Generalised Witt Vectors and the Hill-Hopkins-Ravenel Norm

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Classical Witt Vectors

Fix $n \in \mathbb{N}$ and p prime.

Definition (n-truncated p-typical Witt vectors)

There is a unique functor $W_{p,n} : \mathrm{CRing} \to \mathrm{CRing}$ such that:

- The underlying set of $W_{p,n}(R)$ is $\prod_{0 \le i < n} R$
- For all $0 \le j < n$, the map

$$w_j : W_{p,n}(R) \to R$$
$$(a_i) \mapsto \sum_{0 \le i \le j} p^i a_i^{p^{j-i}}$$

  is a ring homomorphism

π_0 of Topological Hochschild Homology

Theorem (Hesselholt and Madsen, 1997)

For E a connective commutative ring spectrum,

$$\pi_0^{C_{p^n}}(THH(E)) \cong W_{p,n+1}(\pi_0 E)$$

π_0 of TR with coefficients

Generalised by work of Dotto, Krause, Nikolaus and Patchkoria

Definition (n-truncated p-typical Witt vectors with coefficients)

Let Mod the category of all modules over commutative rings. Then there is a functor

$$W_{p,n} : \mathrm{Mod} \to \mathrm{Ab}$$

generalising the classical Witt vectors via $W_{p,n}(R; R) \cong W_{p,n}(R)$.

Theorem (Dotto et al. 2023)

For E a connective commutative ring spectrum and X a connective E-module spectrum,

$$\pi_0(TR^{n+1}(E; X)) \cong W_{p,n+1}(\pi_0 E; \pi_0 X)$$

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The Hill-Hopkins-Ravenel norm

Definition (Hill-Hopkins-Ravenel norm)

For G a finite group and H a subgroup, there is a "multiplicative induction" functor

$$N_H^G : \mathrm{Sp}_H \to \mathrm{Sp}_G$$

- When $H = \{e\}$ is trivial and $X \in \mathrm{Sp}$ is a (cofibrant orthogonal) spectrum, $N_{\{e\}}^G X$ is just $X^{\wedge |G|}$ with the obvious G-action

Connection to Witt vectors

- For $E = \mathbb{S}$ the sphere spectrum and X any connective spectrum,

$$\mathrm{TR}^{n+1}(\mathbb{S}; X) \simeq \left(N_{\{e\}}^{C_{p^n}}(X)\right)^{C_{p^n}}$$

  hence

$$\pi_0^{C_{p^n}}(N_{\{e\}}^{C_{p^n}}(X)) \cong W_{p,n+1}(\mathbb{Z}; \pi_0 X))$$

- So we may ask, what is

$$\pi_0^G(N_{\{e\}}^G(X))$$

  for an arbitrary finite group G? Can we describe it as some version of Witt vectors?

A hint we are on the right track: G-typical Witt vectors

Definition (Dress and Siebeneicher, 1988)

For G a (pro)finite group, there is a functor

$$W_G : \mathrm{CRing} \to \mathrm{CRing}$$

generalising the classical Witt vectors via $W_{C_{p^n}}(R) \cong W_{p,n+1}(R)$. The construction defines $W_G(\mathbb{Z})$ to be (a completed version of) the Burnside ring of G, then extends to other rings.

G-typical Witt vectors with coefficients

Definition (G-typical Witt vectors with coefficients)

For G a (pro)finite group, there is a functor

$$W_G : \mathrm{Mod} \to \mathrm{Ab}$$

simultaneously generalising the p-typical Witt vectors with coefficients of Dotto et al. (via $W_{C_{p^n}}(R; M) \cong W_{p,n+1}(R; M)$) and the G-typical Witt vectors of Dress and Siebeneicher (via $W_G(R; R) \cong W_G(R)$)

Uniqueness result for G-typical Witt vectors with coefficients

Theorem (R., 2023)

The functor $W_G : Mod \to Ab$ is essentially unique such that:

- *There is a natural quotient map of underlying sets*

$$q : \prod_{V \lesssim_o G} M^{\otimes_R G/V} \twoheadrightarrow W_G(R; M).$$

- *We define certain maps $w_U : \prod_{V \lesssim_o G} M^{\otimes_R G/V} \to M^{\otimes_R G/U}$ for each $U \leq_o G$. The product of these maps descends to an additive map $W_G(R; M) \to \prod_{U \leq_o G} M^{\otimes_R G/U}$.*
- *For $(T; Q)$ free, this map out of $W_G(T; Q)$ is an injection.*
- *The functor W_G preserves reflexive coequalisers.*

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G-typical Witt vectors with coefficients

Theorem (R., 2023)

For X a connective spectrum and G a finite group,

$$\pi_0^G(N_{\{e\}}^G(X)) \cong W_G(\mathbb{Z}; \pi_0 X)$$

Mackey functors

Definition (G-typical Witt vectors with coefficients)

For G finite, there is a (strong monoidal) functor

$$\underline{W}_G : \mathrm{Mod} \to \mathrm{Mack}_G(\mathrm{Ab})$$

Theorem (R., 2023)

For X a connective spectrum and G a finite group,

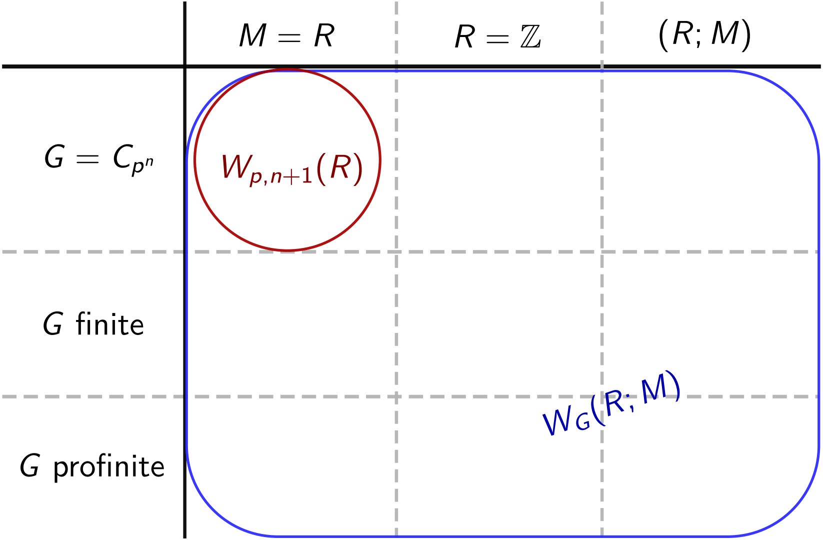
$$\underline{\pi}_0(N_{\{e\}}^G(X)) \cong \underline{W}_G(\mathbb{Z}; \pi_0 X)$$

Recap: generalisations of Witt vectors

	$M = R$	$R = \mathbb{Z}$	$(R; M)$
$G = C_{p^n}$			
G finite			
G profinite			

$W_G(R; M)$

Recap: generalisations of Witt vectors



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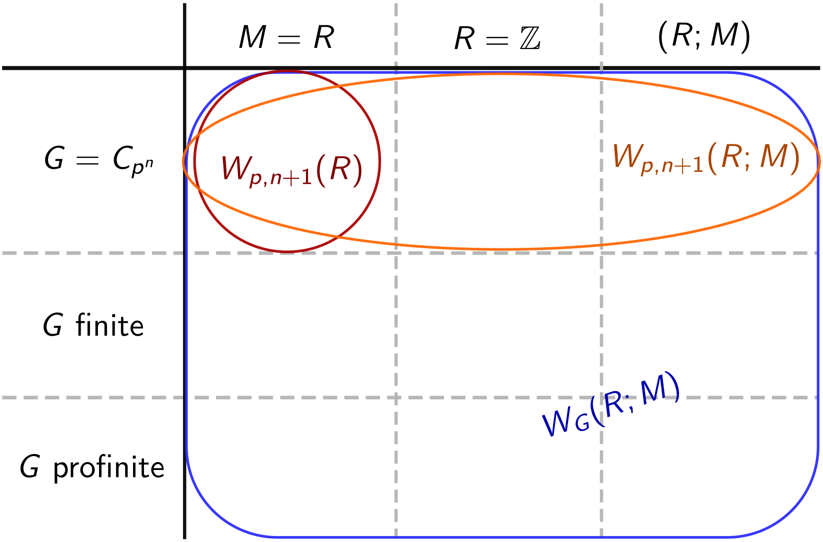

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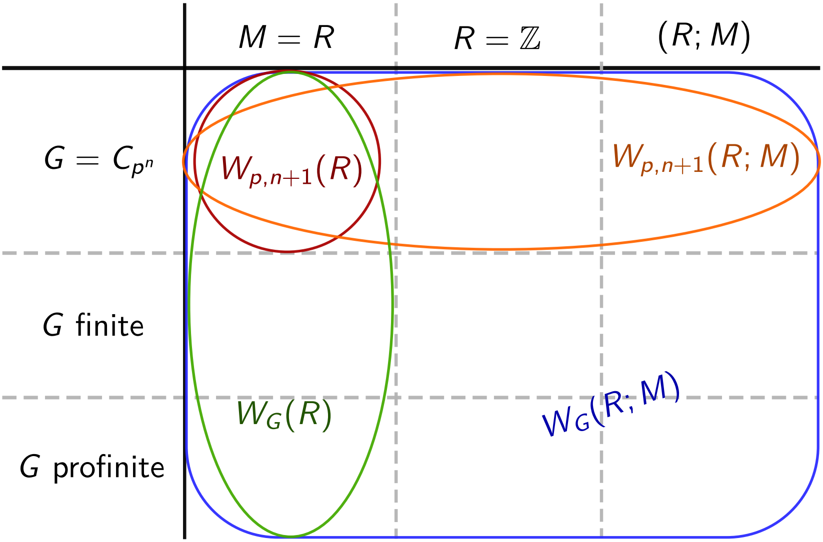

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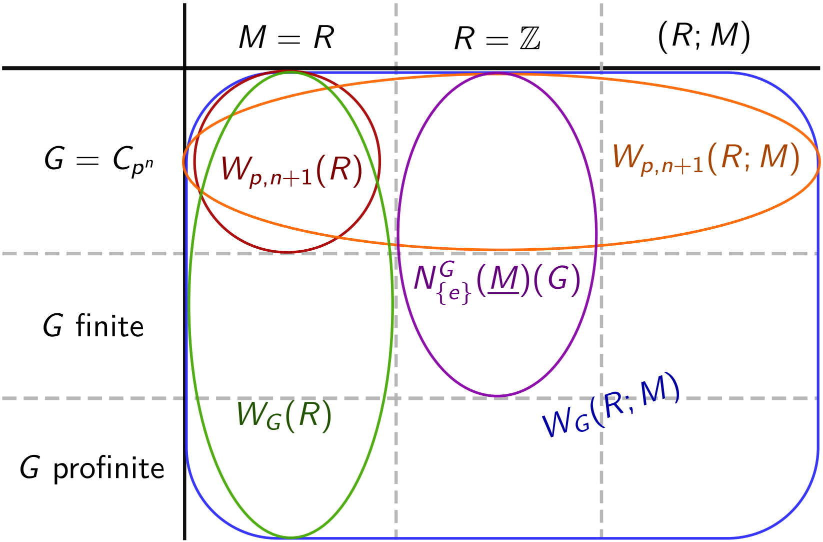

Recap: topology

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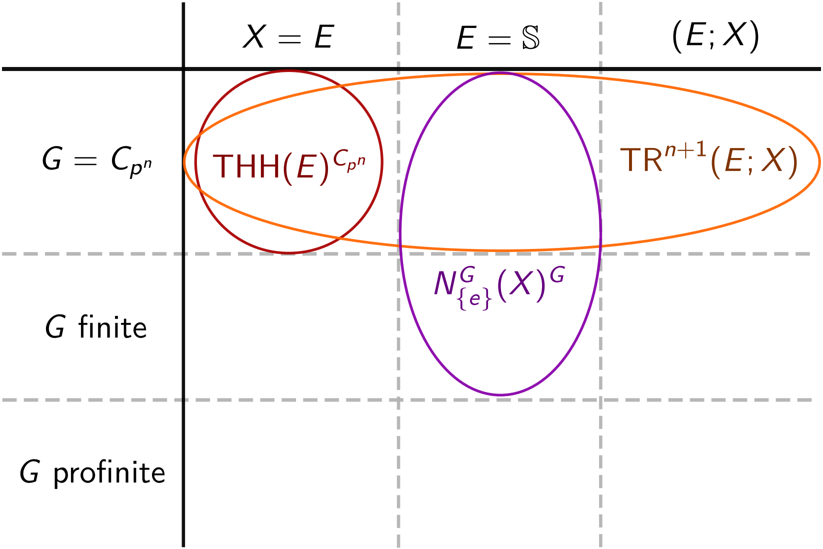

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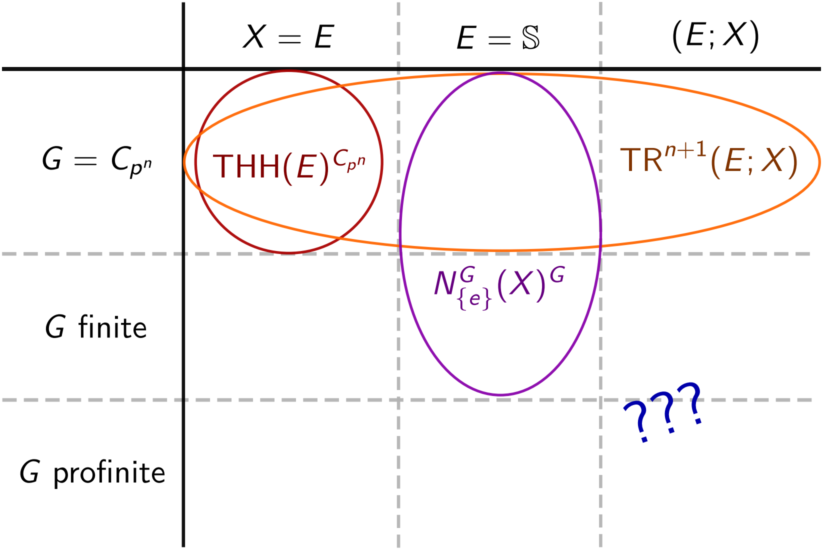

Future directions

- Can we define $N_{\{e\}}^G X$ for profinite G, such that $\pi_0^G(N_{\{e\}}^G X) \cong W_G(\mathbb{Z}; \pi_0 X)$? Think this should work in the setting of quasifinitely genuine G-spectra (Kaledin, Krause et al.)
- Is there a topological interpretation of $W_G(R; M)$ for $R \neq \mathbb{Z}$ and $G \neq C_{p^n}$?

Any questions?

Computations

$$W_{D_6}(\mathbb{Z}; \mathbb{Z}/3) \cong (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/9$$