Derivatives of Hecke Eigenvalues in p-adic Families of Modular Forms

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07/09/23
YRANT 2023, Cambridge

p-adic Families

Definition (Informal)

A p-adic family of modular forms of level pN is a q-expansion $\sum_{n\geq 0} a_n q^n$, such that each a_n is an analytic p-adic function, and for suitable integers, k, $\sum_{n\geq 0} a_n(k) q^n$ is a modular form of weight k and level pN.

Theorem (Hida, Coleman)

Let f be an eigenform of weight $k \geq 2$ with pth Fourier coefficient a_p. If $v_p(a_p) < k - 1$, then there exists a unique p-adic family through f.

Examples of p-adic Families

Example (Eisenstein Series)

Let

$$E_k^{(p)} = \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}^*(n) q^n$$

where $\sigma_k^*(n) = \sum_{d|n, p\nmid d} d^k$. This forms a family, since $\sigma_k^*(n)$ is an analytic function of k, as is $\zeta(1-k)$.

Example

Let $\Delta = q - 24q^2 + 252q^3 + \ldots$ be the unique modular form of level 1 and weight 12, and $f = q - 2q^2 - q^3 + \ldots$ be the unique modular form of level 11 and weight 2. There is a unique p-adic family through both forms, and this 'explains' their congruence modulo 11.

Modular Symbols

Definition

Let V_k be polynomials of degree k. This has a right action of $\mathrm{SL}_2(\mathbb{Z})$ via

$$\left(f \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = (cz+d)^k f\left(\frac{az+b}{cz+d}\right)$$

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A modular symbol of level Γ and weight k is an element of $\mathrm{Symb}_\Gamma(V_k) = \mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), V_k)^\Gamma$, where $(f \cdot \sigma)(D) = f(\sigma D) \cdot \sigma$

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Theorem (Eichler-Shimura)

For a congruence subgroup Γ, there is an isomorphism of Hecke modules $\mathrm{Symb}_\Gamma(V_k) \cong M_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)}$

p-adic Distributions

Definition

For a normed $\mathbb{Q}_p$-algebra R, define $\mathcal{A}(R)$ to be power series $\sum_i a_i z^i$ such that $a_i \to 0$ as $i \to 0$.
Moreover, this comes with a left action of $\Gamma_0(p)$ for any character $\kappa : R^\times \to \mathbb{Q}_p^\times$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot_\kappa f \right)(z) = \kappa(a - cz) f\left(\frac{dz - b}{a - cz} \right)$$

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Definition

Define $\mathcal{D}(R) = \mathrm{Hom}(\mathcal{A}(R), \mathbb{Q}_p) \hat{\otimes} R$, and equip this with an action given by $(\mu \cdot_\kappa \sigma)(f) = \mu(\sigma \cdot_\kappa f)$

For R finite dimensional over $\mathbb{Q}_p$, $\mathcal{D}(R) = \mathrm{Hom}(\mathcal{A}(R), R)$

Overconvergent Modular Symbols

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If $R = \mathbb{Q}_p$, denote $\mathcal{D}(R)$ with the action given by the character $z \mapsto z^k$, by $\mathcal{D}_k(R)$.

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Theorem (Stevens' Control Theorem)

For any $k \geq 0$, there is an isomorphism

$$\mathrm{Symb}_\Gamma(\mathcal{D}_k(\mathbb{Q}_p))^{<k+1} \cong \mathrm{Symb}_\Gamma(V_k)^{<k+1}$$

Another Viewpoint on Families

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Idea

Let $\mathcal{C}$ be a curve over weight space, where above a point, $\chi : \mathbb{Z}_p^\times \to R^\times$, the points are the Hecke eigenvalue systems appearing in $\mathrm{Symb}_\Gamma(\mathcal{D}(R))$ with weight χ.

This should give a geometric interpretation of p-adic families, where R corresponds to a disc in weight space.

Derivatives in Families

Idea

Compute tangent spaces to $\mathcal{C}$ like in algebraic geometry: find $\mathbb{Q}_p[\epsilon] := \mathbb{Q}_p[x]/(x^2)$-points that specialise to the classical point.

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Lemma

Let $\chi : \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times$ be a character. Then for any $t \in \mathbb{Q}_p$, $\chi(z)(1 + t\epsilon \log_p(z))$ is a $\mathbb{Q}_p[\epsilon]$-valued character extending χ, and this is all such.

So study $\mathrm{Symb}_\Gamma(\mathcal{D}(\mathbb{Q}_p[\epsilon]))$ with character $\chi(z)(1 + \epsilon \log_p z)$, and the map $\mathrm{Symb}_\Gamma(\mathcal{D}(\mathbb{Q}_p[\epsilon])) \to \mathrm{Symb}_\Gamma(\mathcal{D}(\mathbb{Q}_p))$ coming from by $\epsilon \mapsto 0$.

Computing Modular Symbols

Theorem (Pollack-Stevens)

The Γ-module $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is finitely presented. Moreover, there exists a fundamental domain for Γ acting on $\mathcal{H}$ such that $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is generated by the boundary paths, and the relations are given by the identifications of the edges, and the sum of all the boundary paths is zero.

The only hard relation to satisfy is the last. If the width of the cusp at ∞ is 1, this becomes, for a symbol ϕ:

$$\phi(\gamma_i)\cdot\sum_i(1-\sigma_i^{-1})+\sum_i\phi(\gamma_i')+\sum_i\phi(\gamma_i'')=\phi((\infty)-(0))\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}-1\right)$$

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Theorem (Pollack-Stevens)

The functional equation $\mu\cdot\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)-1\right)=\nu$ has a solution for μ if and only if $\nu(1)=0$.

Computing Derivatives

Write the deformed symbol as $\phi + \epsilon\psi$, and $\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. The boundary relation becomes

$$\sum_i (\psi(\gamma_i) \cdot (1 - \sigma_i^{-1}))(f(z)) + \sum_i \psi(\gamma_i')(f(z)) + \sum_i \psi(\gamma_i'')(f(z))$$

$$-\sum_i \phi(\gamma_i)(\log(a - cz)f(z)) = \left(\psi((\infty) - (0)) \cdot \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1\right)\right)(f(z))$$

It is enough to make the left hand side zero on $f = 1$.
If $-\sum_i \phi(\gamma_i)(\log(a - cz)f(z)) = 0$, this is immediate. Otherwise, as $\psi(\gamma_i)(1 - \sigma_i^{-1})(1) = \psi(\gamma_i)(1) - \psi(\gamma_i)(\chi(d_i + c_i z))$. If $\psi(\gamma_i)(1) = 1$ and $\psi(\gamma_i)(z^j) = 0$, $j \neq 0$, this is $(1 - \chi(d_i))$, so after rescaling, this makes the left hand side zero.

Computing Derivatives

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Eliminate the classical components of θ - use Hecke operators that kill the symbols that aren't ϕ.

Some Examples

Eigenvalues for the cusp form of level 11 and weight 2

ℓ	Deformed T_ℓ eigenvalue
2	$-2+11(2+11+2\times 11^2+8\times 11^3+10\times 11^4+\mathcal{O}(11^5))\epsilon$
3	$-1+11(10+6\times 11+11^2+10\times 11^3+7\times 11^4+\mathcal{O}(11^5))\epsilon$
5	$1+11(1+10\times 11+11^3+9\times 11^4+\mathcal{O}(11^5))\epsilon$
7	$-2+11(4+4\times 11^2+11^3+3\times 11^4+\mathcal{O}(11^5))\epsilon$
11	$1+11(8+2\times 11+7\times 11^2+11^3+7\times 11^4+\mathcal{O}(11^5))\epsilon$
13	$4+11(8+9\times 11+11^3+5\times 11^4+\mathcal{O}(11^5))\epsilon$

Some Examples

Eigenvalues for the cusp form of level 4 and weight 4 after 3-stabilizing

ℓ	Deformed T_ℓ eigenvalue
2	$-4 + 3(1 + 2 \times 3^4 + \mathcal{O}(3^7))\epsilon$
3	$\alpha + 3(2 + 2 \times 3 + 2 \times 3^4 + 3^5 + \mathcal{O}(3^7))\epsilon$
5	$-5 + 3(2 + 2 \times 3^3 + 2 \times 3^6 + \mathcal{O}(3^7))\epsilon$
7	$6 + 3(1 + 3^2 + 3^3 + 2 \times 3^5 + \mathcal{O}(3^7))\epsilon$
11	$32 + 3^2(2 + 2 \times 3^2 + 3^3 + 2 \times 3^4 + \mathcal{O}(3^6))\epsilon$
13	$-38 + 3^2(2 + 3 + 2 \times 3^2 + 3^3\mathcal{O}(3^6))\epsilon$

Further Work

- Interpretation of the derivatives? a'_p is the $\mathcal{L}$-invariant.
- Other groups? Does this work for Bianchi modular forms, or Hilbert, or ...? Computations ongoing (Bianchi).