CRITICAL EXPONENTS FOR GROUPS OF ISOMETRIES

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Abstract. Let Γ be a convex co-compact group of isometries of a CAT(-1) space X and let Γ_0 be a normal subgroup of Γ. We show that, provided Γ is a free group, a sufficient condition for Γ and Γ_0 to have the same critical exponent is that Γ/Γ_0 is amenable.

0. Introduction and Results

Let Γ be a group of isometries acting freely and properly discontinuously on a CAT(-1) space X. Roughly speaking, a CAT(-1) space is a path metric space for which every geodesic triangle is more pinched than a congruent triangle in the hyperbolic plane; see [5] for a formal definition. Prototypical examples of CAT(-1) spaces are simply connected Riemannian manifold with sectional curvatures bounded above by -1 and (simplicial or non-simplicial) $\mathbb{R}$-trees.

A fundamental quantity associated to Γ is its critical exponent $\delta(\Gamma)$. This is defined to be the abscissa of convergence of the Poincaré series

$$\wp_\Gamma(s) = \sum_{\gamma \in \Gamma} e^{-s d_X(o,\gamma o)}, \tag{0.1}$$

where $o \in X$ and $d_X(\cdot,\cdot)$ denotes the distance in X. In other words, the series converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$. An equivalent definition is that

$$\delta(\Gamma) = \limsup_{T \to +\infty} \frac{1}{T} \log \#\{\gamma \in \Gamma \,:\, d_X(o,\gamma o) \le T\}. \tag{0.2}$$

A simple calculation shows that $\delta(\Gamma)$ is independent of the choice of $x \in X$.

Let ∂X denote the ideal boundary of X. The set $\{\gamma o \,:\, \gamma \in \Gamma\}$ accumulates on a subset $\Lambda_\Gamma \subset \partial X$ (independent of o) called the limit set of Γ. Let $\mathcal{C}_\Gamma = \text{c.h.}(\Lambda_\Gamma) \cap X$, where c.h.$(\Lambda_\Gamma)$ is the geodesic convex hull of Λ_Γ. We say that Γ is convex co-compact if $\mathcal{C}_\Gamma/\Gamma$ is compact. (If Γ is a Kleinian group, this agrees with the classical notion of convex co-compactness.) In addition, we say that Γ is non-elementary if it is not a finite extension of a cyclic group. These two conditions ensure that $\delta(\Gamma) > 0$ and the limit in (0.2) exists.

Now suppose that Γ_0 is a normal subgroup of a convex co-compact group Γ. Then Γ_0 itself has a critical exponent $\delta(\Gamma_0)$ and, clearly, $\delta(\Gamma_0) \le \delta(\Gamma)$. Our main result addresses the question of when we have equality.

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Theorem 1. *If Γ/Γ_0 is amenable then $\delta(\Gamma_0) = \delta(\Gamma)$.*

The definition of amenable group is given in the next section.

Remark. Equality of $\delta(\Gamma_0)$ and $\delta(\Gamma)$ was previously known to hold when Γ/Γ_0 is finite or abelian [15]. (In fact, the results in [15] are stated in the case where X is real hyperbolic space but the proofs given there apply more generally.)

Since obtaining the results in this paper, we have learned that Theorem 1 has been proved by Roblin [16], without the restriction that Γ is a free group, using completely different methods. However, we feel that our alternative approach, based on approximating $\delta(\Gamma)$ and $\delta(\Gamma_0)$ by quantities related to random walks on graphs, has independent interest. It is worth remarking that the equality of the two critical exponents has been used recently in [10].

We shall now outline the contents of the paper. In section 1, we give definition of amenable groups and introduce Grigorchuk's co-growth criterion, interpreting it in terms of a graph. In section 2, we describe how to write the Poincaré series $\wp_\Gamma(s)$ and $\wp_{\Gamma_0}(s)$ in terms of a subshift of finite type. We also introduce sequences of matrices which are used to approximate $\delta(\Gamma)$ and $\delta(\Gamma_0)$. In section 3, we use ideas from the theory of random walks on graphs, in particular [12], to show that, if Γ/Γ_0 is amenable then the respective approximations to $\delta(\Gamma)$ and $\delta(\Gamma_0)$ agree at each stage, from which Theorem 1 follows. In the final section, we consider that special case of $X = \mathbb{H}^{n+1}$.

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1. Amenable Groups and Co-Growth

Amenable groups were defined by von Neumann. A group G is said to be amenable if there is an invariant mean on $L^\infty(G, \mathbb{R})$, i.e., a bounded linear functional $\mu : L^\infty(G, \mathbb{R}) \to \mathbb{R}$ such that, for any $f \in L^\infty(G, \mathbb{R})$,

(i) $\inf_{g\in G} f(g) \le \mu(f) \le \sup_{g\in G} f(g)$; and
(ii) for all $g \in G$, $\mu(g \cdot f) = \mu(f)$, where $g \cdot f(x) = f(g^{-1}x)$.

It is immediate from the definition that any finite group is amenable by setting

$$\mu(f) = \frac{1}{|G|} \sum_{g\in G} f(g).$$

The situation for infinite groups is more subtle and we shall restrict our discussion to finitely generated groups.

A group with subexponential growth is amenable [2],[7]. In particular, any abelian or nilpotent group is amenable. However, there are examples of amenable groups with exponential growth (e.g. the lamplighter groups [8]). In contrast, non-abelian free groups and, more generally, non-elementary Gromov hyperbolic groups are not amenable. It was conjectured by von Neumann that a group fails to be amenable only if it contains the free group on two generators; however, a counterexample to this was constructed by Ol'shanskii [11].

Grigorchuk related amenability to the property of co-growth of subgroups of free groups. Let Γ (considered as an abstract group) be the free group on k generators $\{a_1, \ldots, a_k\}$

and let $|\gamma|$ denote the word length of γ, i.e., the length of the shortest representation of γ as a word in $a_1^{\pm 1}, \ldots, a_k^{\pm 1}$. Clearly, we have that

$$\lim_{n \to +\infty} \left(\#\{\gamma \in \Gamma : |g| = n\}\right)^{1/n} = 2k - 1.$$

Now suppose that Γ_0 is a normal subgroup of Γ. Grigorchuk showed that the co-growth $c(\Gamma_0)$, defined by

$$c(\Gamma_0) := \limsup_{n \to +\infty} \left(\#\{g \in \Gamma_0 : |g| = n\}\right)^{1/n},$$

is equal to $2k - 1$ if and only if $G = \Gamma/\Gamma_0$ is amenable [6] (see also [4]).

Grigorchuk's result may be reinterpreted in terms of graphs. Let $\mathcal{G}$ denote the graph consisting of one vertex and k oriented edges, labelled by $a_1, \ldots, a_k$. The same edges with the reverse orientation will be labelled $a_1^{-1}, \ldots, a_k^{-1}$, respectively. Write $\mathcal{T}$ for the universal cover of $\mathcal{G}$; then $\mathcal{T}$ is a $2k$-regular tree. It is an easy observation that Γ acts freely on $\mathcal{T}$ with quotient $\mathcal{G}$. Furthermore, we may identify elements of word length n in Γ with non-backtracking paths of length n in $\mathcal{G}$. (A path $(e_1, \ldots, e_n)$ is said to be non-backtracking if, for each $i = 2, \ldots, n$, the edge e_i is not equal to e_{i-1} with the reversed orientation.)

Now consider the action of the subgroup Γ_0 on $\mathcal{T}$ and write $\widetilde{\mathcal{G}} = \mathcal{T}/\Gamma_0$ for the quotient graph; this is a G-cover of $\mathcal{G}$. (In fact, $\widetilde{\mathcal{G}}$ is the Cayley graph of G with respect to the generators obtained from $a_1, \ldots, a_k$.) Then we may identify elements of word length n in Γ_0 with non-backtracking paths of length n in $\widetilde{\mathcal{G}}$ starting from and ending at some fixed vertex. Grigorchuk's result may then be reformulated as saying that the growth rate of the number of paths of length n in $\widetilde{\mathcal{G}}$, starting from and ending at a fixed vertex, is equal to the corresponding growth rate for paths in $\mathcal{G}$ if and only if Γ/Γ_0 is amenable.

The parallels between equality of these growth rates and equality of the critical exponents is apparent. However, the "lengths" are different: word length $|\gamma|$ in one setting and the displacement $d(o, \gamma o)$ for the action on X in the other. Nevertheless, this will provide the basis for our approach. In this context, we note that there exists $A > 1$ such that

$$A^{-1}|\gamma| \leq d(o, \gamma o) \leq A|\gamma|. \tag{1.1}$$

We shall use several properties of the graph $\widetilde{\mathcal{G}}$. Firstly, provided it is not itself a tree (which only occurs if Γ_0 is trivial) $\widetilde{\mathcal{G}}$ has the property that "small cycles are dense" [12]: there exists $R > 0$ such that, for each vertex u in $\widetilde{\mathcal{G}}$, the set $B(u, R) = \{v : d_{\widetilde{\mathcal{G}}}(u, v) \leq R\}$ contains a cycle. We also note that there is a number $L(R) > 0$ such that, for every vertex u in $\widetilde{\mathcal{G}}$, $\#B(u, R) \leq L(R)$.

Later we shall need to find paths joining vertices in $\widetilde{\mathcal{G}}$. Let $c_n(u, v)$ denote the number of non-backtracking paths of length n in $\widetilde{\mathcal{G}}$ from u to v.

Lemma 1.1 [17]. *Let u, v be vertices of $\widetilde{\mathcal{G}}$. Then either*

$$\lim_{n \to +\infty} c_n(u, v)^{1/n} = c(\Gamma_0)$$

or

$$\lim_{n \to +\infty} c_{2n+\delta(u,v)}(u, v)^{1/2n} = c(\Gamma_0) \quad \text{and} \quad c_{2n+\delta(u,v)-1}(u, v) = 0,$$

where $\delta(u, v) = 0$ if $d_{\widetilde{\mathcal{G}}}(u, v)$ is even and $\delta(u, v) = 1$ if $d_{\widetilde{\mathcal{G}}}(u, v)$ is odd.

Corollary 1.1.1. *Suppose that G is amenable (or even that $c(\Gamma_0) > 0$) and let u, v be vertices of $\widetilde{\mathcal{G}}$. Then there exists $l(u,v) > 0$ such that either $c_{l(u,v)}(u,v) > 0$ or $c_{l(u,v)-1}(u,v) > 0$.*

2. Shifts of Finite Type and Approximation

Recall that the free group Γ is given in terms of generators $\mathcal{A} = \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}$. We shall form a subshift of finite type $\sigma : \Sigma \to \Sigma$, where

$$\Sigma = \{x = (x_i)_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{Z}^+} : x_{i+1} \neq x_i^{-1}, \ \forall i \in \mathbb{Z}^+\}$$

and σ is the shift map: $(\sigma x)_i = x_{i+1}$. We call $(x_0, \ldots, x_{n-1}) \in \mathcal{A}^n$ an allowed string of length n if $x_{i+1} \neq x_i^{-1}$, $i = 0, \ldots, n-2$. We write Σ_n for the set of all allowed strings of length n, $\Sigma_{\leq n} = \bigcup_{m=0}^{n} \Sigma_m$ and $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$, where Σ_0 is defined to be a singleton consisting of an "empty string" ω. There is an obvious bijection between Σ_n and elements of Γ with word length n (and hence between Γ and Σ^*.

We make $\Sigma \cup \Sigma^*$ into a metric space by setting $d(x,y) = 2^{-n(x,y)}$, where

$$n(x,y) = \begin{cases} 0 \text{ if } x_0 \neq y_0, \\ \sup\{n \geq 0 : x_m = y_m, 0 \leq m \leq n\} \text{ otherwise.} \end{cases}$$

If $f : \Sigma \cup \Sigma^* \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\alpha > 0$ then we write

$$|f|_\alpha = \sup\left\{\frac{f(x) - f(y)}{d(x,y)^\alpha} : x \neq y\right\}.$$

If we define $\sigma(\omega) = \omega$, the shift map extends to $\sigma : \Sigma \cup \Sigma^* \to \Sigma \cup \Sigma^*$ and $\sigma(\Sigma_n) = \Sigma_{n-1}$, $n \geq 1$. For a function $f : \Sigma \cup \Sigma^* \to \mathbb{R}$, we write $f^n(x) = f(x) + f(\sigma x) + \cdots + f(\sigma^{n-1}x)$.

Proposition 2.1 [9],[13],[14]**.** *There is a strictly positive Hölder continuous function $r : \Sigma \cup \Sigma^* \to \mathbb{R}$ such that, if $\gamma = x_0 \cdots x_{n-1}$ then*

$$r^n(x_0, \ldots, x_{n-1}) = d_X(o, \gamma o).$$

Remark. An examination of the proof in [14] shows that what is essential for the proof is that X satisfies the Aleksandrov-Toponogov Comparison property. Thus the result holds if X is a CAT(-1) space.

An easy calculation then shows that

$$\wp_\Gamma(s) = 1 + \sum_{n=1}^{\infty} \sum_{x \in \sigma^{-n}(\omega) \setminus \{\omega\}} e^{-sr^n(x)}.$$

Let $\psi : \Gamma \to G = \Gamma/\Gamma_0$ be the natural homomorphism and, for $x = (x_0, \ldots, x_{n-1}) \in \Sigma_n$, write $\psi_n(x) = \psi(x_0) \cdots \psi(x_{n-1})$. We have

$$\wp_{\Gamma_0}(s) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{x \in \sigma^{-n}(\omega) \setminus \{\omega\} \\ \psi_n(x) = e}} e^{-sr^n(x)}.$$

We shall study the abscissas of convergence of the above two series via a sequence of approximations to r. We define

$$r_N(x) = \begin{cases} r(x) \text{ if } x \in \Sigma_n, n \leq N; \\ r(x_0, \dots, x_{N-1}) \text{ otherwise.} \end{cases}$$

Then $\|r - r_N\|_\infty \leq |r|_\alpha 2^{-\alpha(N+1)}$, where $\alpha > 0$ is the Hölder exponent of r. Hence, given $\epsilon > 0$, we can choose N sufficiently large so that, for each $x \in \Sigma \cup \Sigma^*$ and $n \geq 1$, $|r^n(x) - r_N^n(x)| < n\epsilon$.

We define δ_N and δ_N^0 to be the abscissas of convergence of $\wp_N(s)$ and $\wp_N^0(s)$, respectively, where

$$\wp_N(s) = 1 + \sum_{n=1}^{\infty} \sum_{x \in \sigma^{-n}(\omega) \setminus \{\omega\}} e^{-s r_N^n(x)}, \ \wp_N^0(s) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{x \in \sigma^{-n}(\omega) \setminus \{\omega\} \\ \psi_n(x) = e}} e^{-s r_N^n(x)}.$$

Lemma 2.1. *We have* $\lim_{N \to +\infty} \delta_N = \delta(\Gamma)$ *and* $\lim_{N \to +\infty} \delta_N^0 = \delta(\Gamma_0)$.

Proof. For $\gamma = x_0 \cdots x_{|\gamma|-1} \in \Gamma$, let $x_\gamma = (x_0, \dots, x_{|\gamma|-1}) \in \Sigma^*$. Then, $r^{|\gamma|}(x_\gamma) = d(o, \gamma o)$, so, using this notation,

$$\delta(\Gamma) = \limsup_{T \to +\infty} \frac{1}{T} \log \#\{\gamma : r^{|\gamma|}(x_\gamma) \leq T\}, \ \delta_N = \limsup_{T \to +\infty} \frac{1}{T} \log \#\{\gamma : r_N^{|\gamma|}(x_\gamma) \leq T\}.$$

Fix $\epsilon > 0$ sufficiently small that $A\epsilon < 1$, where A is given by (1.1). Then, provided N is sufficiently large, $r^{|\gamma|}(x_\gamma) \leq r_N^{|\gamma|}(x_\gamma) + |\gamma|\epsilon \leq r_N^{|\gamma|}(x_\gamma) + A r^{|\gamma|}(x_\gamma)\epsilon$ and so

$$r^{|\gamma|}(x_\gamma) \leq \frac{r_N^{|\gamma|}(x_\gamma)}{1 - A\epsilon}.$$

Hence

$$\#\{\gamma : r_N^{|\gamma|}(x_\gamma) \leq T\} \leq \#\{\gamma : r^{|\gamma|}(x_\gamma) \leq (1 - A\epsilon)^{-1} T\}$$

and so $\delta_N \leq (1 - A\epsilon)^{-1} \delta(\Gamma)$. Since we may take ϵ arbitrarily small, we conclude that $\limsup_{N \to +\infty} \delta_N \leq \delta(\Gamma)$. A similar argument gives the corresponding lower bound, so we have $\lim_{N \to +\infty} \delta_N = \delta(\Gamma)$. The same proof gives the result for δ_N^0.

Hence, to prove Theorem 1, it suffices to show that if G is amenable then $\delta_N = \delta_N^0$, for each $N \geq 1$. We shall do this in the next section. First we need to rewrite $\wp_N(s)$ and $\wp_N^0(s)$ in matrix form.

For $N \geq 1$, define matrices P_N, indexed by $\Sigma_N \times \Sigma_N$, by

$$P_N(x, y) = \begin{cases} e^{-\delta_N r_N(x_0, x_1, \dots, x_{N-1}, y_{N-1})} \text{ if } x_n = y_{n-1}, \ n = 1, \dots, N-1; \\ 0 \text{ otherwise,} \end{cases}$$

where $x = (x_0, x_1, \dots, x_{N-1})$, $y = (y_0, y_1, \dots, y_{N-1})$. (For $N = 1$, we set $P_1(x_0, y_0) = 0$ whenever $y_0 = x_0^{-1}$. For $N \geq 2$ this is automatically avoided.) Each P_N is irreducible (and aperiodic). Also define another sequence of matrices Q_N, indexed by $\Sigma_{\leq N} \times \Sigma_{\leq N}$, by

$$Q_N(x, y) = \begin{cases} e^{-\delta_N r_N(x_0, x_1, \dots, x_{N-1}, y_{N-1})} \text{ if } x_n = y_{n-1}, \ n = 1, \dots, N-1; \\ 0 \text{ otherwise,} \end{cases}$$

where, for $x \in \Sigma_m$, we write $x = (x_0, \dots, x_{m-1}, \underbrace{\omega, \dots, \omega}_{N-m})$. The matrices Q_N are not irreducible. Note that P_N is the restriction of Q_N to $\Sigma_N \times \Sigma_N$.

From the definition of Q_N, we have that, for $n > N$,

$$\sum_{x \in \sigma^{-n}(\omega) \setminus \{\omega\}} e^{-\delta_N r_N^n(x)} = \sum_{x \in \Sigma_N} \sum_{a \in \Sigma_1} Q_N^n(x, (a, \omega, \dots, \omega)).$$

Now, since P_N is irreducible, the value of $\limsup_{n \to +\infty} (P_N^n(x,y))^{1/n}$ is independent of $x, y \in \Sigma_N$ (in fact it is the spectral radius of P_N).

Lemma 2.2. *For any $x, y \in \Sigma_N$ and $a \in \Sigma_1$,*

$$\limsup_{n \to +\infty} (P_N^n(x,y))^{1/n} = \limsup_{n \to +\infty} (Q_N^n(x, (z, \omega, \dots, \omega)))^{1/n}.$$

Proof. We have

$$\begin{aligned} Q_N^n(x, (a, \omega, \dots, \omega)) &= \sum_{y \in \Sigma_N} Q_N^{n-N}(x,y)\ Q_N^N(y, (a, \omega, \dots, \omega)) \\ &= \sum_{y \in \Sigma_N} P_N^{n-N}(x,y)\ Q_N^N(y, (a, \omega, \dots, \omega)). \end{aligned}$$

Since δ_N is the abscissa of convergence of $\wp_N(s)$, we deduce that, for each $x, y \in \Sigma_N$, $\limsup_{n \to +\infty} (P_N^n(x,y))^{1/n} = 1$.

By the Perron-Frobenius Theorem, P_N has 1 as an eigenvalue and an associated strictly positive (row) eigenvector v_N: $v_N P_N = v_N$. In addition, we may suppose that P_N is normalized so that

$$\sum_{y \in \Sigma_N} P_N(x,y) = 1.$$

In other words, P_N may be regarded as a matrix of transition probabilities between elements of Σ_N.

Now we define another sequence of (infinite) matrices $\widetilde{P}_N$, $N \geq 1$, indexed by $(\Sigma_N \times G) \times (\Sigma_N \times G)$, by

$$\widetilde{P}_N((x,g),(y,h)) = \begin{cases} P_N(x,y) \text{ if } \psi(x_0) = g^{-1}h; \\ 0 \text{ otherwise.} \end{cases}$$

(Note that the exponent in the entries of $\widetilde{P}_N$ is δ_N not δ_N^0.) Each $\widetilde{P}_N$ is locally finite in the sense that, for each (x,g), there are only finitely many (y,h) such that $\widetilde{P}_N((x,g),(y,h)) > 0$.

We also define a corresponding sequence of infinite matrices $\widetilde{Q}_N$, $N \geq 1$, indexed by $(\Sigma_{\leq N} \times G) \times (\Sigma_{\leq N} \times G)$, by

$$\widetilde{Q}_N((x,g),(y,h)) = \begin{cases} Q_N(x,y) \text{ if } \psi(x_0) = g^{-1}h; \\ 0 \text{ otherwise.} \end{cases}$$

We have

$$\sum_{\substack{x\in\sigma^{-n}(\omega)\setminus\{\omega\}\\ \psi(x)=e}} e^{-sr_N^n(x)} = \sum_{x\in\Sigma_N}\sum_{y\in\Sigma_1} \widetilde{Q}_N^n((x,e),((y,\omega,\dots,\omega),e)).$$

In section 4, we shall prove the following lemma.

Lemma 2.3. *G is amenable if and only if* $\limsup_{n\to+\infty}(\widetilde{P}_N^n((x,e),(y,e)))^{1/n}=1$.

This lemma implies that, provided G is amenable, $\delta_N=\delta_N^0$, $N\geq 1$. Combining this with Lemma 2.1 gives Theorem 1.

3. An Auxiliary Estimate

In this section we establish an estimate needed to complete the proof of Lemma 2.3 in section 4.

Write $\mathrm{Fix}_n=\{x\in\Sigma:\sigma^n x=x\}$. If $x=(x_0,x_1,\dots,x_{n-1},x_0,\dots)\in\mathrm{Fix}_n$, write $x^{-1}=(x_{n-1}^{-1},\dots,x_1^{-1},x_0^{-1},x_{n-1}^{-1},\dots)\in\mathrm{Fix}_n$.

Lemma 3.1. *For each $N\geq 1$, $r_N^n(x)=r_N^n(x^{-1})$ whenever $x\in\mathrm{Fix}_n$, $n\geq 1$.*

Proof. For $n\geq N$,

$$\begin{aligned} r_N^n(x) &= r(x_0,x_1,\dots,x_{N-1})+r(x_1,x_2,\dots,x_N)+\cdots+r(x_{n-1},x_0,\dots,x_{N-2})\\ &= d(o,x_0x_1\cdots x_{N-1}o)-d(o,x_1\cdots x_{N-1}o)\\ &+d(o,x_1x_2\cdots x_No)-d(o,x_2\cdots x_No)\\ &+\cdots+d(o,x_{n-1}x_0\cdots x_{N-2}o)-d(o,x_0\cdots x_{N-2}o). \end{aligned}$$

On the other hand,

$$\begin{aligned} r_N^n(x^{-1}) &= r(x_{n-1}^{-1},x_{n-2}^{-1},\dots,x_{n-N}^{-1})+r(x_{n-2}^{-1},x_{n-3}^{-1},\dots,x_{n-N-1}^{-1})\\ &+\cdots+r(x_0^{-1},x_{n-1}^{-1},\dots,x_{n-N+1}^{-1})\\ &= d(o,x_{n-1}^{-1}x_{n-2}^{-1}\cdots x_{n-N}^{-1}o)-d(o,x_{n-2}^{-1}\cdots x_{n-N}^{-1}o)\\ &+d(o,x_{n-2}^{-1}x_{n-3}^{-1}\cdots x_{n-N-1}^{-1}o)-d(o,x_{n-3}^{-1}\cdots x_{n-N-1}^{-1}o)\\ &+\cdots+d(o,x_0^{-1}x_{n-1}^{-1}\cdots x_{n-N+1}^{-1}o)-d(o,x_{n-1}^{-1}\cdots x_{n-N+1}^{-1}o)\\ &= d(o,x_{n-N}\cdots x_{n-2}x_{n-1}o)-d(o,x_{n-N}\cdots x_{n-2}o)\\ &+d(o,x_{n-N-1}\cdots x_{n-3}x_{n-2}o)-d(o,x_{n-N-1}\cdots x_{n-3}o)\\ &+\cdots+d(o,x_{n-N+1}\cdots x_{n-1}x_0o)-d(o,x_{n-N+1}\cdots x_{n-1}o)\\ &= r_N^n(x). \end{aligned}$$

If $n<N$, the calculations become easier.

Consider the restriction $r_N:\Sigma_N\to\mathbb{R}$. We can define another function $\check{r}_N:\Sigma_N\to\mathbb{R}$ by $\check{r}_N(x_0,\dots,x_{N-1})=r_N(x_{N-1}^{-1},\dots,x_0^{-1})$. Applying Livsic's theorem for finite directed graphs to the above result, we may deduce:

Corollary 3.1.1. *There exists $u : \Sigma_{N-1} \to \mathbb{R}$ such that*

$$r_N(x_0, x_1, \ldots, x_{N-1}) = r_N(x_{N-1}^{-1}, \ldots, x_1^{-1}, x_0^{-1}) + u(x_1, \ldots, x_{N-1}) - u(x_0, \ldots, x_{N-2}).$$

Lemma 3.2. *There exists a constant $C_0 > 0$ such that, for all $(x, g), (y, h) \in \Sigma_N \times G$ and $n \geq 1$,*

$$P_N^n((x, g), (y, h)) \leq C_0 P_N^n((\check{y}, h^{-1}), (\check{x}, g^{-1})),$$

where, if $x = (x_0, x_1, \ldots, x_{N-1})$ and $y = (y_0, y_1, \ldots, y_{N-1})$, we use the notation $\check{x} = (x_{N-1}^{-1}, \ldots, x_1^{-1}, x_0^{-1})$ and $\check{y} = (y_{N-1}^{-1}, \ldots, y_1^{-1}, y_0^{-1})$.

We may take

$$C_0 = \exp\left(2\delta_N \sup\{|u(x)| : x \in \Sigma_{N-1}\}\right).$$

4. Random Walks on Graphs

In order to prove Lemma 2.3, we shall adapt work of Ortner and Woess on non-backtracking random walks on graphs contained in [12].

For each $N \geq 1$, we define an (undirected) graph $\mathcal{S}_N$ with vertex set $\Sigma_N \times G$. Two vertices (x, g) and (y, h) will be joined by an edge if and only if either $\widetilde{P}_N((x, g), (y, h)) > 0$ or $\widetilde{P}_N((y, h), (x, g)) > 0$. We note that $\mathcal{S}_N$ is connected and that each vertex has degree $2k$.

We may think of $\widetilde{P}_N$ as defining a Markov process on $\mathcal{S}_N$. As part of the proof of Lemma 2.3, we will show that $\widetilde{P}_N$ has the following three properties [12]:

(1) $\widetilde{P}_N$ has bounded range, i.e., there exists $R > 0$ such that if $\widetilde{P}_N((x, g), (y, h)) > 0$ then (x, g) and (y, h) are at distance $\leq R$ in $\mathcal{S}_N$.

(2) $\widetilde{P}_N$ has a bounded invariant measure; i.e., there exists a function $\nu : \Sigma_N \times G \to \mathbb{R}^+$, bounded above and below away from zero, such that, for all $(y, h) \in \Sigma_N \times G$,

$$\sum_{(x,g) \in \Sigma_N \times G} \widetilde{P}_N((x, g), (y, h))\, \nu((x, g)) = \nu((y, h)).$$

(3) $\widetilde{P}_N$ is *uniformly irreducible*, i.e., there exist constants $K > 0$, $\epsilon > 0$ such that, for any pair of neighbouring vertices $(x, g), (y, h)$ in $\mathcal{S}_N$, one can find $k \leq K$ such that $\widetilde{P}_N^k((x, g), (y, h)) \geq \epsilon$.

We note that (1) holds immediately with $R = 1$.

To show (2), let recall that there is a strictly positive row vector $v_N = (v_N(x))_{x \in \Sigma_N}$ such that $v_N P_N = v_N$. Define ν by $\nu((x, g)) = v_N(x)$. Clearly this is bounded above and below away from zero. A simple calculation shows it has the desired $\widetilde{P}_N$-invariance.

Finally, we show that $\widetilde{P}_N$ is uniformly irreducible.

Lemma 4.1. *$\widetilde{P}_N$ is uniformly irreducible.*

Proof. Fix a number K (to be determined later). Let $\epsilon_0 < 1$ denote the smallest positive entry of $\widetilde{P}_N$ and let $\epsilon = \epsilon_0^K$; then, for every $k \leq K$, each positive entry of $\widetilde{P}_N^k$ is greater than or equal to ϵ. Let (x, g) and (y, h) be neighbouring vertices in $\mathcal{S}_N$. Without lose of generality, $\widetilde{P}_N((x,g),(y,h)) > \epsilon$ and $\widetilde{P}_N((y,h),(x,g)) = 0$. To complete the proof we need to find a positive probability path of length at most K from (y,h) to (x,g).

Observe that we can identify $\Sigma_N \times G$ with the set of non-backtracking paths of length N in $\widetilde{\mathcal{G}}$ and a positive probability path of length k in $\mathcal{S}_N$ corresponds to a non-backtracking path of length $N+k$ in $\widetilde{\mathcal{G}}$. We therefore need to show that, for any two non-backtracking paths (given by sequences of vertices) $(u_0, u_1, \ldots, u_N)$ and $(v_0, v_1, \ldots, v_N)$ in $\widetilde{\mathcal{G}}$, there exists $k \leq K$ such that there is a non-backtracking path of length k joining them to give a non-backtracking path from u_0 to v_N. It follows from Corollary 1.1.1 that there is a non-backtracking path $(u_N, w_1, \ldots, w_{\kappa-1}, v_0)$, with $\kappa \leq l(u_N, v_0)$, joining u_N to v_0. However, it is possible then when this is inserted between the other two paths, backtracking occurs. To avoid this we shall use the "small cycles are dense" property of $\widetilde{\mathcal{G}}$. (The following part of the proof is adapted from the proof of Lemma 3.7 in [12].)

First we consider the beginning of the inserted path. If $w_1 \neq u_{N-1}$ there is nothing to do, so suppose that $w_1 = u_{N-1}$. Choose a neighbour z_1 of u_N which is not equal to u_{N-1}. By Lemma 3.3 of [12], (u_N, z_1) may be extended into non-backtracking paths which reach infinitely many vertices. Since $B(u_{N-1}, R)$ is finite, we may choose one of these paths, $(u_N, z_1, \ldots, z_r)$, so that $z_r \notin B(u_{N-1}, R)$ but $z_i \in B(u_{N-1}, R)$, $i = 1, \ldots, r-1$ (with $r \leq L(R)+1$. By the "small cycles are dense" property, there is a cycle $(c_0, c_1, \ldots, c_{p-1}, c_0)$ in $B(z_r, R)$ (with $p \leq L(R)$). Either

(a) $z_r = c_i$ for some $i = 0, 1, \ldots, p-1$, or,

(b) by the definition of $B(z_r, R)$, there is a non-backtracking path $(z_r, a_1, \ldots, a_{q-1}, c_0)$ $(a_1 \neq z_{r-1})$ joining z_r to c_0 (with $q \leq R$).

In case (a), we insert

$$(u_N, z_1, \ldots, z_r, c_{i+1}, \ldots, c_{p-1}, c_0, \ldots, c_{i-1}, z_r, z_{r-1}, \ldots, z_1, u_N)$$

and in case (b), we insert

$$(u_N, z_1, \ldots, z_r, a_1, \ldots, a_{q-1}, c_0, c_1, \ldots, c_{p-1}, c_0, a_{q-1}, \ldots, a_1, z_r, z_{r-1}, \ldots, z_1, u_N)$$

between $(u_0, u_1, \ldots, u_N)$ and $(u_N, w_1, \ldots, w_{\kappa-1}, v_0)$.

Now consider the end of the path $(u_N, w_1, \ldots, w_{\kappa-1}, v_0)$. If $w_{k-1} \neq v_1$ there is nothing to do. On the other hand, if $w_{k-1} = v_1$ then we carry out a similar construction to that in the paragraph above.

In this way, we have obtained a non-backtracking path starting with $(u_0, u_1, \ldots, u_N)$ and ending with $(v_0, v_1, \ldots, v_N)$ with u_N and v_0 being joined in at most $l(u_N, v_0) + 4(L(R)+1) + 4R + 4L(R)$ steps.

To complete the proof, we need to show that this number may be bounded independently of our initial choice of (x,g) and (y,h) (which determine u_N and v_0). First, we note that there are only finitely many x and y in Σ_N. Second, we observe that, for any $a \in G$, $\widetilde{P}_N((x,ag),(y,ah)) = \widetilde{P}_N((x,g),(y,h))$, so, without loss of generality, we may suppose that $g = e$. Since (y,h) is a neighbour of (x,g) in $\mathcal{S}_N$, this forces h to be one of the finitely many elements $\psi(a_1^{\pm 1}), \ldots, \psi(a_k^{\pm 1})$. Therefore, we may choose K to be the maximum of $l(u_N, v_0) + 8L(R) + 4R + 4$, taken over this finite number of choices.

Since $\widetilde{P}_N$ has an invariant measure ν, it acts on the Hilbert space $l^2(\mathcal{S}_N, \nu)$. Let $\rho_2(\widetilde{P}_N)$ denote the spectral radius. Also, since $\widetilde{P}_N$ is irreducible,

$$\rho(\widetilde{P}_N) = \limsup_{n\to+\infty} (\widetilde{P}_N^n((x,g),(y,h)))^{1/n}$$

is independent of (x,g) and (y,h) and $\rho(\widetilde{P}_N) \le \rho_2(\widetilde{P}_N)$.

To complete the proof of Lemma 2.3 (and hence of Theorem 1) we use the following results from [12]. (See page 112 of [18] for the definition of an amenable graph.)

Proposition 4.1 [12, Theorem 3.6]**.** *If $\mathcal{S}_N$ is connected with bounded vertex degrees and $\widetilde{P}_N$ satisfies (1),(2) and (3) then $\rho_2(\widetilde{P}_N) = 1$ if and only if $\mathcal{S}_N$ is amenable.*

We have already seen that the hypotheses used in Proposition 4.1 are satisfied. The next result relates $\rho_2(\widetilde{P}_N)$ and $\rho(\widetilde{P}_N)$.

Proposition 4.2. *$\rho(\widetilde{P}_N) = \rho_2(\widetilde{P}_N)$.*

Proof. The proof is a simple modification of the proof of Proposition 1.6 in [12]. The hypothesis there is that one has a graph for which "small cycles are dense"; since this holds for $\widetilde{\mathcal{G}}$, it also holds for $\mathcal{S}_N$. There are two differences from the proof in [12]:

(1) we consider a matrix $\overline{P}_N = \frac{1}{2}(I + \widetilde{P}_N)$, where I is the identity matrix, and observe that $\overline{P}_N$ preserves ν (rather than the counting measure as in [12]);

(2) we use Lemma 3.2: there exists a constant $C_0 > 0$ such that, for all $(x,g),(y,h) \in \Sigma_N \times G$ and $n \ge 1$,

$$P_N^n((x,g),(y,h)) \le C_0 P_N^n((\check{y},h^{-1}),(\check{x},g^{-1})),$$

where, if $x = (x_0, x_1, \ldots, x_{N-1})$ and $y = (y_0, y_1, \ldots, y_{N-1})$, we use the notation $\check{x} = (x_{N-1}^{-1}, \ldots, x_1^{-1}, x_0^{-1})$ and $\check{y} = (y_{N-1}^{-1}, \ldots, y_1^{-1}, y_0^{-1})$. (In [12], the inequality is an equality with $C_0 = 1$.)

Neither of these affect the proof.

Together, these two results show that $\rho(\widetilde{P}_N) = 1$ if and only if $\mathcal{S}_N$ is amenable. To finish things off, we show that the latter condition is equivalent to the amenability of G.

Recall that a map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometry* if there exist $A \ge 1$, $B, C \ge 0$ such that,

(i) for all $x, x' \in X$, $A^{-1}d_X(x,x') - B \le d_Y(f(x), f(x')) \le A d_X(x,x') + B$; and

(ii) for every $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) \le C$.

Proposition 4.3. *$\mathcal{S}_N$ is amenable if and only if G is amenable.*

Proof. We identify G with its Cayley graph $\mathcal{C}(G)$; G is an amenable group if and only if $\mathcal{C}(G)$ is an amenable graph. Define a map $f_N : \mathcal{S}_N \to \mathcal{C}(G)$ on the vertices by $f_N(x,g) = g$ and extend it to the edges by $f_N((x,g),(y,h)) = (g,h)$. This map is clearly a quasi-isometry. Since, for graphs with bounded vertex degree, amenability is an invariant of quasi-isometry [18, Theorem 4.7], the result is proved.

In this section we shall discuss the relevance of our results for Kleinian groups acting on the hyperbolic space $\mathbb{H}^{n+1}$ and, in particular, for finitely generated Fuchsian results. (These results are subsumed by those in [16].)

We begin be describing the results of Brooks on amenability and the spectrum of the Laplacian. Let N be a complete Riemannian manifold and let Δ_N denote the Laplace-Beltrami operator acting on $L^2(N)$. Then $-\Delta_N$ is a positive self-adjoint operator on $L^2(N)$. If $\sigma(-\Delta_N)$ denotes the spectrum of $-\Delta_N$ then $\sigma(-\Delta_N) \subset [0, +\infty)$. Let $\lambda_0(N)$ denote the bottom of the spectrum, i.e.,

$$\lambda_0(N) = \inf \sigma(-\Delta_N).$$

If $\widetilde{N}$ is a Riemannian cover of N then $\lambda_0(\widetilde{N}) \geq \lambda_0(N)$.

Theorem (Brooks [3]**).** *Suppose that $\widetilde{N}$ is a Riemannian cover of N. If $\pi_1(N)/\pi_1(\widetilde{N})$ is amenable then $\lambda_0(\widetilde{N}) = \lambda_0(N)$.*

Remark. Subject to certain conditions, in particular, if N is compact, Brooks also showed the converse.

Let Γ be a Kleinian group, i.e., a discrete group of isometries of the real $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. We say that Γ is geometrically finite if it is possible to choose a fundamental domain which is a finite sided polyhedron. We shall suppose that Γ acts freely so that $\mathbb{H}^{n+1}/\Gamma$ is a smooth manifold and that Γ is non-elementary. Then $0 < \delta(\Gamma) \leq n$, with equality if and only if $\mathbb{H}^{n+1}/\Gamma$ has finite volume. As before, Γ_0 will be a normal subgroup of Γ.

In this setting, $\delta(\Gamma)$ is related to $\lambda_0(\mathbb{H}^{n+1}/\Gamma)$ by the formula

$$\lambda_0(\mathbb{H}^{n+1}/\Gamma) = \begin{cases} \delta(\Gamma)(n - \delta(\Gamma)) & \text{if } \delta(\Gamma) > n/2 \\ n^2/4 & \text{if } \delta(\Gamma) \leq n/2, \end{cases}$$

with an identical formula holding for Γ_0. Thus, in the range $\delta(\Gamma) > n/2$, the critical exponent may be read off from the λ_0 and vice versa, while for $\delta(\Gamma) \leq n/2$ the critical exponent is a more subtle quantity.

Using the above relation, Brooks was able to deduce that, if Γ is geometrically finite and $\delta(\Gamma) > n/2$ then amenability of Γ/Γ_0 implies that $\delta(\Gamma_0) = \delta(\Gamma)$ [3]. In the case where Γ is a free group, we can remove the restriction that $\delta(\Gamma) > n/2$. In particular, this gives a complete result for finitely generated Fuchsian groups.

Theorem 2. *Let Γ be a finitely generated Fuchsian group and let Γ_0 be a normal subgroup. If Γ/Γ_0 is amenable then $\delta(\Gamma_0) = \delta(\Gamma)$.*

Proof. First we note that, for Fuchsian groups, if Γ is finitely generated then it is geometrically finite. If $\mathbb{H}^2/\Gamma$ is compact then $\delta(\Gamma) = 1$, so Brooks's result applies. If $\mathbb{H}^2/\Gamma$ is not compact then Γ is a free group. If $\mathbb{H}^2/\Gamma$ has a cusp then $\delta(\Gamma) > 1/2$ [1], so again Brooks's result applies. In the remaining case, the result follows from Theorem 1.

References

1. A. Beardon, *The exponent of convergence of Poincaré series*, Proc. London Math. Soc. **18** (1968), 461-483.
2. R. Brooks, *The fundamental group and the spectrum of the Laplacian*, Comment. Math. Helv. **56** (1981), 581-598.
3. R. Brooks, *The bottom of the spectrum of a Riemannian covering*, J. Reine Angew. Math. **357** (1985), 101-114.
4. J. Cohen, *Cogrowth and amenability in discrete groups*, J. Funct. Anal. **48** (1982), 301-309.
5. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, 83, Birkhauser, Boston, 1990.
6. R. Grigorchuk, *Symmetrical random walks on discrete groups*, Multicomponent Random Systems, Adv. Probab. Related Topics, 6, Dekker, New York, 1980, pp. 285-325.
7. M. Hirsch and W. Thurston, *Foliated bundles, invariant measures and flat manifolds*, Ann. of Math. **101** (1975), 369-390.
8. V. Kaimanovich and A. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), 457-490.
9. S. Lalley, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits*, Acta Math. **163** (1989), 1-55.
10. F. Ledrappier and O. Sarig, *Invariant measures for the horocycle flow on periodic hyperbolic surfaces*, to appear, Israel J. Math.
11. A. Yu. Ol'shanskii, *On a geometric method in the combinatorial group theory*, Proceedings of the International Congress of Mathematicians, Vol. 1,, (Warsaw, 1983), PWN, Warsaw, 1984, pp. 415-424.
12. R. Ortner and W. Woess, *Non-backtracking random walks and cogrowth of graphs*, to appear, Canadian J. Math.
13. M. Pollicott and R. Sharp, *Comparison theorems and orbit counting in hyperbolic geometry*, Trans. Amer. Math. Soc. **350** (1998), 473-499.
14. M. Pollicott and R. Sharp, *Poincaré series and comparison theorems for variable negative curvature*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, 202, Amer. Math. Soc., Providence, RI, 2001, pp. 229-240.
15. M. Rees, *Checking ergodicity of some geodesic flows with infinite Gibbs measure*, Ergodic Theory Dynamical Systems **1** (1981), 107-133.
16. T. Roblin, *Un théorème de Fatou pour les densités conformes avec applications aux revêtements galoisiens en courbure négative*, Israel J. Math. **147** (2005), 333-357.
17. W. Woess, *Cogrowth of groups and simple random walks*, Arch. Math. (Basel) **41** (1983), 363–370.
18. W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics 138, Cambridge University Press, Cambridge, 2000.

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