathcal{D}) \hookrightarrow Fun(\mathcal{C}, \mathcal{D})$$

is left exact (preserves finite limits) hence right exact (preserves finite colimits) so exact. That gives us the desired. □

Sketch of intuitive proof for $n = 2$. The main arguments is again that limits commutes with limits. Let $F : I \to Fun(\mathcal{C}, \mathcal{D})$ such that F is 2-excisive. Then we wish to prove that $lim_{i \in I} F_i$ is 2-excisive again. So let a 2-dimensional strongly cocartesian cube

$$\begin{tikzcd} X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\ X_{21} \arrow[r] & X_{22} \end{tikzcd}$$

be given. Then we want

$$\begin{array}{ccc} lim_{i\in I}(F_i(X_{11})) & \longrightarrow & lim_{i\in I}(F_i(X_{12})) \\ \downarrow & & \downarrow \\ lim_{i\in I}(F_i(X_{21})) & \longrightarrow & lim_{i\in I}(F_i(X_{22})) \end{array}$$

to be a limit diagram. We know that each diagram

$$\begin{array}{ccc} F_i(X_{11}) & \longrightarrow & F_i(X_{12}) \\ \downarrow & & \downarrow \\ F_i(X_{21}) & \longrightarrow & F_i(X_{22}) \end{array}$$

is a limit digram, so since limits commutes with limits, the desired diagram is indeed a limit diagram. □

Proof of lemma 0.6. First we wish to prove that $(\Delta_n F)^{t\Sigma_n}$ is $(n-1)$-excisive. We know that $(\Delta_n F)$ is $(1,...,1)$-excisive, so it's a result that it then is n-excsisive, so since the tate-construction is a colimit it follows by lemma 0.7 that $(\Delta_n F)^{t\Sigma_n}$ is n-excisive as well. That means $P_n((\Delta_n F)^{t\Sigma_n}) \simeq (\Delta_n F)^{t\Sigma_n}$. We have the cofibration sequence

$$D_n((\Delta_n F)^{t\Sigma_n}) \to P_n((\Delta_n F)^{t\Sigma_n}) \simeq (\Delta_n F)^{t\Sigma_n} \to P_{n-1}((\Delta_n F)^{t\Sigma_n}),$$

so to show $P_{n-1}((\Delta_n F)^{t\Sigma_n}) \simeq (\Delta_n F)^{t\Sigma_n}$ it is sufficient to prove that $D_n((\Delta_n F)^{t\Sigma_n} \simeq *$. Using that $\Delta_n G(X)_{h\Sigma_n} \simeq D_n G(X)$ for any n-excisive functor $G : Sp \to Sp$, we have that

$$D_n((\Delta_n F)^{t\Sigma_n}) \simeq P_{(1,...,1)}(cr_n((\Delta_n F)^{t\Sigma_n})_{h\Sigma_n},$$

so if we show that $cr_n((\Delta_n F)^{t\Sigma_n}) \simeq *$ it will follow that $D_n((\Delta_n F)^{t\Sigma_n}) \simeq *$. Since we have a weak natural equivalence between the n'th cross effect and it's dual, it is sufficent that we prove that $cr^n((\Delta_n F)^{t\Sigma_n}) \simeq *$:

$$cr^n((\Delta_n F)^{t\Sigma_n}) \simeq (cr^n(\Delta_n F))^{t\Sigma_n} \simeq (Map_{Sp}(\Sigma_{n+}, cr^n F))^{t\Sigma_n} \simeq *.$$

Here the first equivalence follows from limits commuting with limits, the second equivalence follows by lemma 0.5(2) and the third equivalence follows from Lemma 2.5 in [Kuh03] which says that since Σ_{n+} is a finite free Σ_n-CW complex we get that exact equivalence. This proves that $(\Delta_n F)^{t\Sigma_n}$ is $(n-1)$-excisive.

Now, let's prove the second statement. Using that $(n-1)$-excisive approsixmation preserves cofiber sequences in spectra, we get

$$\begin{array}{ccccc} (\Delta_n F)_{h\Sigma_n} & \xrightarrow{Nm_{\Sigma_n}} & (\Delta_n F)^{h\Sigma_n} & \longrightarrow & (\Delta_n F)^{t\Sigma_n} \\ \downarrow & & \downarrow & & \downarrow \\ P_{n-1}((\Delta_n F)_{h\Sigma_n}) & \longrightarrow & P_{n-1}((\Delta_n F)^{h\Sigma_n}) & \longrightarrow & P_{n-1}((\Delta_n F)^{t\Sigma_n}) \end{array}$$

consisting of two cofiber sequences. Using that $(\Delta_n F)_{h\Sigma_n}$ is n-homogenous we get that the fiber $P_{n-1}((\Delta_n F)_{h\Sigma_n}) \simeq *$ is trivial, hence

$$P_{n-1}((\Delta_n F)^{h\Sigma_n}) \xrightarrow{\simeq} P_{n-1}((\Delta_n F)^{t\Sigma_n})$$

is an equivalence. We have just proved that $(\Delta_n F)^{t\Sigma_n}$ is $(n-1)$-excisive, hence $(\Delta_n F)^{t\Sigma_n} \xrightarrow{\simeq} P_{n-1}((\Delta_n F)^{t\Sigma_n})$ is an equivalence, so we can define an equivalence $P_{n-1}((\Delta_n F)^{h\Sigma_n}) \xrightarrow{\simeq} (\Delta_n F)^{t\Sigma_n}$ as the composition

$$\begin{tikzcd}
 & (\Delta_n F)^{t\Sigma_n} \arrow[dl, "\simeq"'] \arrow[d, "\simeq"] \\
P_{n-1}((\Delta_n F)^{h\Sigma_n}) \arrow[r, "\simeq"'] & P_{n-1}((\Delta_n F)^{t\Sigma_n}).
\end{tikzcd}$$

Now, using this equivalence, and that $(\Delta_n F)^{h\Sigma_n}$ is n-excisive, so $(\Delta_n F)^{h\Sigma_n} \to P_n((\Delta_n F)^{h\Sigma_n})$ is an equivalence, we get the following diagram

$$\begin{tikzcd}
D_n((\Delta_n F)^{h\Sigma_n} \arrow[r] & P_n((\Delta_n F)^{h\Sigma_n}) \arrow[r] & P_{n-1}((\Delta_n F)^{h\Sigma_n}) \\
(\Delta_n F)_{h\Sigma_n} \arrow[r] & (\Delta_n F)^{h\Sigma_n} \arrow[r] \arrow[u, "\simeq"'] & (\Delta_n F)^{t\Sigma_n}, \arrow[u, "\simeq"']
\end{tikzcd}$$

where both columns are cofibration sequences (since we are in a stable category). We wish to prove that the right square commutes. This is done by considering $(\Delta_n F)^{h\Sigma_n} \to P_{n-1}((\Delta_n F)^{h\Sigma_n}$:

$$\begin{tikzcd}
P_n((\Delta_n F)^{h\Sigma_n}) \arrow[r] & P_{n-1}((\Delta_n F)^{h\Sigma_n}) \\
(\Delta_n F)^{h\Sigma_n} \arrow[r] \arrow[u, "\simeq"'] \arrow[ur, dotted] & (\Delta_n F)^{t\Sigma_n}. \arrow[u, "\simeq"']
\end{tikzcd} \tag{1}$$

We know that the lower triangle commutes since we constructed $(\Delta_n F)^{t\Sigma_n} \to P_{n-1}((\Delta_n F)^{h\Sigma_n})$ such that the triangle to the right in the diagram below and the entire diagram commutes:

$$\begin{tikzcd}
P_{n-1}((\Delta_n F)^{h\Sigma_n} \arrow[r, "\simeq"] & P_{n-1}((\Delta_n F)^{t\Sigma_n}) \\
(\Delta_n F)^{h\Sigma_n} \arrow[r] \arrow[u] & (\Delta_n F)^{t\Sigma_n}. \arrow[u] \arrow[ul, "\simeq"']
\end{tikzcd}$$

We see that the left triangle in (1) commutes since this is the splitting from the Tayler tower.

□

We are now ready to prove the main theorem.

Proof of theorem 0.1. Again, since $\Delta_n F \to \Delta_n P_n F$ is always an equivalence we may assume that F is n-excisive. That lets us apply lemma 0.6 which together with the natural transformation

$$\alpha^n(X) : F(X) \to (\Delta_n F)(X)^{h\Sigma_n}$$

gives us the following commutative diagram of weak natural transformations:

$$\begin{tikzcd}
D_nF(X) \arrow[r, "D_n(\alpha^n)"] \arrow[d] & D_n((\Delta_nF)(X)^{h\Sigma_n}) \arrow[r, "\simeq"] \arrow[d] & (\Delta_nF)(X)_{h\Sigma_n} . \arrow[d] \\
P_nF(X) \arrow[r, "P_n(\alpha^n)"] \arrow[d] & P_n((\Delta_nF)(x)^{h\Sigma_n}) \arrow[r, "\simeq"] \arrow[d] & (\Delta_nF)(X)^{h\Sigma_n} \arrow[d] \\
P_{n-1}F(X) \arrow[r, "P_{n-1}(\alpha^n)"] & P_{n-1}((\Delta_nF)(X)^{h\Sigma_n}) \arrow[r, "\simeq"] & (\Delta_nF)(X)^{t\Sigma_n}
\end{tikzcd}$$

The two outer vertical columns are homotopy fibration sequences. From lemma 0.5(1) we have that $D_n(\alpha^n)$ is an equivalence, hence the square

$$\begin{tikzcd}
P_nF(X) \arrow[r] \arrow[d] & (\Delta_nF)(X)^{h\Sigma_n} \arrow[d] \\
P_{n-1}F(X) \arrow[r] & (\Delta_nF)(X)^{t\Sigma_n}
\end{tikzcd}$$

is a homotopy pullback square as desired. □

In [AC19] this theorem is stated a bit different in theorem 2.9. We have a one-to-one correspondence between spectra with an Σ_n action and symmetric linear functors, and for D_nF this spectra is called the n'th differential and is denoted by $\partial^{(n)}F$. When F preserves filtered colimits this satiesfies the equivalence $D_nF(X) \simeq (\partial^{(n)} \wedge X^{\wedge n})_{h\Sigma_n}$, which says that $D_nF(X)$ is weakly equivalent to a homotopy orbit spectrum. We will be using that $\partial^{(n)}F \simeq (D^{(n)}F)(S^0, ..., S^0)$.

Corollary 0.8. *For a functor $F : Sp \to Sp$ that preserves filtered colimits, there is a natural homotopy pullback square of the form*

$$\begin{tikzcd}
P_nF(X) \arrow[r] \arrow[d] & (\partial_nF \wedge X^{\wedge n})^{h\Sigma_n} \arrow[d] \\
P_{n-1}F(X) \arrow[r] & (\partial_nF \wedge X^{\wedge n})^{t\Sigma_n}.
\end{tikzcd}$$

Proof. Due to theorem 0.1 it is sufficient to prove that $\Delta_nF \simeq (\partial^{(n)}F \wedge X^{\wedge n})$. Theorem 6.1 in [Goo03] gives us the following

$$\Delta_nF(X) = P_{(1,...,1)}(cr_nF)(X, ..., X) \simeq D^{(n)}F(X, ..., X) = cr_nD_nF(X, ..., X).$$

On the other hand, using that $\partial^{(n)}F \simeq (D^{(n)}F)(S^0, ..., S^0)$ we get that

$$\begin{aligned}\partial^{(n)}F \wedge X^{\wedge n} &\simeq (D^{(n)}F)(S^0, ..., S^0) \wedge X^{\wedge n} \\ &= (D^{(n)}F)(X, ..., X) \\ &= (cr_n D_n F)(X, ..., X).\end{aligned}$$

□

References

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