ABELIAN LIVSIČ THEOREMS FOR ANOSOV FLOWS

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ABSTRACT. We give two short proofs of the abelian Livsič theorem of Gogolev and Rodriguez Hertz. We show that these proofs may be extended to give new abelian Livsic theorems for positive density sets of null-homologous orbits and for amenable covers.

1. INTRODUCTION

Let M be a compact smooth Riemannian manifold and let $X^t : M \to M$ be a transitive Anosov flow, generated by the vector field X. Let $\mathcal{P}$ denote the set of prime periodic orbits of the flow and let $\ell(\gamma)$ denote the least period of $\gamma \in \mathcal{P}$. For $f : M \to \mathcal{R}$ and $\gamma \in \mathcal{P}$, write

$$\int_\gamma f = \int_0^{\ell(\gamma)} f(X^t(x_\gamma))\, dt$$

for any x_γ on γ.

A classical result is the Livsič (or Livshits) periodic orbit theorem: if a Hölder continuous function $f : M \to \mathbb{R}$ satisfies

$$\int_\gamma f = 0 \quad \forall \gamma \in \mathcal{P} \tag{1.1}$$

then $f = L_X u$, where $u : M \to \mathbb{R}$ is a Hölder continuous function (with the same exponent as f) which is continuously differentiable along flow lines and L_X is the Lie derivative [10]. A more recent addition to this theory is the beautiful abelian Livsič theorem of Andrey Gogolev and Federico Rodriguez Hertz, which characterises Hölder functions f for which it is only assumed that (1.1) holds for *null-homologous* periodic orbits. For $\gamma \in \mathcal{P}$ (or more generally for any closed curve on M), let $[\gamma] \in H_1(M, \mathbb{Z})$ denote the homology class of γ. Write $\mathcal{P}_0 = \{\gamma \in \mathcal{P} : [\gamma] = 0\}$.

Theorem 1.1 (Gogolev and Rodriguez Hertz [8]). *Let $X^t : M \to M$ be a homologically full transitive Anosov flow. If $f : M \to \mathbb{R}$ satisfies*

$$\int_\gamma f = 0 \quad \forall \gamma \in \mathcal{P}_0 \tag{1.2}$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1*-form* ω *and some Hölder continuous* $u : M \to \mathbb{R}$ *which is continuously differentiable along flow lines.*

Remark 1.2. If the first Betti number of M is zero, so that $H_1(M, \mathbb{Z})$ is finite, then M admits no non-zero closed 1-forma and it is easy to see that Theorem 1.1 reduces to the classical Livsič theorem.

We give two new and short proofs of Theorem 1.1, one based on the weighted equidistribution theorems for null-homologous periodic orbits in [4] and the other on older asymptotic counting results of Lalley [9], Sharp [17] and Babillot and Ledrappier [2]. Our second proof also gives an abelian Livsič theorems for sets of null-homologous orbits with positive density (analogous to the positive density version of the classical Livsič theorem obtained recently by Dilsavor and Reber [5]).

Theorem 1.3. *Let* $X^t : M \to M$ *be a homologically full transitive Anosov flow. If* $f : M \to \mathbb{R}$ *satisfies*

$$\limsup_{T \to \infty} \frac{\#\left\{\gamma \in \mathcal{P}_0 \,:\, T < \ell(\gamma) \leq T + \Delta,\ \int_\gamma f = 0\right\}}{\#\{\gamma \in \mathcal{P}_0 \,:\, T < \ell(\gamma) \leq T + \Delta\}} > 0 \tag{1.3}$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1*-form* ω *and some Hölder continuous* $u : M \to \mathbb{R}$ *which is continuously differentiable along flow lines.*

We also have an abelian Livsič theorem for amenable covers. For the next result, $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ is the lifted flow on a regular cover of M. Let G be the covering group, with identity element e. For each $\gamma \in \mathcal{P}$, we can associated its *Frobenius class* $\langle \gamma \rangle$, which is a conjugacy class in G. If $\tilde{\gamma}$ is any lift of $\gamma \in \mathcal{P}$ then the initial and final endpoints of $\tilde{\gamma}$ are related by the action of some $g \in \langle \gamma \rangle$, and $\tilde{\gamma}$ is itself a periodic orbit if and only if $\langle \gamma \rangle = \{e\}$. Write $\widetilde{\mathcal{P}}_0 = \{\gamma \in \mathcal{P} \,:\, \langle \gamma \rangle = \{e\}\}$.

Theorem 1.4. *Let* $X^t : M \to M$ *be a homologically full transitive Anosov flow and let* $\widetilde{M}$ *be a regular cover of* M *such that*

- *the lifted flow* $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ *is topologically transitive, and*
- *the covering group* G *is amenable.*

If $f : M \to \mathbb{R}$ *satisfies*

$$\int_\gamma f = 0 \quad \forall \gamma \in \widetilde{\mathcal{P}}_0 \tag{1.4}$$

then

$$f = \omega(X) + L_X u,$$

for some smooth closed 1*-form* ω *and some Hölder continuous* $u : M \to \mathbb{R}$ *which is continuously differentiable along flow lines.*

In the next section we give some background on Anosov flows and give two proofs of Theorem 1.1 (the second of which also implies Theorem 1.3). In sction 3, we discuss how to extend the abelian Livsič theorem to amenable covers and prove Theorem 1.4.

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2. Anosov flows and cohomology

Let M be a compact Riemannian manifold and $X^t : M \to M$ be a C^1 transitive Anosov flow [1] (for a comprehensive modern treatment, see [7]). We have $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^b \oplus \mathfrak{F}$, where $b = b_1(M) \geq 0$ is the first Betti number of M and $\mathfrak{F}$ is a finite abelian group.

We say that X^t is *homologically full* if the map $\mathcal{P} \to H_1(M, \mathbb{Z}) : \gamma \mapsto [\gamma]$ is a surjection. This automatically implies that the flow is weak-mixing (since an Anosov flow fails to be weak-mixing only when it is a constant suspension of an Anosov diffeomorphism [12], it which case it can have no null-homologous periodic orbits). From now on, we assume that $b \geq 1$ and ignore any torsion in $H_1(M, \mathbb{Z})$ (so we interpret $[\gamma]$ as an element of $H_1(M, \mathbb{Z})/\mathfrak{F}$).

We interpret the real cohomology group $H^1(M, \mathbb{R})$ as the de Rham cohomology group. i.e. the quotient of the space of smooth closed 1-forms on M by the space of smooth exact 1-forms. We write $[\omega]$ for the cohomology class determined by the closed 1-form ω, i.e. if $[\omega] = [\omega']$ then $\omega - \omega'$ is an exact form, and we say that ω has integral periods if $\int_c \omega \in \mathbb{Z}$ for every smooth closed curve in M. We can choose forms $\omega_1, \ldots, \omega_b$ with integral periods so that $[\omega_1], \ldots, [\omega_b]$ is a basis for $H^1(M, \mathbb{R})$. Then, for $\gamma \in \mathcal{P}$ the map

$$\gamma \mapsto \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_b \right) = \left(\int_\gamma \omega_1(X), \ldots, \int_\gamma \omega_b(X) \right)$$

induces an isomorphism between $H_1(M, \mathbb{Z})/\mathfrak{F}$ and $\mathbb{Z}^b$.

Let $\mathcal{M}(X)$ denote the set of X^t-invariant Borel probability measures on M. Given $\nu \in \mathcal{M}(X)$, we define the associated winding cycle (or asymptotic cycle) $\Phi_\nu \in H_1(M, \mathbb{R})$ by

$$\langle \Phi_\nu, [\omega] \rangle = \int \omega(X)\, d\nu,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing (Schwartzman [16], Verjovsky and Vila Freyer [18]).

In addition to the cohomology of the manifold M, we will also consider the dynamical cohomology associated to the flow $X^t : M \to M$. For $0 < \theta < 1$, let $C^\theta(M, \mathbb{R})$ denote the space of Hölder continuous functions from M to $\mathbb{R}$ with exponent θ. A function $f : M \to \mathbb{R}$ is called a (dynamical) coboundary if $f = L_X u$, for some function $u : M \to \mathbb{R}$ which is differentialble along flow lines, and two functions are cohomologous if their difference is a coboundary. We write $B^\theta(M, \mathbb{R})$ for the intersection of the set of flow coboundaries with

$C^\theta(M,\mathbb{R})$. Finally we define the dynamical cohomology group $\mathcal{H}^1(X,\mathbb{R})$ by $\mathcal{H}^1(X,\mathbb{R}) := C^\theta(M,\mathbb{R})/B^\theta(M,\mathbb{R})$ and this is independent of the choice of θ. Given a Hölder continuous function $f : M \to \mathbb{R}$, we write $[f]$ for its flow cohomology class.

Let $\varphi : M \to \mathbb{R}$ be Hölder continuous. We define its pressure $P(\varphi)$ by

$$P(\varphi) = \sup\left\{h(\nu) + \int \varphi \, d\nu \, : \, \nu \in \mathcal{M}(X)\right\}$$

and there is a unique $\mu_\varphi \in \mathcal{M}(X)$, called the equilibrium state for φ, at which the supremum is attained. The following result is fundamental.

Lemma 2.1. *If $\varphi, \psi : M \to \mathbb{R}$ are Hölder continuous then $\mu_\varphi = \mu_\psi$ if and only if $\varphi - \psi$ is a cohomologous to a constant.*

Proof. This is a standard result but it is hard to find a clear reference. First, we note that the "if" direction is trivial. For the other direction, we can proceed using symbolic dynamics. Suppose $\mu_\varphi = \mu_\psi$. Without loss of generality, we can add constants to φ and ψ so that $P(\varphi) = P(\psi) = 0$. By the classical results of Bowen [3] and Ratner [13], we can find a suspension flow $\sigma^t : \Sigma^r \to \Sigma^r$ over a mixing subshift of finite type $\sigma : \Sigma \to \Sigma$, where $r : \Sigma \to \mathbb{R}^{>0}$ is Hölder continuous, and a Hölder continuous surjection $\pi : \Sigma^r \to M$ that is one-to-one on a residual set and satisfies $X^t \circ \pi = \pi \circ \sigma^t$. Furthermore, $\pi_*(\mu_\varphi) = \mu_{\varphi\circ\pi}$ and $\pi_*(\mu_\psi) = \mu_{\psi\circ\pi}$, where the measures on the right are equilibrium states with respect to the suspension flow, and $P(\varphi \circ \pi) = P(\psi \circ \pi) = 0$. Thus, if $\mu_\varphi = \mu_\psi$ then $\mu_{\varphi\circ\pi} = \mu_{\psi\circ\pi}$. If we define $\Phi, \Psi : \Sigma \to \mathbb{R}$ by

$$\Phi(x) = \int_0^{r(x)} \varphi \circ \pi(\sigma^t(x,0)) \, dt \text{ and } \Psi(x) = \int_0^{r(x)} \psi \circ \pi(\sigma^t(x,0)) \, dt$$

then (by Proposition 6.1 of [11]) $P(\Phi) = P(\Psi) = 0$ and

$$\mu_{\varphi\circ\pi} = \frac{m_\Phi \times \mathrm{Leb}}{\int r \, dm_\Phi} \text{ and } \mu_{\psi\circ\pi} = \frac{m_\Psi \times \mathrm{Leb}}{\int r \, dm_\Psi},$$

where m_Φ and m_Ψ are the equilibrium states of Φ and Ψ, respectively, with respect to $\sigma : \Sigma \to \Sigma$. We can then deduce that $m_\Phi = m_\Psi$. By Proposition 3.6 of [11], Φ and Ψ are cohomologous. (Here, we have use that $P(\Phi) = P(\Psi) = 0$ to ensure the additive constant is zero.) We can then deduce that Φ and Ψ have the same sums around each σ-periodic orbit and hence that $\int_\gamma \varphi - \int_\gamma \psi$ for all $\gamma \in \mathcal{P}$. Applying, Livsič's theorem, φ and ψ are cohomologous. □

We will now prove Theorem 1.1.

First proof of Theorem 1.1. Following section 5 of [4], given a Hölder continuous function $\varphi : M \to \mathbb{R}$, we can fund a unique $\xi(\varphi) \in \mathbb{R}^b$ such that the equilibrium state of $\varphi + \sum_{i=1}^b \xi_i(\varphi)\omega_i(X)$, which we shall denote by $\mu(\varphi)$, satisfies $\Phi_{\mu(\varphi)} = 0$ and

$$h(\mu(\varphi)) = \int \varphi \, d\mu(\varphi) = \sup \left\{ h(\nu) + \int \varphi \, d\nu \, : \, \nu \in \mathcal{M}_X \text{ and } \Phi_\nu = 0 \right\}.$$

Furthermore, from Theorem 6.7 of [4], we see that μ is a weak* limit of averages of null-homologous orbital measures. More precisely, for all continuous functions $\psi : M \to \mathbb{R}$, we have

$$\lim_{T\to\infty} \left(\sum_{\substack{\gamma\in\mathcal{P}_0 \\ T<\ell(\gamma)\le T+1}} e^{\int_\gamma \varphi} \right)^{-1} \sum_{\substack{\gamma\in\mathcal{P}_0 \\ T<\ell(\gamma)\le T+1}} e^{\int_\gamma \varphi} \frac{\int_\gamma \psi}{\ell(\gamma)} = \int \psi \, d\mu(\varphi). \tag{2.1}$$

Now let $f : M \to \mathbb{R}$ be a Hölder continuous function satisfying $\int_\gamma f = 0$ for all $\gamma \in \mathcal{P}_0$. If we compare the cases $\varphi = 0$ and $\varphi = f$ then the corresponding terms on the left hand side of (2.1) are equal and so we obtain that $\mu(0) = \mu(f)$. Applying Lemma 2.1, we see that

$$f = L_X u + \sum_{i=1}^b (\xi_i(0) - \xi_i(f))\omega_i(X) + c,$$

where $u : M \to \mathbb{R}$ is a Hölder function which continuously differentiable along flow lines and $c \in \mathbb{R}$. We can see from (2.1) (with $\psi = f$) that $\int f \, d\mu(0) = 0$ and, combined with $\Phi_{\mu(0)} = 0$, this gives $c = 0$, completing the proof. □

Second proof of Theorem 1.1. An alternative, slightly longer, argument is to show that the failure of Theorem 1.1 is inconsistent with the periodic orbit counting results of [2], [9], [17]. Given a Hölder continuous function $f : M \to \mathbb{R}$ satisfying (1.4), we have

$$\#\{\gamma \in \mathcal{P}_0 \, : \, T < \ell(\gamma) \le T+1\} = \# \left\{ \gamma \in \mathcal{P}_0 \, : \, T < \ell(\gamma) \le T+1, \, \int_\gamma f = 0 \right\}.$$

Now, we have from [17] that

$$\#\{\gamma \in \mathcal{P}_0 \, : \, T < \ell(\gamma) \le T+1\} \sim C \frac{e^{\alpha T}}{T^{1+b/2}}, \qquad \text{as } T \to \infty,$$

for some $C > 0$ and $\alpha > 0$. On the other hand, if $[f]$ is not in the span of $[\omega_1(X)], \ldots, [\omega_b(X)]$ then the results of [2], [9] give that

$$\# \left\{ \gamma \in \mathcal{P}_0 \, : \, T < \ell(\gamma) \le T+1, \, \int_\gamma f = 0 \right\} \sim C' \frac{e^{\alpha T}}{T^{1+(b+1)/2}}, \qquad \text{as } T \to \infty,$$

for some $C' > 0$, a contradiction. □

One sees that the second proof immediately gives a proof of Theorem 1.3. (I am grateful to Andrey Gogolev for this observation.)

Remark 2.2. Let $\overline{M}$ be any regular abelian cover of M, with covering group A (of rank at least one). The homology class $[c]$ of a closed curve on M projects to an element $[c]_A \in A$. We say that $X^t : M \to M$ is A-full if the map $\mathcal{P} \to A : \gamma \mapsto [\gamma]_A$ is a surjection. The above analysis also works in this setting and an abelian Livsič theorem holds. (Note that this statement is not trivially true as the flow might be A-full but not homologically full.)

3. Amenable covers

Let $\widetilde{M}$ be a regular cover of M such that $G = \pi_1(M)/\pi_1(\widetilde{M})$ is amenable. Let $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ be the lift of $X^t : M \to M$ to the cover.

Theorem 3.1. *Suppose that $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ is topologically transitive. Then for sufficiently large $\Delta > 0$ and any Hölder continuous function $\varphi : M \to \mathbb{R}$, we have*

$$\lim_{T\to\infty} \left(\sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T+\Delta}} e^{\int_\gamma \varphi} \right)^{-1} \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T+\Delta}} e^{\int_\gamma \varphi} \frac{\int_\gamma \psi}{\ell(\gamma)} = \int \psi \, d\mu(\varphi),$$

for all continuous functions $\psi : M \to \mathbb{R}$.

Following the large deviations approach of section 9 of [6], Theorem 3.1 follows once one has shown the following.

Proposition 3.2. *For sufficiently large $\Delta > 0$,*

$$\lim_{T\to\infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \leq T+\Delta}} e^{\int_\gamma \varphi} = P\left(\varphi + \sum_{i=1}^b \xi_i(\varphi)\omega_i(X) \right).$$

Before we prove this, we need to introduce some more ideas from thermodynamic formalism, particularly that of Gurevič pressure. Recall the notation introduced in the proof of Lemma 2.1. The flow $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ may be modelled by a suspension flow $\tilde{\sigma}^t$ over a skew-product system $T_\alpha : \Sigma \times G \to \Sigma \times G$ defined by $T_\alpha(x, g) = (\sigma x, g\alpha(x))$, where $\alpha : \Sigma \to G$ is some continuous function depending on two co-ordinates, with a roof function $\tilde{r} : \Sigma \times G \to \mathbb{R}^{>0}$ satisfying $\tilde{r}(x, g) = r(x)$; to simplify notation, we will write r instead of $\tilde{r}$. The function $\sum_{i=1}^b \xi_i(\varphi)\omega_i(X) : M \to \mathbb{R}$ induces a function $\Xi : \Sigma \to \mathbb{R}$, defined by

$$\Xi(x) := \int_0^{r(x)} \left(\varphi + \sum_{i=1}^b \xi_i(\varphi)\omega_i(X) \right) \circ \pi(\sigma^t(x, 0)) \, dt.$$

It is a standard result (Proposition 6.1 of [11]) that

$$P\left(-P\left(\varphi + \sum_{i=1}^b \xi_i(\varphi)\omega_i(X) \right) r + \Phi + \Xi \right) = 0.$$

Now let $\overline{G}$ be the torsion free part of the abelianization of G and let $\mathfrak{a} : G \to \overline{G}$ be the natural projection homomorphism. This gives a regular abelian cover of M and transitivity of $\widetilde{X}^t : \widetilde{M} \to \widetilde{M}$ gives that $X^t : M \to M$ is $\overline{G}$-full. As indicated in Remark 2.2, we could work in this geberal setting but, for simplicity, suppose that

$$\overline{G} = H_1(M, \mathbb{Z})/\text{torsion} \cong \mathbb{Z}^b.$$

There is also a skew-product $T_{\bar{\alpha}} : \Sigma \times \mathbb{Z}^b \to \Sigma \times \mathbb{Z}^b$ defined by $T_{\bar{\alpha}}(x, m) = (\sigma x, m + \bar{\alpha}(x))$, where $\bar{\alpha} = \mathfrak{a} \circ \alpha$. Both skew-products T_α and $T_{\bar{\alpha}}$ are topologically transitive countable state Markov shifts (where transitivity is given by Lemma 7.3 of [6]), so we can define the *Gurevič pressure* of locally Hölder continuous potentials, see Sarig's original paper [14] or his survey [15].

A Hölder continuous function $F : \Sigma \to \mathbb{R}$ induces functions $\tilde{F} : \Sigma \times G \to \mathbb{R}$ and $\bar{F} : \Sigma \times \mathbb{Z}^b \to \mathbb{R}$ by $\tilde{F}(x, g) = \bar{F}(x, m) = F(x)$. It will not cause any confusion to denote all three functions by F. For such functions, the Gurevič pressure of F with respect to T_α and $T_{\bar{\alpha}}$, is defined by

$$P_{\mathrm{G}}(F, T_\alpha) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \alpha_n(x) = e}} e^{F^n(x)}$$

and

$$P_{\mathrm{G}}(F, T_{\bar{\alpha}}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \bar{\alpha}^n(x) = 0}} e^{F^n(x)},$$

respectively. (Here we have used that $T_\alpha^n(x, g) = (\sigma^n x, g\alpha^n(x))$, where $\alpha^n(x) := \alpha(x)\alpha(\sigma x) \cdots \alpha(\sigma^{n-1}(x).)$

The following result of [6] is key to our analysis.

Proposition 3.3 (Theorem 5.1 of [6])**.** *If T_α is topologically transitive and G is amenable then*

$$P_{\mathrm{G}}(F, T_\alpha) = P_{\mathrm{G}}(F, T_{\bar{\alpha}}).$$

Fix $\Delta > 0$ and define

$$\overline{\mathfrak{P}}(\varphi) := \lim_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \mathcal{P}_0 \\ T < \ell(\gamma) \le T + \Delta}} e^{\int_\gamma \varphi}.$$

We know from Corollary 6.2 of [4] that

$$\overline{\mathfrak{P}}(\varphi) = P\left(\varphi + \sum_{i=1}^b \xi_i(\varphi)\omega_i(X)\right),$$

so that

$$P\left(-\overline{\mathfrak{P}}(\varphi) r + \Phi + \Xi\right) = 0.$$

Now let

$$\mathfrak{P}(\varphi) := \limsup_{T \to \infty} \frac{1}{T} \log \sum_{\substack{\gamma \in \widetilde{\mathcal{P}}_0 \\ T < \ell(\gamma) \le T + \Delta}} e^{\int_\gamma \varphi}.$$

Clearly, $\mathfrak{P}(\varphi) \le \overline{\mathfrak{P}}(\varphi)$ and we claim that we have equality.

Lemma 3.4. *We have*

$$\mathfrak{P}(\varphi) = \overline{\mathfrak{P}}(\varphi).$$

Proof. Consider the series

$$S_1(s) := \sum_{\gamma \in \widetilde{\mathcal{P}}_0} e^{-s\ell(\gamma) + \int_\gamma \varphi}.$$

The series $S_1(s)$ has abscissa of convergence $\mathfrak{P}(\varphi)$, and so $\mathfrak{P}(\varphi) = \overline{\mathfrak{P}}(\varphi)$ if $\overline{\mathfrak{P}}(\varphi)$ is the abscissa of convergence of $S_1(s)$.

Now consider the corresponding series for T_α,

$$S_2(s) := \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{\sigma^n x = x \\ \alpha^n(x) = e}} e^{-sr^n(x) + \Phi^n(x)}.$$

This may involve some overcounting compared to $S_1(s)$ but it is standard that $S_1(s) - S_2(s)$ converges for $\mathrm{Re}(s) > \overline{\mathfrak{P}}(\varphi) - \epsilon$, for some $\epsilon > 0$. Thus the abscissa of convergence of $S_1(s)$ is $\overline{\mathfrak{P}}(\varphi)$ if and only if the abscissa of convergence of $S_2(s)$ is $\overline{\mathfrak{P}}(\varphi)$.

The abscissa of convergence of $S_2(s)$ is given by the value c for which $P_{\mathrm{G}}(-cr + \Phi, T_\alpha) = 0$. By Proposition 3.3,

$$P_{\mathrm{G}}(-cr + \Phi, T_\alpha) = P_{\mathrm{G}}(-cr + \Phi, T_{\bar{\alpha}})$$

and so $c = \overline{\mathfrak{P}}(\varphi)$, as required. □

To complete the proof of Proposition 3.2 we observe that the arguments in section 2 of the correction to [6] show that, provided $\Delta > 0$ is sufficiently large, the limsup defining $\mathfrak{P}(\varphi)$ is a limit.

The proof of Theorem 1.4 now follows from Theorem 3.1 exactly as in the first proof of Theorem 1.1.

Remark 3.5. An interesting example of an amenable cover is that associated to the second commutator of $\pi_1(M)$. Let M be a quotient $M = U/\Gamma$, where U is the universal cover of M and $\Gamma \cong \pi_1(M)$ is a group of isometries acting freely on U. The universal abelian cover of M is the regular cover with covering group $\Gamma/\Gamma' \cong H_1(M, \mathbb{Z})$, where $\Gamma' = [\Gamma, \Gamma]$ is the commutator subgroup (derived subgroup) of Γ, generated by the set of all commutators in Γ. The second commutator subgroup (second derived subgroup) Γ'' is the subgroup generated by all commutators of commutators, i.e. by all elements of the form $[[a,b],[c,d]]$ for $a,b,c,d \in \Gamma$. The quotient Γ/Γ'' is metabelian and hence amenable. In terms of a flow on M, a periodic orbit has trivial Frobenius class for this cover if and only if it is null-homologous and lifts to a null-homologous periodic orbit on the universal abelian cover. If $X^t : M \to M$ is a geodesic flow over a compact manifold with negative sectional curvatures then the lifted flow on the Γ/Γ'' cover is topologically transitive and so Theorem 1.4 applies.

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