ABELIAN LIVŠIC THEOREMS FOR ANOSOV FLOWS

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ABSTRACT. We give two short proofs of the abelian Livšic theorem of Gogolev and Rodriguez Hertz. We show that these proofs may be extended to give new abelian Livšic theorems for positive density sets of null-homologous orbits and for amenable covers.

1. INTRODUCTION

Let M be a compact smooth Riemannian manifold and let $X^t : M \to M$ be a transitive Anosov flow, generated by the vector field X. (An Anosov flow is transitive if it has a dense orbit.) Let $\mathcal{P}$ denote the set of prime periodic orbits of the flow and let $\ell(\gamma)$ denote the least period of $\gamma \in \mathcal{P}$. (A periodic orbit is called prime if it traverses its image only once.) For $f : M \to \mathbb{R}$ and $\gamma \in \mathcal{P}$, write

$$\int_\gamma f = \int_0^{\ell(\gamma)} f(X^t(x_\gamma)) \, dt$$

for any x_γ on γ.

A classical result is the Livšic (or Livshits) periodic orbit theorem: if a Hölder continuous function $f : M \to \mathbb{R}$ satisfies

$$\int_\gamma f = 0 \quad \forall \gamma \in \mathcal{P} \tag{1.1}$$

then $f = L_X u$, where $u : M \to \mathbb{R}$ is a Hölder continuous function (with the same exponent as f) which is continuously differentiable along flow lines and L_X is the Lie derivative [17]. A more recent addition to this theory is the beautiful abelian Livšic theorem of Andrey Gogolev and Federico Rodriguez Hertz, which characterises Hölder functions f for which it is only assumed that (1.1) holds for *null-homologous* periodic orbits. For $\gamma \in \mathcal{P}$ (or more generally for any closed curve on M), let $[\gamma] \in H_1(M, \mathbb{Z})$ denote the homology class of γ. We say that X^t is *homologically full* if every class in $H_1(M, \mathbb{Z})$ is represented by an element of $\mathcal{P}$. Write $\mathcal{P}_0 = \{\gamma \in \mathcal{P} : [\gamma] = 0\}$.

Theorem 1.1 (Gogolev and Rodriguez Hertz [13]). *Let $X^t : M \to M$ be a homologically full transitive Anosov flow. If $f : M \to \mathbb{R}$ is Hölder continuous and satisfies*

$$\int_\gamma f = 0 \quad \forall \gamma \in \mathcal{P}_0 \tag{1.2}$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1-form ω and some Hölder continuous $u : M \to \mathbb{R}$
which is continuously differentiable along flow lines.

Remark 1.2. If the first Betti number of M is zero, so that $H_1(M, \mathbb{Z})$ is
finite, then M admits no non-zero closed 1-form and it is easy to see that
Theorem 1.1 reduces to the classical Livšic theorem. (There are many ex-
amples of transitive Anosov flows on manifolds with zero first Betti number.
The simplest are given by geodesic flows over two-dimensional hyperbolic
orbifolds of genus zero. Such orbifolds have the form $S = \mathbb{H}^2/\Gamma$, where $\mathbb{H}^2$
is the hyperbolic plane and $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian group such that S
has p cone points satisfying $p \geq 5$ or $p = 4$ and the orders of these points are
not all 2, or $p = 3$ and the sum of the reciprocals of the orders is less than 1.
Then we can define the geodesic flow on $M = \mathrm{PSL}(2, \mathbb{R})/\Gamma$, which has zero
first Betti number (Lemma 2.1 of [6]), and the flow is transitive Anosov.
More examples are given by surgery, see [14] and [12]. These flows are au-
tomatically homologically full since the periodic orbits are equidistributed
with respect to any finite group [18], [26].)

We give two new and short proofs of Theorem 1.1, one based on the
weighted equidistribution theorems for null-homologous periodic orbits in
[5] and the other on older asymptotic counting results of Lalley [16], Sharp
[25] and Babillot and Ledrappier [2]. Our second proof also gives an abelian
Livšic theorems for sets of null-homologous orbits with positive density
(analogous to the positive density version of the classical Livšic theorem
obtained recently by Dilsavor and Marshall Reber [7]).

Theorem 1.3. *Let $X^t : M \to M$ be a homologically full transitive Anosov*
flow. If, for some $\Delta > 0$, a Hölder continuous function $f : M \to \mathbb{R}$ satisfies

$$(1.3) \qquad \limsup_{T \to \infty} \frac{\# \left\{ \gamma \in \mathcal{P}_0 \, : \, T < \ell(\gamma) \leq T + \Delta, \ \int_\gamma f = 0 \right\}}{\#\{\gamma \in \mathcal{P}_0 \, : \, T < \ell(\gamma) \leq T + \Delta\}} > 0$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1-form ω and some Hölder continuous $u : M \to \mathbb{R}$
which is continuously differentiable along flow lines.

We also have an abelian Livšic theorem for amenable covers. For the next
result, $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ is the lifted flow on a regular cover of M. Let G be the
covering group, with identity element e. For each $\gamma \in \mathcal{P}$, we can associated
its *Frobenius class* $\langle \gamma \rangle$, which is a conjugacy class in G. If $\tilde{\gamma}$ is any lift of
$\gamma \in \mathcal{P}$ then the initial and final endpoints of $\tilde{\gamma}$ are related by the action of
some $g \in \langle \gamma \rangle$, and $\tilde{\gamma}$ is itself a periodic orbit if and only if $\langle \gamma \rangle = \{e\}$. Write
$\widetilde{\mathcal{P}}_0 = \{\gamma \in \mathcal{P} \, : \, \langle \gamma \rangle = \{e\}\}$.

Theorem 1.4. *Let $X^t : M \to M$ be a homologically full transitive Anosov flow and let $\widetilde{M}$ be a regular cover of M such that*

- *the lifted flow $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ is topologically transitive, and*
- *the covering group G is amenable.*

If $f : M \to \mathbb{R}$ is Hölder continuous and satisfies

$$(1.4) \qquad \int_\gamma f = 0 \quad \forall \gamma \in \widetilde{\mathcal{P}}_0$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1-form ω and some Hölder continuous $u : M \to \mathbb{R}$ which is continuously differentiable along flow lines.

In the next section we give some background on Anosov flows and cohomology. In section 3, we give two proofs of Theorem 1.1 (the second of which also implies Theorem 1.3). In section 4, we consider general abelian covers. In section 5, we discuss how to extend the abelian Livšic theorem to amenable covers and prove Theorem 1.4.

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2. ANOSOV FLOWS AND COHOMOLOGY

Let M be a compact Riemannian manifold and $X^t : M \to M$ be a C^1 transitive Anosov flow [1] (for a comprehensive modern treatment, see [11]). We have $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^b \oplus \mathfrak{F}$, where $b = b_1(M) \geq 0$ is the first Betti number of M and $\mathfrak{F}$ is a finite abelian group.

We say that X^t is *homologically full* if the map $\mathcal{P} \to H_1(M, \mathbb{Z}) : \gamma \mapsto [\gamma]$ is a surjection. (See Remark 2.1 for examples.) This automatically implies that the flow is weak-mixing (since an Anosov flow fails to be weak-mixing only when it is a constant suspension of an Anosov diffeomorphism [20], it which case it can have no null-homologous periodic orbits). From now on, we assume that $b \geq 1$ and ignore any torsion in $H_1(M, \mathbb{Z})$ (so we interpret $[\gamma]$ as an element of $H_1(M, \mathbb{Z})/\mathfrak{F}$).

We interpret the real cohomology group $H^1(M, \mathbb{R})$ as the de Rham cohomology group. i.e. the quotient of the space of smooth closed 1-forms on M by the space of smooth exact 1-forms. We write $[\omega]$ for the cohomology class determined by the closed 1-form ω, i.e. if $[\omega] = [\omega']$ then $\omega - \omega'$ is an exact form, and we say that ω has integral periods if $\int_c \omega \in \mathbb{Z}$ for every smooth closed curve in M. We can choose forms $\omega_1, \ldots, \omega_b$ with integral periods so that $[\omega_1], \ldots, [\omega_b]$ is a basis for $H^1(M, \mathbb{R})$. This choice determines a fixed isomorphism between $H_1(M, \mathbb{Z})/\mathfrak{F}$ and $\mathbb{Z}^b$ given by

$$[c] \mapsto \left(\int_c \omega_1, \ldots, \int_c \omega_b \right),$$

where c is a smooth closed curve on M and $[c] \in H_1(M, \mathbb{Z})/\mathfrak{F}$ is its homology class. If $\gamma \in \mathcal{P}$ then we have

$$\left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_b \right) = \left(\int_\gamma \omega_1(X), \ldots, \int_\gamma \omega_b(X) \right).$$

Let $\mathcal{M}(X)$ denote the set of X^t-invariant Borel probability measures on M. Given $\nu \in \mathcal{M}(X)$, we define the associated winding cycle (or asymptotic cycle) $\Phi_\nu \in H_1(M, \mathbb{R})$ by

$$\langle \Phi_\nu, [\omega] \rangle = \int \omega(X) \, d\nu,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing (Schwartzman [24], Verjovsky and Vila Freyer [27]).

Remark 2.1. There are many examples of homologically full transitive Anosov flows. Most obviously, geodesic flows $X^t : T^1N \to T^1N$, where $M = T^1N$ is the unit-tangent bundle over a compact Riemannian manifold N with negative sectional curvatures. Homological fullness holds here because every non-trivial free homotopy class in N contains a closed geodesic (see Theorem 2.2 in Chapter 12 of [8]) and closed geodesics on N are in natural one-to-one correspondence with periodic orbits for the geodesic flow, while $H_1(N, \mathbb{Z})$ and $H_1(T^1N, \mathbb{Z})$ are isomorphic if $\dim(M) \geq 3$ and satisfy $H_1(T^1N, \mathbb{Z}) \cong H_1(N, \mathbb{Z}) \oplus \mathbb{Z}/(-\chi(N))\mathbb{Z}$, where $\chi(N)$ is the Euler characteristic of N. More generally, any contact Anosov flow is transitive and homologically full. (Recall that $X^t : M \to M$ is a contact flow if $\dim(M) = 2k + 1$ and there exists a 1-form α on M such that the volume form $\alpha \wedge (d\alpha)^k$ is X^t-invariant.) To see this, we use the characterization in the proof of Theorem 1 of [25] that X^t is homologically full if and only if $\Phi_{\mu_\varphi} = 0$ for some Hölder continuous function $\varphi : M \to \mathbb{R}$. Since the volume measure m corresponding to $\alpha \wedge (d\alpha)^k$ is the equilibrium state of a Hölder continuous function, Corollary 4.10 of [20] gives that $\Phi_m = 0$.

In addition to the cohomology of the manifold M, we will also consider the dynamical cohomology associated to the flow $X^t : M \to M$. For $0 < \theta < 1$, let $C^\theta(M, \mathbb{R})$ denote the space of Hölder continuous functions from M to $\mathbb{R}$ with exponent θ. The theorems we prove in this paper concern Hölder continuous functions $f : M \to \mathbb{R}$ and, given a particular f, we fix θ to be its Hölder exponent. A function $g \in C^\theta(M, \mathbb{R})$ is called a (C^θ dynamical) coboundary if $g = L_X u$, for some function $u : M \to \mathbb{R}$ which is differentiable along flow lines, and two functions are cohomologous if their difference is a coboundary. We write $B^\theta(M, \mathbb{R})$ for the intersection of the set of flow coboundaries with $C^\theta(M, \mathbb{R})$, i.e. $B^\theta(M, \mathbb{R})$ is the set of functions $L_X u$ such that $u \in C^\theta(M, \mathbb{R})$ with u C^θ-differentiable along flow lines. Finally we define the dynamical cohomology group $\mathcal{H}^1_\theta(X, \mathbb{R})$ by $\mathcal{H}^1_\theta(X, \mathbb{R}) := C^\theta(M, \mathbb{R})/B^\theta(M, \mathbb{R})$. Given a Hölder continuous function $g \in C^\theta(M, \mathbb{R})$, we write $[g]$ for its (C^θ dynamical) cohomology class (where

the notation suppresses the dependence on θ). We can define a linear map $\iota : H^1(M, \mathbb{R}) \to \mathcal{H}^1_\theta(X, \mathbb{R})$ as follows. Given $w \in H^1(M, \mathbb{R})$, choose a smooth closed 1-form ω in this class. Then define $\iota(w) = [\omega(X)]$, i.e. the (C^θ dynamical) cohomology class of the function $\omega(X) \in C^\theta(M, \mathbb{R})$.

Lemma 2.2. *The map $\iota : H^1(M, \mathbb{R}) \to \mathcal{H}^1_\theta(X, \mathbb{R})$ is an injection.*

Proof. Suppose that $\iota([\omega]) = 0$. Then

$$\langle [\gamma], [\omega] \rangle = \int_\gamma \omega = \int_\gamma \omega(X) = 0$$

for every $\gamma \in \mathcal{P}$. Since $\{[\gamma] : \gamma \in \mathcal{P}\}$ generates $H_1(M, \mathbb{Z})$ as a group (this is true even without the assumption that the flow is homologically full [18], [26]), we can conclude that $[\omega] = 0$, so ι is an injection. $\qquad\square$

We will also use the Bruschlinsky theory of cohomology [4]. Let $K = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle. Then we can interpret $H^1(M, \mathbb{Z})$ as the set of continuous functions $u : M \to K$ modulo functions homotopic to the identity (i.e. modulo functions of the form $e^{2\pi i h(x)}$, where $h \in C(M, \mathbb{R})$), and we let $[u]$ be the cohomology class represented by u. Furthermore, it is shown in [24] that we can assume the representative u is continuously differentiable along flow lines, in which case we have

$$\langle [\gamma], [u] \rangle = \frac{1}{2\pi i} \int_\gamma \frac{L_X u}{u}$$

for all $\gamma \in \mathcal{P}$. We will use the following lemma, which appeared as Proposition 3.7 in [13].

Lemma 2.3. *Let $f : M \to \mathbb{R}$ be a Hölder continuous function with Hölder exponent θ such that the set of f-periods $\left\{ \int_\gamma f : \gamma \in \mathcal{P} \right\}$ is contained in a discrete subgroup of $\mathbb{R}$. Then $[f] \in \iota(H^1(M, \mathbb{R}))$, i.e. f is cohomologous (in $C^\theta(M, \mathbb{R})$) to a function of the form $\omega(X)$, for some smooth closed 1-form ω.*

Proof. Suppose that the set of f-periods is contained in $c\mathbb{Z}$, for some $c > 0$. Then $e^{2\pi i \int_\gamma c^{-1} f} = 1$ for all $\gamma \in \mathcal{P}$. Applying the Livšic theorem for K, there is a function $u : M \to K$, C^1 along flow lines, such that

$$u(X^t x) = u(x) e^{2\pi i \int_0^t c^{-1} f(X^s x)\, ds},$$

for all $x \in M$. Hence

$$c^{-1} f = \frac{1}{2\pi i} \frac{L_X u}{u}.$$

Now choose a smooth closed 1-form η in the cohomology class $[u] \in H^1(M, \mathbb{Z})$. For each $\gamma \in \mathcal{P}$, we then have

$$\int_\gamma c^{-1} f = \frac{1}{2\pi i} \int_\gamma \frac{L_X}{u} = \langle [\gamma], [u] \rangle = \int_\gamma \eta = \int_\gamma \eta(X).$$

It follows from the classical Livšic theorem that f is cohomologous $\omega(X)$, where $\omega = c\eta$. $\square$

Let $\varphi : M \to \mathbb{R}$ be Hölder continuous. We define its pressure $P(\varphi)$ by

$$P(\varphi) = \sup \left\{ h(\nu) + \int \varphi \, d\nu : \nu \in \mathcal{M}(X) \right\}$$

and there is a unique $\mu_\varphi \in \mathcal{M}(X)$, called the equilibrium state for φ, at which the supremum is attained. The following result is fundamental.

Lemma 2.4. *If $\varphi, \psi : M \to \mathbb{R}$ are Hölder continuous then $\mu_\varphi = \mu_\psi$ if and only if $\varphi - \psi$ is a cohomologous to a constant.*

Proof. This is a standard result but it is hard to find a clear reference. First, we note that the "if" direction is trivial. For the other direction, we can proceed using symbolic dynamics. Suppose $\mu_\varphi = \mu_\psi$. Without loss of generality, we can add constants to φ and ψ so that $P(\varphi) = P(\psi) = 0$. By the classical results of Bowen [3] and Ratner [21], we can find a suspension flow $\sigma^t : \Sigma^r \to \Sigma^r$ over a mixing subshift of finite type $\sigma : \Sigma \to \Sigma$, where $r : \Sigma \to \mathbb{R}^{>0}$ is Hölder continuous, and a Hölder continuous surjection $\pi : \Sigma^r \to M$ that is one-to-one on a residual set and satisfies $X^t \circ \pi = \pi \circ \sigma^t$. Furthermore, $\pi^*(\mu_\varphi) = \mu_{\varphi \circ \pi}$ and $\pi^*(\mu_\psi) = \mu_{\psi \circ \pi}$ (and π is a.e. one-to-one between the respective pairs of measures), where the measures on the right are equilibrium states with respect to the suspension flow, and $P(\varphi \circ \pi) = P(\psi \circ \pi) = 0$. Thus, if $\mu_\varphi = \mu_\psi$ then $\mu_{\varphi \circ \pi} = \mu_{\psi \circ \pi}$. If we define $\Phi, \Psi : \Sigma \to \mathbb{R}$ by

$$\Phi(x) = \int_0^{r(x)} \varphi \circ \pi(\sigma^t(x, 0)) \, dt \text{ and } \Psi(x) = \int_0^{r(x)} \psi \circ \pi(\sigma^t(x, 0)) \, dt$$

then (by Proposition 6.1 of [19]) $P(\Phi) = P(\Psi) = 0$ and

$$\mu_{\varphi \circ \pi} = \frac{m_\Phi \times \text{Leb}}{\int r \, dm_\Phi} \text{ and } \mu_{\psi \circ \pi} = \frac{m_\Psi \times \text{Leb}}{\int r \, dm_\Psi},$$

where m_Φ and m_Ψ are the equilibrium states of Φ and Ψ, respectively, with respect to $\sigma : \Sigma \to \Sigma$. We can then deduce that $m_\Phi = m_\Psi$. By Proposition 3.6 of [19], Φ and Ψ are cohomologous. (Here, we have use that $P(\Phi) = P(\Psi) = 0$ to ensure the additive constant is zero.) We can then deduce that Φ and Ψ have the same sums around each σ-periodic orbit and hence that $\int_\gamma \varphi - \int_\gamma \psi = 0$ for all $\gamma \in \mathcal{P}$. Applying, Livšic's theorem, φ and ψ are cohomologous. $\square$

3. Two proofs of Theorem 1.1

We will now prove Theorem 1.1.

First proof of Theorem 1.1. Following section 5 of [5], given a Hölder continuous function $\varphi : M \to \mathbb{R}$, we can find a unique $\xi(\varphi) \in \mathbb{R}^b$ such that the

equilibrium state of $\varphi + \sum_{i=1}^{b} \xi_i(\varphi)\omega_i(X)$, which we shall denote by $\mu(\varphi)$, satisfies $\Phi_{\mu(\varphi)} = 0$ and

$$h(\mu(\varphi)) = \int \varphi \, d\mu(\varphi) = \sup \left\{ h(\nu) + \int \varphi \, d\nu : \nu \in \mathcal{M}_X \text{ and } \Phi_\nu = 0 \right\}.$$

Furthermore, from Theorem 6.7 of [5], we see that μ is a weak* limit of averages of null-homologous orbital measures. More precisely, for all continuous functions $\psi : M \to \mathbb{R}$, we have

$$(3.1) \qquad \lim_{T \to \infty} \left(\sum_{\substack{\gamma \in \mathcal{P}_0 \\ T < \ell(\gamma) \leq T+1}} e^{\int_\gamma \varphi} \right)^{-1} \sum_{\substack{\gamma \in \mathcal{P}_0 \\ T < \ell(\gamma) \leq T+1}} e^{\int_\gamma \varphi} \frac{\int_\gamma \psi}{\ell(\gamma)} = \int \psi \, d\mu(\varphi).$$

Now let $f : M \to \mathbb{R}$ be a Hölder continuous function satisfying $\int_\gamma f = 0$ for all $\gamma \in \mathcal{P}_0$. If we compare the cases $\varphi = 0$ and $\varphi = f$ then the corresponding terms on the left hand side of (3.1) are equal and so we obtain that $\mu(0) = \mu(f)$. Applying Lemma 2.4, we see that

$$f = L_X u + \sum_{i=1}^{b} (\xi_i(0) - \xi_i(f))\omega_i(X) + c,$$

where $u : M \to \mathbb{R}$ is a Hölder function which continuously differentiable along flow lines and $c \in \mathbb{R}$. We can see from (3.1) (with $\psi = f$) that $\int f \, d\mu(0) = 0$ and, combined with $\Phi_{\mu(0)} = 0$, this gives $c = 0$, completing the proof. $\qquad \square$

Second proof of Theorem 1.1. An alternative, slightly longer, argument is to show that the failure of Theorem 1.1 is inconsistent with the periodic orbit counting results of [2], [16], [25]. We fix a Hölder continuous function $f : M \to \mathbb{R}$ satisfying (1.2). Then, clearly,

$$\#\{\gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T+1\} = \# \left\{ \gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T+1, \int_\gamma f = 0 \right\}.$$

Now, we have from [25] that

$$(3.2) \qquad \#\{\gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T+1\} \sim C \frac{e^{\alpha T}}{T^{1+b/2}}, \qquad \text{as } T \to \infty,$$

for some $C > 0$ and $\alpha > 0$. We aim to show that if f is not cohomologous to $\omega(X)$, for some closed 1-form ω (i.e. that the conclusion of Theorem 1.1 fails to hold), then the asymptotic (3.2) cannot be true, giving a contradiction.

Suppose that f is not cohomologous to $\omega(X)$, for some closed 1-form ω. Then, in particular, by Lemma 2.3, the set of f-periods $\{\int_\gamma f \ \gamma \in \mathcal{P}\}$ is not contained in a discrete subgroup of $\mathbb{R}$. We will use Theorem 1.2 of [2] and define a function $F : M \to \mathbb{R}^{b+1}$ by $F = (f, \omega_1(X), \ldots, \omega_b(X))$, so that

$$\int_\gamma F = \left(\int_\gamma f, [\gamma] \right),$$

and define

$$\mathcal{C}(F) = \left\{ \int F \, d\nu \, : \, \nu \in (X) \right\} \subset \mathbb{R}^{b+1}.$$

Let Γ denote the smallest closed subgroup of $\mathbb{R}^{b+1}$ containing $\{\int_\gamma F : \gamma \in \mathcal{P}\}$; clearly, $\Gamma = \mathbb{R} \times \mathbb{Z}^b$. We also let $\widetilde{\Gamma}$ denote the smallest closed subgroup of $\mathbb{R}^{b+2}$ containing $\{(\ell(\gamma), \int_\gamma F) : \gamma \in \mathcal{P}\}$.

Let $g_0 : \mathbb{R} \to \mathbb{R}^+$ be the indicator function of the interval $(-1, 0]$ and let $g : \Gamma \to \mathbb{R}^+$ be defined by $g(y_0, y_1, \ldots, y_b) = \kappa(y_0)\delta_0(y_1, \ldots, y_b)$, where $\delta_0 : \mathbb{Z}^b \to \mathbb{R}^+$ is the indicator function of $\{0\}$ and $\kappa : \mathbb{R} \to \mathbb{R}^+$ is any continuous compactly supported function satisfying $\kappa(0) = 1$. Then

$$\# \left\{ \gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + 1, \, \int_\gamma f = 0 \right\} = \sum_{\gamma \in \mathcal{P}} g_0(\ell(\gamma) - T) \, g \left(\int_\gamma F \right).$$

The right hand side is of the form considered in Theorem 1.2 of [2] (the theorem requires g_0 to be continuous but an easy approximation argument allows one to pass from continuous functions to the indicator function of an interval). Thus we will have that

$$(3.3) \qquad \sum_{\gamma \in \mathcal{P}} g_0(\ell(\gamma) - T) \, g \left(\int_\gamma F \right) \sim C' \frac{e^{\alpha' T}}{T^{1+(b+1)/2}}, \qquad \text{as } T \to \infty,$$

for some $0 < \alpha' \leq \alpha$ and $C' > 0$, provided we can show that $0 \in \text{int}(\mathcal{C}(F))$ and that $\widetilde{\Gamma} = \mathbb{R} \times \Gamma$ (Assumption (A) in [2]). (Actually, one can show that $\alpha = \alpha'$ but this is not required for the argument.)

Justification of $0 \in \text{int}(\mathcal{C}(F))$. Suppose that $\langle a, F \rangle = a_0 f + \sum_{i=1}^b a_i \omega_i(X)$ is cohomologous to a constant c for some $a = (a_0, a_1, \ldots, a_b) \in \mathbb{R}^{b+1}$. As in the first proof above, $\int f \, d\mu(0) = 0$ and $\Phi_{\mu(0)} = 0$, so that $\int F \, d\mu(0) = 0$ and $c = 0$. If $a_0 \neq 0$, then f is cohomologous to $a_0^{-1}(\sum_{i=1}^b a_i \omega_i(X))$, which is ruled out by our initial assumption. So $a_0 = 0$ and hence

$$\langle (a_1, \ldots, a_b), [\gamma] \rangle = 0 \quad \forall \gamma \in \mathcal{P}.$$

Since $\{[\gamma] : \gamma \in \mathcal{P}\}$ generates $H_1(M, \mathbb{Z})$, we must have $(a_1, \ldots, a_b) = 0$ and hence $a = 0$. If we define $\mathfrak{p}_F : \mathbb{R}^{b+1} \to \mathbb{R}$ by $\mathfrak{p}_F(w) = P(\langle w, F \rangle)$, then it is a standard result that $\nabla \mathfrak{p}_F$ is a diffeomorphism from $\mathbb{R}^{b+1}$ to its image $\text{Im}(\nabla \mathfrak{p}_F)$ and that

$$\nabla \mathfrak{p}_F(w) = \int F \, d\mu^w,$$

where μ^w is the equilibrium state for $\langle w, F \rangle$ (see page 164 of [16]). Then

$$\text{Im}(\nabla \mathfrak{p}_F) = \left\{ \int F \, d\mu^w \, : \, w \in \mathbb{R}^{b+1} \right\}$$

is an open subset of $\mathcal{C}(F)$, and hence is contained in its interior. The claimed result then follows from

$$\nabla \mathfrak{p}_F((0, \xi_1(0), \ldots, \xi_b(0))) = \int F \, d\mu(0) = 0.$$

Justification of $\widetilde{\Gamma} = \mathbb{R} \times \Gamma$. Clearly, $\widetilde{\Gamma}$ is a closed subgroup of $\mathbb{R} \times \Gamma$. If it is a proper subgroup, then there will be a non-trivial character of $\mathbb{R} \times \Gamma$ which constantly takes the value 1 on $\widetilde{\Gamma}$. So, to prove the claim, we will show that no such non-trivial character can exist.

A character of $\mathbb{R} \times \Gamma$ has the form $\Xi_{\alpha,a}(\ell, x_0, x_1, \ldots, x_b) := e^{2\pi i (\alpha \ell + \sum_{j=0}^b a_j x_j)}$, with $\alpha \in \mathbb{R}$ and $a = (a_0, a_1, \ldots, a_b) \in \mathbb{R} \times (\mathbb{R}/\mathbb{Z})^b$. If $\Xi_{\alpha,a}$ is trivial on $\widetilde{\Gamma}$ then

$$(3.4) \qquad \Xi_{\alpha,a}\left(\ell(\gamma), \int_\gamma f, \int_\gamma \omega_1(X), \ldots, \int_\gamma \omega_b(X)\right) = 1 \quad \forall \gamma \in \mathcal{P}$$

For brevity, we will write $\omega = \sum_{i=1}^b a_i \omega_i$. Let $K = \{z \in \mathbb{C} : |z| = 1\}$. As in the proof of Lemma 2.3, applying the Livšic theorem for the circle K gives us a function $u : M \to K$, C^1 along flow lines, such that

$$u(X^t x) = u(x) e^{2\pi i \alpha t + 2\pi i \int_0^t (a_0 f(X^s x) + \omega(X)(X^s x)) \, ds},$$

for all $x \in M$, and hence

$$\frac{1}{2\pi i} \frac{L_X u}{u} = \alpha + a_0 f + \omega(X).$$

Since f and $\omega(X)$ integrate to zero with respect to $\mu(0)$, we obtain

$$\alpha = \int \frac{1}{2\pi i} \frac{L_X u}{u} \, d\mu(0).$$

However, as in the proof of the lemma, u also represents a Bruschlinsky cohomology class $[u]$ in $H^1(M, \mathbb{Z})$ ([4], [24]) and we have

$$0 = \langle \Phi_{\mu(0)}, [u] \rangle = \int \frac{1}{2\pi i} \frac{L_X u}{u} \, d\mu(0),$$

so that $\alpha = 0$. Hence, (3.4) reduces to

$$\exp\left(2\pi i \left(a_0 \int_\gamma f + \langle (a_1, \ldots, a_b), [\gamma] \rangle\right)\right) = 1 \quad \forall \gamma \in \mathcal{P},$$

which, by the definition of Γ, implies that $a = 0$ also. So $\Xi_{\alpha,a} = \Xi_{0,0}$, the trivial character and hence $\widetilde{\Gamma} = \mathbb{R} \times \Gamma$.

We have now shown (3.3), which contradicts (3.2). $\qquad \square$

Remark 3.1. In the justification that 0 is in the interior of $\mathcal{C}(F)$, one might notice that, for any $a \in \mathbb{R}$, one has

$$\nabla \mathfrak{p}_F((a, \xi_1(0), \ldots, \xi_b(0))) = \int F \, d\mu(af) = 0,$$

which contradicts results on the strict convexity of pressure. This leads to a third proof of Theorem 1.1.

One sees that the second proof leads to a proof of Theorem 1.3. (I am grateful to Andrey Gogolev for this observation.)

Proof of Theorem 1.3. Suppose that $f : M \to \mathbb{R}$ is a Hölder continuous function satisfying the condition (1.3). Under this hypothesis, we have

$$\# \left\{ \gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + 1, \ \int_\gamma f = 0 \right\} \leq \sum_{\gamma \in \mathcal{P}} g_0(\ell(\gamma) - T) \, g \left(\int_\gamma F \right),$$

where g_0 and g are as in the second proof of Theorem 1.1. For a contradiction, suppose that f s not cohomologous to $\omega(X)$, for some closed 1-form ω. If $\int f \, d\mu(0) = 0$ then we can use the arguments above to show that $\#\{\gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + \Delta, \ \int_\gamma f = 0\} = o(\#\{\gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + \Delta\})$, contradicting (1.3). On the other hand, if $\int f \, d\mu(0) \neq 0$ then we use a large deviations argument to show that, for any $\epsilon > 0$,

$$\lim_{T \to \infty} \frac{\# \left\{ \gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + \Delta, \ \left| \frac{1}{\ell(\gamma)} \int_\gamma f - \int f \, d\mu(0) \right| \geq \epsilon \right\}}{\#\{\gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + \Delta\}} = 0,$$

which, if ϵ is chosen sufficiently small, also contradicts (1.3). $\square$

Remark 3.2. In [7], the authors also establish a weighted result where, for a Hölder continuous function $\varphi : M \to \mathbb{R}$, one replaces counting with summing the terms $\exp(\int_\gamma \varphi)$. This type of result also holds in our setting: writing $\mathcal{P}_0(T, \Delta) = \{\gamma \in \mathcal{P}_0 : T < \ell(\gamma) \leq T + \Delta\}$, if

$$\limsup_{T \to \infty} \frac{\sum_{\gamma \in \mathcal{P}_0(T,\Delta), \int_\gamma f = 0} \exp(\int_\gamma \varphi)}{\sum_{\gamma \in \mathcal{P}_0(T,\Delta)} \exp(\int_\gamma \varphi)} > 0$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1-form ω and some Hölder continuous $u : M \to \mathbb{R}$ which is continuously differentiable along flow lines. Following the model of the second proof of Theorem 1.1, this can be shown using arguments from the proof of the weighted counting result (Theorem 6.1) in [5].

4. General abelian covers

Let $\overline{M}$ be any regular abelian cover of M, with covering group A (of rank $k \geq 1$). For simplicity of exposition, we assume that A is torsion-free. The homology class $[c]$ of a closed curve on M projects to an element $[c]_A \in A$. We say that $X^t : M \to M$ is A-full if the map $\mathcal{P} \to A : \gamma \mapsto [\gamma]_A$ is a surjection. If X^t is homologically full then (since $[\gamma] = 0$ implies $[\gamma]_A = 0$) it is trivial that the conclusion of Theorem 1.1 holds if f integrates to zero over every periodic orbit satisfying $[\gamma]_A = 0$. However, if the flow is A-full (but not homologically full) then the situation is slightly more subtle.

To discuss this further, we consider the vector spaces $A \otimes \mathbb{R}$ (which corresponds to a quotient of $H_1(M, \mathbb{R})$) and $(A \otimes \mathbb{R})^*$ (which corresponds to

a subspace of $H^1(M, \mathbb{R})$). To each $\nu \in \mathcal{M}(X)$, we can associate a *relative winding cycle* $\Phi_\nu^A \in A \otimes \mathbb{R}$, defined by

$$\langle \Phi_\nu^A, [\omega] \rangle = \int \omega(X) \, d\nu,$$

where ω is a closed 1-form with $[\omega] \in (A \otimes \mathbb{R})^*$. If X^t is A-full then we can find equilibrium states $\mu(\varphi)$ as above with $\Phi_{\mu(\varphi)}^A = 0$.

Writing $\mathcal{P}_0^A = \{\gamma \in \mathcal{P} : [\gamma]_A = 0\}$, we have the following theorem.

Theorem 4.1. *Let $X^t : M \to M$ be an A-full transitive Anosov flow. If $f : M \to \mathbb{R}$ is Hölder continuous and satisfies*

$$(4.1) \qquad \int_\gamma f = 0 \quad \forall \gamma \in \mathcal{P}_0^A$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1-form ω in $(A \otimes \mathbb{R})^*$ and some Hölder continuous $u : M \to \mathbb{R}$ which is continuously differentiable along flow lines.

Proof. The first proof of Theorem 1.1 will apply here provided the weighted equidistribution result (3.1) holds with $\mathcal{P}_0$ replaced by $\mathcal{P}_0^A$. If we look at the proof of the weighted equidistribution result in [5] (Theorem 6.7), we see that (rather than a detailed asymptotic) we only require the exponential growth rate result given in Corollary 6.2 of [5], which tells us that

$$(4.2) \qquad \lim_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \mathcal{P}_0^A \\ T < \ell(\gamma) \leq T+1}} e^{\int_\gamma \varphi} = \beta(\varphi) := P\left(\varphi + \sum_{i=1}^k \xi_i(\varphi) \omega_i(X)\right),$$

where now $k = \dim(A \otimes \mathbb{R})^*$, $[\omega_1], \ldots, [\omega_k]$ form an integral basis for $(A \otimes \mathbb{R})^*$ and the $\xi_i(\varphi)$ minimizes the pressure function on the right hand side.

We outline proof of (4.2). Suppose first that $\beta(\varphi) > 0$. Following the analysis in [25], we see that $\beta(\varphi)$ is the abscissa of convergence of the Dirichlet series

$$\sum_{\gamma \in \mathcal{P}_0^A} \ell(\gamma)^q e^{\int_\gamma \varphi - s\ell(\gamma)},$$

where $q = [k/2]$, and hence

$$\beta(\varphi) = \limsup_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \mathcal{P}_0^A \\ \ell(\gamma) \leq T}} \ell(\gamma)^q e^{\int_\gamma \varphi}.$$

Since $\beta(\varphi) > 0$, we can replace $\ell(\gamma) \leq T$ with $T < \ell(\gamma) \leq T+1$ and remove the polynomial term $\ell(\gamma)^q$, i.e. we have

$$\beta(\varphi) = \limsup_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \mathcal{P}_0^A \\ T < \ell(\gamma) \leq T+1}} e^{\int_\gamma \varphi}.$$

Furthermore, following the arguments in [15], one can show that the limsup is a limit. Finally, if $\beta(\varphi) \leq 0$, choose $c > 0$ such that $\beta(\varphi+c) = \beta(\varphi)+c > 0$; we can then deduce the result for φ from the result for $\varphi + c$. $\square$

Remark 4.2. It is instruction to consider why the proof in the preceding section do not immediately apply. If we wished to use the first proof of Theorem 1.1 without further argument, then we would need to be able to apply Theorem 6.1 of [5] in the A-full setting. The key step in proving this would be to show that $\widetilde{\Gamma_A} = \mathbb{R} \times \Gamma_A = \mathbb{R} \times \mathbb{Z}^k$, where Γ_A is the smallest closed subgroup of $\mathbb{R}^k$ containing $\{[\gamma]_A : \gamma \in \mathcal{P}\}$ (which is equal to $\mathbb{Z}^k$ by hypothesis) and $\widetilde{\Gamma_A}$ is the smallest closed subgroup of $\mathbb{R}^{k+1}$ containing $\{(\ell(\gamma), [\gamma]_A) : \gamma \in \mathcal{P}\}$. Similarly, if we wanted to use the second proof, we would need to show that $\widetilde{\Gamma_{A,f}} = \mathbb{R} \times \Gamma_{A,f}$, where $\Gamma_{A,f}$ is the smallest closed subgroup of $\mathbb{R}^k$ containing $\{(\int_\gamma f, [\gamma]_A) : \gamma \in \mathcal{P}\}$ and $\widetilde{\Gamma_{A,f}}$ is the smallest closed subgroup of $\mathbb{R}^{k+1}$ containing $\{(\ell(\gamma), \int_\gamma f, [\gamma]_A) : \gamma \in \mathcal{P}\}$.

Let us try to show $\widetilde{\Gamma_A} = \mathbb{R} \times \Gamma_A$. If we follow the argument that $\widetilde{\Gamma} = \mathbb{R} \times \Gamma$ above, we obtain the equation

$$(4.3) \qquad \frac{1}{2\pi i} \frac{L_X u}{u} = \alpha + \omega(X),$$

for some closed 1-form ω with $[\omega] \in (A \otimes \mathbb{R})^*$ and some $u : M \to K$ which is C^1 along flow lines. To proceed , one needs to show that $\alpha = 0$. If X^t is homologically full then we can integrate with respect to a measure ν satisfying $\Phi_\nu = 0$ to show this and thus obtain the desired conclusion. However, if we only assume A-full then integrating with respect to a measure ν satisfying $\Phi_\nu^A = 0$ will be enough to kill the $\omega(X)$ term (which corresponds to a cohomology class in $(A \otimes \mathbb{R})^*$) but not enough to kill the u term (which corresponds to a general cohomology class).

Nevertheless, we can show that $\alpha = 0$ for large classes of Anosov flows. First, we note that we will have $\alpha = 0$ is there is *any* $\nu \in \mathcal{M}(X)$ for which $\Phi_\nu = 0$, since then integrating with respect to ν will kill both cohomological terms. However, the existence of such a measure is equivalent not having a global cross-section (see Proposition 7 of [25] and section 7 of [24]). Thus, the first proof of Theorem 1.1 holds if the flow $X^t : M \to M$ does *not* have a global cross-section. If the flow has a global cross-section then (up to a velocity change) it is the suspension of an Anosov diffeomorphism. We observe that the only currently known examples of Anosov diffeomorphisms are topologically conjugate to algebraic examples on infranilmanifolds (with tori as special cases) and the induced action on homology is hyperbolic. In consequence, in these examples, $H_1(M, \mathbb{Z})$ has rank 1 and so there are no infinite abelian covers to consider apart from the universal homology cover.

From another point of view, passing to symbolic dynamics and modelling the flow by a suspension flow over a subshift of finite type, we can use (4.3) to conclude that if $\alpha \neq 0$ then the roof function is cohomologous to a locally constant function. However, the condition that the roof function

is *not* cohomologous to a locally constant function. is generic [10] and, for example, is always satisfied when X^t is a co-dimension one flow or when X^t is exponentially mixing.

5. AMENABLE COVERS

Let $\widetilde{M}$ be a regular cover of M such that $G = \pi_1(M)/\pi_1(\widetilde{M})$ is amenable. Let $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ be the lift of $X^t : M \to M$ to the cover. We then have a weighted equidistribution theorem, which generalises Theorem 9.2 in [9] (which covers the case $\varphi = 0$).

Theorem 5.1. *Let $X^t : M \to M$ be a transitive Anosov flow and let $\widetilde{M}$ be a regular cover of M such that*

- *the lifted flow $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ is topologically transitive, and*
- *the covering group G is amenable.*

Then for sufficiently large $\Delta > 0$ and any Hölder continuous function $\varphi : M \to \mathbb{R}$, we have

$$\lim_{T \to \infty} \left(\sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T + \Delta}} e^{\int_\gamma \varphi} \right)^{-1} \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T + \Delta}} e^{\int_\gamma \varphi} \frac{\int_\gamma \psi}{\ell(\gamma)} = \int \psi \, d\mu(\varphi),$$

for all continuous functions $\psi : M \to \mathbb{R}$.

We note that the hypothesis $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ is topologically transitive implies that the maximal abelian subcover of $\widetilde{M}$ is full. Using the large deviations approach of section 9 of [9], Theorem 5.1 follows once one has shown the following.

Proposition 5.2. *For sufficiently large $\Delta > 0$,*

$$\lim_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T + \Delta}} e^{\int_\gamma \varphi} = P\left(\varphi + \sum_{i=1}^b \xi_i(\varphi) \omega_i(X) \right).$$

Before we prove this, we need to introduce some more ideas from thermodynamic formalism, particularly that of Gurevič pressure. Recall the notation introduced in the proof of Lemma 2.4. The flow $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ may be modelled by a suspension flow $\tilde{\sigma}^t$ over a skew-product system $T_\alpha : \Sigma \times G \to \Sigma \times G$ defined by $T_\alpha(x, g) = (\sigma x, g\alpha(x))$, where $\alpha : \Sigma \to G$ is some continuous function depending on two co-ordinates, with a roof function $\tilde{r} : \Sigma \times G \to \mathbb{R}^{>0}$ satisfying $\tilde{r}(x, g) = r(x)$; to simplify notation, we will write r instead of $\tilde{r}$. The function $\sum_{i=1}^b \xi_i(\varphi) \omega_i(X) : M \to \mathbb{R}$ induces a function $\Xi : \Sigma \to \mathbb{R}$, defined by

$$\Xi(x) := \int_0^{r(x)} \left(\varphi + \sum_{i=1}^b \xi_i(\varphi) \omega_i(X) \right) \circ \pi(\sigma^t(x, 0)) \, dt.$$

It is a standard result (Proposition 6.1 of [19]) that

$$P\left(-P\left(\varphi + \sum_{i=1}^{b} \xi_i(\varphi)\omega_i(X)\right)r + \Phi + \Xi\right) = 0.$$

Now let $\overline{G}$ be the torsion free part of the abelianization of G and let $\mathfrak{a} : G \to \overline{G}$ be the natural projection homomorphism. This gives a regular abelian cover of M and transitivity of $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ gives that $X^t : M \to M$ is $\overline{G}$-full. There is also a skew-product $T_{\bar{\alpha}} : \Sigma \times \mathbb{Z}^b \to \Sigma \times \mathbb{Z}^b$ defined by $T_{\bar{\alpha}}(x, m) = (\sigma x, m + \bar{\alpha}(x))$, where $\bar{\alpha} = \mathfrak{a} \circ \alpha$. Both skew-products T_{α} and $T_{\bar{\alpha}}$ are topologically transitive countable state Markov shifts (where transitivity is given by Lemma 7.3 of [9]), so we can define the *Gurevič pressure* of locally Hölder continuous potentials, see Sarig's original paper [22] or his survey [23].

A Hölder continuous function $F : \Sigma \to \mathbb{R}$ induces functions $\tilde{F} : \Sigma \times G \to \mathbb{R}$ and $\bar{F} : \Sigma \times \mathbb{Z}^b \to \mathbb{R}$ by $\tilde{F}(x, g) = \bar{F}(x, m) = F(x)$. It will not cause any confusion to denote all three functions by F. For such functions, the Gurevič pressure of F with respect to T_{α} and $T_{\bar{\alpha}}$, is defined by

$$P_{\mathrm{G}}(F, T_{\alpha}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \alpha_n(x) = e}} e^{F^n(x)}$$

and

$$P_{\mathrm{G}}(F, T_{\bar{\alpha}}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \bar{\alpha}^n(x) = 0}} e^{F^n(x)},$$

respectively. (Here we have used that $T_{\alpha}^n(x, g) = (\sigma^n x, g\alpha^n(x))$, where $\alpha^n(x) := \alpha(x)\alpha(\sigma x) \cdots \alpha(\sigma^{n-1}(x))$.)

The following result of [9] is key to our analysis.

Proposition 5.3 (Theorem 5.1 of [9]). *If T_{α} is topologically transitive and G is amenable then*

$$P_{\mathrm{G}}(F, T_{\alpha}) = P_{\mathrm{G}}(F, T_{\bar{\alpha}}).$$

Fix $\Delta > 0$ and define

$$\overline{\mathfrak{P}}(\varphi) := \lim_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \mathcal{P}_0 \\ T < \ell(\gamma) \leq T + \Delta}} e^{\int_{\gamma} \varphi}.$$

We know from Corollary 6.2 of [5] that

$$\overline{\mathfrak{P}}(\varphi) = P\left(\varphi + \sum_{i=1}^{b} \xi_i(\varphi)\omega_i(X)\right),$$

so that

$$P\left(-\overline{\mathfrak{P}}(\varphi)r + \Phi + \Xi\right) = 0.$$

Now let

$$\mathfrak{P}(\varphi) := \limsup_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T + \Delta}} e^{\int_\gamma \varphi}.$$

Clearly, $\mathfrak{P}(\varphi) \leq \overline{\mathfrak{P}}(\varphi)$ and we claim that we have equality.

Lemma 5.4. *We have*

$$\mathfrak{P}(\varphi) = \overline{\mathfrak{P}}(\varphi).$$

Proof. Consider the series

$$S_1(s) := \sum_{\gamma \in \widetilde{\mathcal{P}}_0} e^{-s\ell(\gamma) + \int_\gamma \varphi}.$$

The series $S_1(s)$ has abscissa of convergence $\mathfrak{P}(\varphi)$, and so $\mathfrak{P}(\varphi) = \overline{\mathfrak{P}}(\varphi)$ if $\overline{\mathfrak{P}}(\varphi)$ is the abscissa of convergence of $S_1(s)$.

Now consider the corresponding series for T_α,

$$S_2(s) := \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{\sigma^n x = x \\ \alpha^n(x) = e}} e^{-sr^n(x) + \Phi^n(x)}.$$

This may involve some overcounting compared to $S_1(s)$ but it is standard that $S_1(s) - S_2(s)$ converges for $\mathrm{Re}(s) > \overline{\mathfrak{P}}(\varphi) - \epsilon$, for some $\epsilon > 0$. Thus the abscissa of convergence of $S_1(s)$ is $\overline{\mathfrak{P}}(\varphi)$ if and only if the abscissa of convergence of $S_2(s)$ is $\overline{\mathfrak{P}}(\varphi)$.

The abscissa of convergence of $S_2(s)$ is given by the value c for which $P_{\mathrm{G}}(-cr + \Phi, T_\alpha) = 0$. By Proposition 5.3,

$$P_{\mathrm{G}}(-cr + \Phi, T_\alpha) = P_{\mathrm{G}}(-cr + \Phi, T_{\bar{\alpha}})$$

and so $c = \overline{\mathfrak{P}}(\varphi)$, as required. $\qquad\square$

To complete the proof of Proposition 5.2 we observe that the arguments in section 2 of the correction to [9] show that, provided $\Delta > 0$ is sufficiently large, the limsup defining $\mathfrak{P}(\varphi)$ is a limit.

The proof of Theorem 1.4 now follows from Theorem 5.1 exactly as in the first proof of Theorem 1.1.

Remark 5.5. An interesting example of an amenable cover is that associated to the second commutator of $\pi_1(M)$. Let M be a quotient $M = U/\Gamma$, where U is the universal cover of M and $\Gamma \cong \pi_1(M)$ is a group of isometries acting freely on U. The universal abelian cover of M is the regular cover with covering group $\Gamma/\Gamma' \cong H_1(M, \mathbb{Z})$, where $\Gamma' = [\Gamma, \Gamma]$ is the commutator subgroup (derived subgroup) of Γ, generated by the set of all commutators in Γ. The second commutator subgroup (second derived subgroup) Γ'' is the subgroup generated by all commutators of commutators, i.e. by all elements of the form $[[a, b], [c, d]]$ for $a, b, c, d \in \Gamma$. The quotient Γ/Γ'' is metabelian and hence amenable. In terms of a flow on M, a periodic orbit has trivial Frobenius class for this cover if and only if it is null-homologous

and lifts to a null-homologous periodic orbit on the universal abelian cover. If $X^t : M \to M$ is a geodesic flow over a compact manifold with negative sectional curvatures then the lifted flow on the Γ/Γ'' cover is topologically transitive and so Theorem 1.4 applies.

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