Generalised Witt Vectors and the Hill-Hopkins-Ravenel Norm

Thomas Read
thomas.read@warwick.ac.uk

University of Warwick, United Kingdom

AMS Special Session on Topics in Equivariant Algebra,
Joint Mathematics Meetings 2024
Fix $n \in \mathbb{N}$ and $p$ prime.

**Definition ($n$-truncated $p$-typical Witt vectors)**

There is a unique functor $W_{p,n} : \text{CRing} \to \text{CRing}$ such that:

- The underlying set of $W_{p,n}(R)$ is $\prod_{0 \leq i < n} R$
- For all $0 \leq j < n$, the map $w_j : W_{p,n}(R) \to R$

$$w_j : W_{p,n}(R) \to R$$

$$(a_i) \mapsto \sum_{0 \leq i \leq j} p^i a_i^{p^j - i}$$

is a ring homomorphism.
\[ \pi_0 \text{ of Topological Hochschild Homology} \]

**Theorem (Hesselholt and Madsen, 1997)**

*For \( E \) a connective commutative ring spectrum,*

\[ \pi_0^{C_{p^n}}(\text{THH}(E)) \cong W_{p,n+1}(\pi_0 E) \]
**π₀ of TR with coefficients**

Generalised by work of Dotto, Krause, Nikolaus and Patchkoria

**Definition (n-truncated p-typical Witt vectors with coefficients)**

Let Mod the category of all modules over commutative rings. Then there is a functor

\[ W_{p,n} : \text{Mod} \to \text{Ab} \]

generalising the classical Witt vectors via \( W_{p,n}(R; R) \cong W_{p,n}(R) \).

**Theorem (Dotto et al. 2023)**

For \( E \) a connective commutative ring spectrum and \( X \) a connective \( E \)-module spectrum,

\[ \pi_0(\text{TR}^{n+1}(E; X)) \cong W_{p,n+1}(\pi_0E; \pi_0X) \]
π₀ of TR with coefficients

Generalised by work of Dotto, Krause, Nikolaus and Patchkoria

**Definition (n-truncated p-typical Witt vectors with coefficients)**

Let Mod the category of all modules over commutative rings. Then there is a functor

\[ W_{p,n} : \text{Mod} \rightarrow \text{Ab} \]

generalising the classical Witt vectors via \( W_{p,n}(R; R) \cong W_{p,n}(R) \).

**Theorem (Dotto et al. 2023)**

For \( E \) a connective commutative ring spectrum and \( X \) a connective \( E \)-module spectrum,

\[ \pi_0(\text{TR}^{n+1}(E; X)) \cong W_{p,n+1}(\pi_0 E; \pi_0 X) \]
The Hill-Hopkins-Ravenel norm

Definition (Hill-Hopkins-Ravenel norm)

For $G$ a finite group and $H$ a subgroup, there is a "multiplicative induction" functor

$$N^G_H : \text{Sp}_H \to \text{Sp}_G$$

- When $H = \{e\}$ is trivial and $X \in \text{Sp}$ is a (cofibrant orthogonal) spectrum, $N^G_{\{e\}} X$ is just $X \wedge |G|$ with the obvious $G$-action
Connection to Witt vectors

- For $E = \mathbb{S}$ the sphere spectrum and $X$ any connective spectrum,

  $$\text{TR}^{n+1}(\mathbb{S}; X) \simeq \left( N_{\{e\}}^{C_p^n}(X) \right)^{C_p^n}$$

  hence

  $$\pi_0^{C_p^n}(N_{\{e\}}^{C_p^n}(X)) \simeq W_{p,n+1}(\mathbb{Z}; \pi_0^X)$$

- So we may ask, what is

  $$\pi_0^G(N_{\{e\}}^G(X))$$

  for an arbitrary finite group $G$? Can we describe it as some version of Witt vectors?
A hint we are on the right track: \( G \)-typical Witt vectors

**Definition (Dress and Siebeneicher, 1988)**

For \( G \) a (pro)finite group, there is a functor

\[
W_G : \text{CRing} \rightarrow \text{CRing}
\]

generalising the classical Witt vectors via \( W_{C_{p^n}}(R) \cong W_{p,n+1}(R) \). The construction defines \( W_G(\mathbb{Z}) \) to be (a completed version of) the Burnside ring of \( G \), then extends to other rings.
$G$-typical Witt vectors with coefficients

**Definition ($G$-typical Witt vectors with coefficients)**

For $G$ a (pro)finite group, there is a functor

$$W_G : \text{Mod} \to \text{Ab}$$

simultaneously generalising the $p$-typical Witt vectors with coefficients of Dotto et al. (via $W_{C_p^n}(R; M) \cong W_{p,n+1}(R; M)$) and the $G$-typical Witt vectors of Dress and Siebeneicher (via $W_G(R; R) \cong W_G(R)$)
Uniqueness result for $G$-typical Witt vectors with coefficients

**Theorem (R., 2023)**

The functor $W_G : \text{Mod} \rightarrow \text{Ab}$ is essentially unique such that:

- There is a natural quotient map of underlying sets

\[ q : \prod_{V \leq \circ G} M \otimes_{RG} V \rightarrow W_G(R; M). \]

- We define certain maps $w_U : \prod_{V \leq \circ G} M \otimes_{RG} V \rightarrow M \otimes_{RG} U$ for each $U \leq \circ G$. The product of these maps descends to an additive map $W_G(R; M) \rightarrow \prod_{U \leq \circ G} M \otimes_{RG} U$.

- For $(T; Q)$ free, this map out of $W_G(T; Q)$ is an injection.

- The functor $W_G$ preserves reflexive coequalisers.
Uniqueness result for $G$-typical Witt vectors with coefficients

**Theorem (R., 2023)**

The functor $W_G : \text{Mod} \to \text{Ab}$ is essentially unique such that:

- There is a natural quotient map of underlying sets

$$q : \prod_{V \preceq \circ G} M^\otimes_{R^G/V} \twoheadrightarrow W_G(R; M).$$

- We define certain maps $w_U : \prod_{V \preceq \circ G} M^\otimes_{R^G/V} \to M^\otimes_{R^G/U}$ for each $U \preceq \circ G$. The product of these maps descends to an additive map $W_G(R; M) \to \prod_{U \preceq \circ G} M^\otimes_{R^G/U}$.

- For $(T; Q)$ free, this map out of $W_G(T; Q)$ is an injection.

- The functor $W_G$ preserves reflexive coequalisers.
Uniqueness result for $G$-typical Witt vectors with coefficients

**Theorem (R., 2023)**

The functor $W_G : \text{Mod} \to \text{Ab}$ is essentially unique such that:

- There is a natural quotient map of underlying sets

$$q : \prod_{V \leq_G} M \otimes_{R^G} V \to W_G(R; M).$$

- We define certain maps $w_U : \prod_{V \leq_G} M \otimes_{R^G} V \to M \otimes_{R^G} U$ for each $U \leq_G G$. The product of these maps descends to an additive map $W_G(R; M) \to \prod_{U \leq_G} M \otimes_{R^G} U$.

- For $(T; Q)$ free, this map out of $W_G(T; Q)$ is an injection.

- The functor $W_G$ preserves reflexive coequalisers.
Uniqueness result for $G$-typical Witt vectors with coefficients

**Theorem (R., 2023)**

The functor $W_G : \text{Mod} \to \text{Ab}$ is essentially unique such that:

- There is a natural quotient map of underlying sets
  
  $$q : \prod_{V \leq \circ G} M \otimes_{R^G} V \twoheadrightarrow W_G(R; M).$$

- We define certain maps $w_U : \prod_{V \leq \circ G} M \otimes_{R^G} V \to M \otimes_{R^G} U$ for each $U \leq \circ G$. The product of these maps descends to an additive map
  
  $$W_G(R; M) \to \prod_{U \leq \circ G} M \otimes_{R^G} U.$$

- For $(T; Q)$ free, this map out of $W_G(T; Q)$ is an injection.

- The functor $W_G$ preserves reflexive coequalisers.
$G$-typical Witt vectors with coefficients

**Theorem (R., 2023)**

For $X$ a connective spectrum and $G$ a finite group,

$$\pi^G_0(N^G_{\{e\}}(X)) \cong W_G(\mathbb{Z}; \pi_0X)$$
Definition (\(G\)-typical Witt vectors with coefficients)

For \(G\) finite, there is a (strong monoidal) functor

\[
\mathcal{W}_G : \text{Mod} \to \text{Mack}_G(\text{Ab})
\]

Theorem (R., 2023)

For \(X\) a connective spectrum and \(G\) a finite group,

\[
\pi_0(N^G_e(X)) \cong \mathcal{W}_G(\mathbb{Z}; \pi_0X)
\]
Recap: generalisations of Witt vectors

<table>
<thead>
<tr>
<th></th>
<th>$M = R$</th>
<th>$R = \mathbb{Z}$</th>
<th>$(R; M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = C_{p^n}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$ finite</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$ profinite</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$W_G(R; M)$
Recap: generalisations of Witt vectors

$$M = R$$
$$R = \mathbb{Z}$$
$$(R; M)$$

$$G = C_{p^n}$$

$$W_{p,n+1}(R)$$

$$G \text{ finite}$$

$$WG(R; M)$$

$$G \text{ profinite}$$
Recap: generalisations of Witt vectors

\[ M = R \quad R = \mathbb{Z} \quad (R; M) \]

\[ G = C_p^n \]

\[ W_{p,n+1}(R) \]

\[ W_{p,n+1}(R; M) \]

\[ W_G(R; M) \]
Recap: generalisations of Witt vectors

\[ M = R \quad R = \mathbb{Z} \quad (R; M) \]

\[ G = C_{p^n} \]

\[ W_{p,n+1}(R) \quad W_{p,n+1}(R; M) \]

\[ W_G(R) \quad W_G(R; M) \]

\[ G \text{ finite} \]

\[ G \text{ profinite} \]
Recap: generalisations of Witt vectors

- \( M = R \)
- \( R = \mathbb{Z} \)
- \( (R; M) \)

- \( G = C_p^n \)
  - \( W_{p,n+1}(R) \)
  - \( N^G_{\{e\}}(M)(G) \)
  - \( W_G(R; M) \)

- \( G \) finite
  - \( W_G(R) \)

- \( G \) profinite
Recap: topology

<table>
<thead>
<tr>
<th></th>
<th>$X = E$</th>
<th>$E = S$</th>
<th>$(E; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = C_p^n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$ finite</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$ profinite</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Recap: topology

<table>
<thead>
<tr>
<th>G = $C_{p^n}$</th>
<th>$X = E$</th>
<th>$E = S$</th>
<th>$(E; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$THH(E)^{C_{p^n}}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- $G$ finite
- $G$ profinite
Recap: topology

\[
\begin{array}{ccc}
X = E & E = S & (E; X) \\
\hline
G = C_{p^n} & \text{THH}(E)_{C_{p^n}} & \text{TR}^{n+1}(E; X) \\
G \text{ finite} & & \\
G \text{ profinite} & & \\
\end{array}
\]
Recap: topology

\[
\begin{array}{cccc}
X = E & E = S & (E; X) \\
G = C_{p^n} & \text{THH}(E)_{C_{p^n}} & \text{TR}^{n+1}(E; X) \\
G \text{ finite} & & \\
G \text{ profinite} & \text{N}^G_{\{e\}}(X)^G & \\
\end{array}
\]
Recap: topology

\[ G = C_{p^n} \]

\[ THH(E)^{C_{p^n}} \]

\[ E = S \]

\[ (E; X) \]

\[ TR^{n+1}(E; X) \]

\[ N_{\{e\}}^G(X)^G \]

???
Future directions

- Can we define $N^G_{\{e\}}X$ for profinite $G$, such that 
  $\pi_0^G(N^G_{\{e\}}X) \cong W_G(\mathbb{Z}; \pi_0X)$? Think this should work in the setting of quasifinitely genuine $G$-spectra (Kaledin, Krause et al.)

- Is there a topological interpretation of $W_G(R; M)$ for $R \neq \mathbb{Z}$ and $G \neq C_{p^n}$?
Any questions?
Computations

\[ W_{D_6}(\mathbb{Z}; \mathbb{Z}/3) \cong (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/9 \]