# CONFORMAL MARKOV SYSTEMS, PATTERSON-SULLIVAN MEASURE ON LIMIT SETS AND SPECTRAL TRIPLES

RICHARD SHARP

Abstract. For conformal graph directed Markov systems, we construct a spectral triple from which one can recover the associated conformal measure via a Dixmier trace. As a particular case, we can recover the Patterson-Sullivan measure for a class of Kleinian groups.

## 1. Introduction

One theme in noncommutative geometry is the recovery of classical geometric data from operator algebraic descriptions of various spaces, formalized through the concept of a *spectral triple*. This is defined below but, roughly speaking, consists of a C$^*$-algebra acting on a Hilbert space, giving co-ordinates for the space, and a Dirac operator, which encodes geometric information. As a prototypical example, Connes showed how to recover the volume measure and metric on a spin Riemannian manifold [11, 12].

As well as manifolds, many fractal sets may be described within this framework. Using ideas of Lapidus and Pomerance [21], Connes gave a construction of a spectral triple for a Cantor subset of the real line, from which the Minkowski content may be recovered, and used this to study the limit set of Fuchsian Schottky groups ([11], Chapter IV, §3.$\epsilon$). Using a different approach, Connes and Sullivan constructed Patterson-Sullivan measue for a quasi-Fuchsian group as a noncommutative measure ([11], Chapter IV, §3.$\gamma$).

Connes's original approach was generalized and the subject considerably advanced in the last decade by Guido and Isola [16, 17, 18]. They defined spectral triples associated to fractal subsets of $\mathbb{R}^d$ obtained as a result of a *limit construction*. In particular, this includes the limit sets of finite state iterated function systems (IFSs). If the transformations are similarities (so the limit set is *self-similar*), the $\delta$-dimensional Hausdorff measure, where $\delta$ is the Hausdorff dimension, can be recovered as a noncommutative measure. Similar results have been obtained for more general Gibbs measures by Kesseböhmer and Samuel [20, 32] and the author [35], where the underlying dynamics is an expanding map on a Cantor set or a (one-sided) subshift of finite type. (These should be compared with the constructions in [10].) Falconer and Samuel [15] have begun to investigare multifractal phenomena for Cantor subsets of the real line.

In the self-similar setting, the Hausdorff measure arises as a *conformal measure* and the recovery of conformal measures will be the subject of this paper. We will do this in the context of conformal graph directed Markov systems (CGDMSs) (defined in section 2), which provide a natural generalization of IFSs.

In greater generality, Palmer [24], following work of Pearson and Bellissard [27], has shown how to construct spectral triples from which the Hausdorff measure (in the Hausdorff dimension) may be recovered. However, his construction is not explicit (in the sense that it depends on the choice of an appropriate *resolving sequence* of open subsets, the existence of which depends on a fairly arbitrary construction,

*Key words and phrases.* spectral triple, Dixmier trace, conformal graph directed Markov system, conformal measure, Patterson-Sullivan measure.

unrelated to the geometric structure of the set under consideration – see Theorem 5.3.4 of [24]) and, in the cases where the conformal measure is Hausdorff measure, our results can be viewed as a more explicit (in the sense of being given by the Markov structure) version of the construction of spectral triples in [24].

A particularly important example of this is given by the limit sets of a class of Kleinian groups, described below, and the associated Patterson-Sullivan measure. As a special case of our results, we will show that Patterson-Sullivan measure may be obtained as noncommutative measures via a Dixmier trace associated to an appropriate Dirac operator.

We begin by defining spectral triples.

**Definition 1.1.** A *spectral triple* is a triple $(A, H, D)$, where

- (i) $H$ is a Hilbert space;
- (ii) $A$ is a C$^*$-algebra equipped with a faithful $*$-representation $\pi : A \to B(H)$ (the bounded linear operators on $H$);
- (iii) $D$ is an essentially self-adjoint unbounded linear operator on $H$ with compact resolvent and such that $\{f \in A : \|[D, \pi(f)]\| < +\infty\}$ is dense in $A$, where $[D, \pi(f)] : H \to H$ is the commutator operator $[D, \pi(f)](\xi) = D\pi(f)(\xi) - \pi(f)D(\xi)$. This $D$ is called a Dirac operator.

In this introduction, we state our results in the special (but important) case of Kleinian limit sets and Patterson-Sullivan measure. Let $\Gamma$ be a Kleinian group, i.e. a discrete group of isometries of $d$-dimensional hyperbolic space $\mathbb{H}^d$, $d \geq 2$, which we may identify with the unit ball in $\mathbb{R}^d$ equipped with the Poincaré metric. If $d = 2$, $\Gamma$ is called a Fuchsian group. The limit set $\Lambda_\Gamma \subset S^{d-1}$ is defined to be the accumulation set of the orbit $\Gamma o$ for some $o \in \mathbb{H}^d$ and is independent of the point $o$ chosen. We assume that $\Gamma$ is non-elementary, i.e. $\Lambda_\Gamma$ contains more than two (and hence infinitely many) elements. Write $\mathcal{C}(\Gamma) \subset \mathbb{H}^d \cup S^{d-1}$ for the (hyperbolic) convex hull of $\Lambda_\Gamma$. If the quotient space $\Gamma \backslash (\mathcal{C}(\Gamma) \cap \mathbb{H}^d)$ is compact then we say that $\Gamma$ is *convex co-compact.*

Associated to a Kleinian group $\Gamma$ is its critical exponent

$$\delta = \inf \left\{ s > 0 : \sum_{g \in \Gamma} e^{-s d_{\mathbb{H}^d}(go, o)} < +\infty \right\},$$

which is independent of the choice of $o \in \mathbb{H}^d$. For a non-elementary convex co-compact group, $0 < \delta = \dim \Lambda_\Gamma$, the Hausdorff dimension of $\Lambda_\Gamma$ [37, 38].

Our results will hold for Fuchsian groups and also for the following two classes of Kleinian groups.

**Definition 1.2.** A Kleinian group $\Gamma$ is called quasi-Fuchsian if it is a discrete faithful representation of the fundamental group of a compact surface [7]. These can be obtained by deforming a Fuchsian group $\Gamma_0 < \mathrm{PSL}(2, \mathbb{R})$ to a discrete group $\Gamma < \mathrm{PSL}(2, \mathbb{C})$.

**Definition 1.3.** A Kleinian group $\Gamma$ is said to satisfy the even corners condition if $\Gamma$ admits a fundamental domain $R$ which is a finite sided polyhedron (possibly with infinite volume) such that $\bigcup_{g \in \Gamma} g\partial R$ is a union of hyperplanes. (This definition was introduced by Bowen and Series [9] for the case $d = 2$ and studied by Bourdon [6] for $d \geq 3$.)

**Example 1.4.** Let $K_1, \ldots, K_{2k}$ be $2k$ disjoint $d-1$-dimensional spheres in $\mathbb{R}^d$, each meeting $S^{d-1}$ at right angles. For $i = 1, \ldots, k$, let $g_i$ be the isometry which maps the exterior of $K_i$ onto the interior of $K_{k+i}$. Then the group $\Gamma$ generated in this is called a Schottky group and satisfies the even corners condition. Viewed as an abstract group, it is the free group on $k$ generators. In this case, $\Lambda_\Gamma$ is a Cantor set.

**Example 1.5.** Let $R$ be a polyhedron in $\mathbb{H}^d$ with a finite number of faces and with interior angles all equal to $\pi/k$, $k \in \mathbb{N}$, $k \geq 2$. Let $\Gamma$ be the Kleinian group generated by reflections in the faces of $R$. Then $\Gamma$ satisfies the even corners condition. For instance, let $R$ be a regular tetrahedron in $\mathbb{H}^3$ with infinite volume and with dihedral angles $\pi/4$. In this case, $\Lambda_\Gamma$ is a Sierpiński curve [3, 6].

The limit set of a Kleinian group supports a natural probability measure $\mu$ called Patterson-Sullivan measures [26, 36]. If $\Gamma$ is convex co-compact, this is characterized as the unique non-atomic measures supported on $\Lambda_\Gamma$ satisfying $g^*\mu = |g'|^\delta \mu$, for all $g \in \Gamma$. It is also equal to the normalized restriction of the $\delta$-dimensional Hausdorff measure to $\Lambda_\Gamma$ [37, 38].

**Remark 1.6.** Strictly speaking, $\mu$ depends on a choice of basepoint $o \in \mathbb{H}^d$. One may as well take the $o$ to be the origin (in the unit ball model), in which case distances on $\Lambda_\Gamma$ are measured with respect to the Euclidean metric on $\mathbb{R}^d$. Other choices of $o$ are related to an equivalent measure and a conformally equivalent metric on $\Lambda_\Gamma$. (In particular, the derivative in the scaling factor $|g'|^\delta$ is taken with respect to the appropriate metric.)

**Theorem 1.7.** *Let $\Gamma$ be either*

(i) *a convex co-compact Fuchsian group acting on $\mathbb{H}^2$;*
(ii) *a quasi-Fuchsian group acting on $\mathbb{H}^3$; or*
(iii) *a convex co-compact Kleinian group acting on $\mathbb{H}^d$, $d \geq 3$ with the even corners property*

*and let $\Lambda_\Gamma \subset S^n$ denote its limit set. Then there is a spectral triples $(A, H, D)$ from which the Patterson-Sullivan measure on $\Lambda_\Gamma$ may be recovered.*

For a more precise statement, see Corollary 3.2.

In the next section, we introduce conformal graph directed Markov systems and conformal measures and define our spectral triple. In section 3, we discuss singular traces and state our main result: that conformal measures may be recovered from a spectral triple. In section 4, we discuss some material on transfer operators. In section 5, we prove Theorem 2.13. In section 6, we complete the paper by proving our result on noncommutative measures and conformal measures, Theorem 3.1.

## 2. Conformal Graph Directed Markov Systems

2.1. **Graph directed systems.** Conformal graph directed Markov systems were introduced by Mauldin and Urbanski as a natural generalization of conformal iterated function systems. The theory they developed is described in [23] and the definitions presented below may be found there.

A (finite) conformal graph directed Markov system (CGDMS) is defined as follows. Let $(V, E)$ be a finite directed graph with vertices $V$ and directed edges $E$, where we assume $E$ has cardinality $\#E \geq 2$. For $e \in E$, $\mathfrak{o}(e) \in V$ will denote the initial vertex (the origin) of $e$ and $\mathfrak{t}(e) \in V$ with denote the final vertex (the terminus) of $e$. We define a zero-one matrix $A$, indexed by $E \times E$, by

$$A(e, e') = \begin{cases} 1 & \text{if } \mathfrak{t}(e) = \mathfrak{o}(e') \\ 0 & \text{otherwise.} \end{cases}$$

We suppose that the following condition holds.

(C1) The matrix $A$ is *aperiodic*, i.e. that there exists $N \geq 1$ such that $A^N$ has strictly positive entries.

Let $U \subset \mathbb{R}^d$ be an open set. A $C^1$ map $\phi : U \to \mathbb{R}^d$ is *conformal* if, at each point $x \in U$, $D\phi(x)$ is the product of a scalar $|\phi'(x)|$ with an isometry of $\mathbb{R}^d$. In addition, $\phi$ is a contraction if $\sup_{x \in U} |\phi'(x)| < 1$. If $d = 1$, any monotone $C^1$ map is conformal; in this case we will also need to assume that the derivative $\phi'$ is Hölder continuous. If $d = 2$, a map is conformal if and only if (identifying $\mathbb{R}^2$ with $\mathbb{C}$) it is holomorphic or anti-holomorphic. If $d \geq 3$, a map is conformal if and only if it is a Möbius transformation.

For each $v \in V$, we suppose there is a non-empty compact connected set $X_v \subset \mathbb{R}^d$ and, for each $e \in E$, a conformal contraction $\phi_e : X_{\mathfrak{t}(e)} \to X_{\mathfrak{o}(e)}$, satisfying the following conditions.

(C2) For each $v \in V$, $X_v = \overline{\mathrm{int}(X_v)}$.
(C3) (Open Set Condition.) For all $e, e' \in E$, with $e \neq e'$,

$$\phi_e(\mathrm{int}(X_{\mathfrak{t}(e)}) \cap \phi_{e'}(\mathrm{int}(X_{\mathfrak{t}(e')}) = \varnothing.$$

(C4) For each $v \in V$, there is an open connected set $U_v \supset X_v$ such that, for each $e \in E$ with $\mathfrak{t}(e) = v$, the map $\phi_e$ extends to a $C^1$ conformal diffeomorphism of $U_v$ into $U_{\mathfrak{o}(e)}$. If $d = 1$ then, in addition, $\phi'$ is Hölder continuous.
(C5) (Cone property.) There exists $0 < \gamma < \pi/2$ and $l > 0$ such that for every $x \in \coprod_{v \in V} X_v$ there exists an open cone

$$\mathrm{Con}(x, \gamma, l) \subset \mathrm{int}\left(\coprod_{v \in V} X_v\right)$$

with vertex $x$, central angle of measure $\gamma$, and altitude $l$.

The CGDMS, which we denote by $\mathcal{S} = (V, E, \{X_v\}_{v \in V}, \{\phi_e\}_{e \in E})$, consists of the directed graph $(V, E)$ and the set of maps $\{\phi_e : X_{\mathfrak{t}(e)} \to X_{\mathfrak{o}(e)} : e \in E\}$.

**Example 2.1.** In the special case where $V$ consists of a single vertex (so that $A(e, e') = 1$ for all $e, e' \in E$), $\mathcal{S}$ is called a conformal Iterated Function System (IFS). There is a large literature devoted to the study of IFSs.

2.2. **The limit set and conformal measures.** An ordered $n$-tuple $(w_1, \ldots, w_n)$, with $w_m \in E$, $m = 1, \ldots, n$, is called an *allowed word* of length $n$ if $A(w_m, w_{m+1}) = 1$ for $m = 1, \ldots, n-1$. Let $W_n$ denote the set of allowed words of length $n$ and let

$$W^* = \bigcup_{n=1}^{\infty} W_n.$$

We shall write

$$\Sigma_A^+ = \{(e_n)_{n=1}^{\infty} : A(e_n, e_{n+1}) = 1 \text{ for all } n \geq 1\}.$$

For $w = (w_1, \ldots, w_n) \in W_n$, we write

$$[w] = \{(e_n)_{n=1}^{\infty} \in \Sigma_A^+ : e_m = w_m, \ m = 1, \ldots, n\}$$

and $\mathfrak{t}(w) = \mathfrak{t}(w_n) \in V$. For $w = (w_1, \ldots, w_n) \in W_n$ and $\underline{e} = (e_n)_{n=1}^{\infty} \in \Sigma_A^+$, $w\underline{e}$ will denote the sequence defined by

$$(w\underline{e})_m = \begin{cases} w_m & \text{if } 1 \leq m \leq n \\ e_{m-n} & \text{if } m \geq n+1. \end{cases}$$

Clearly, $w\underline{e} \in \Sigma_A^+$ if and only if $A(w_n, e_1) = 1$.

For $w = (e_1, \ldots, e_n) \in W_n$, we write $\phi_w = \phi_{e_1} \circ \cdots \circ \phi_{e_n}$. Define $\Pi : \Sigma_A^+ \to \coprod_{v \in V} X_v$ by

$$\Pi((e_n)_{n=1}^{\infty}) \in \bigcap_{n=1}^{\infty} \phi_{(e_1, \ldots, e_n)}(X_{\mathfrak{t}(e_n)}).$$

This is well-defined because the intersection is a singleton.

**Definition 2.2.** The *limit set* of the CGDMS $\mathcal{S} = (V, E, \{X_v\}_{v \in V}, \{\phi_e\}_{e \in E})$ is the set $\Lambda = \Pi(\Sigma_A^+)$.

The set $\Sigma_A^+$ may be made into a compact metric space by defining, for $\underline{e} = (e_n)_{n=1}^\infty$ and $\underline{e}' = (e'_n)_{n=1}^\infty$,

$$d(\underline{e}, \underline{e}') = 2^{-\mathfrak{n}(\underline{e}, \underline{e}')},$$

where $\mathfrak{n}(\underline{e}, \underline{e}') = 0$ if $(e_n)_{n=1}^\infty = (e'_n)_{n=1}^\infty$ and

$$\mathfrak{n}(\underline{e}, \underline{e}') = \min\{n \geq 1 : e_n \neq e'_n\}$$

if $\underline{e} \neq \underline{e}'$. We then have the following result, which is a simple consequence of the above definition and the fact that $\{\phi_e\}_{e \in E}$ consists of contractions.

**Proposition 2.3.** *The map $\Pi : \Sigma_A^+ \to \Lambda$ is Hölder continuous.*

For $w = (e_1, \ldots, e_n) \in W_n$, we shall write

$$X_w = \bigcap_{i=1}^n \phi_{(e_1, \ldots, e_i)}(X_{\mathfrak{t}(e_i)}).$$

We also write $X_\emptyset = \coprod_{v \in V} X_v$.

The following two lemmas may be found in [23] (pages 72-73) and are commonly known as the *principle of bounded variation.*

**Lemma 2.4.** *For all $w \in W^*$ and all $x, y \in U_{\mathfrak{t}(w)}$, we have*

$$|\log|\phi'_w(x)| - \log|\phi'_w(y)|| \leq \frac{C}{1-a}\|x - y\|,$$

*where $a = \max_{e \in E} \sup_{x \in U_{\mathfrak{t}(e)}} |\phi'_e(x)| < 1$.*

**Lemma 2.5.** *There exists $K \geq 1$ such that, for all $w \in W^*$ and all $x, y \in X_{\mathfrak{t}(w)}$,*

$$|\phi'_w(y)| \leq K|\phi'_w(x)|.$$

The next result is an easy consequence of Lemma 2.5.

**Lemma 2.6.** *There exists $C > 0$ such that, for all $w \in W^*$ and all $x, y \in X_{\mathfrak{t}(w)}$,*

$$\|\phi_w(x) - \phi_w(y)\| \leq C|\phi'_w(x)|.$$

We shall be interested in probability measures on $\Lambda$. We have the following definition.

**Definition 2.7.** Let $t \geq 0$. A measure $m$ on $\mathbb{R}^d$ is $t$-conformal for $\mathcal{S}$ if $m$ is supported on $\Lambda$ and if

(i) for every $w \in W^*$ and for every Borel set $A \subset X_{\mathfrak{t}(e)}$, we have

$$m(\phi_w(A)) = \int_A |\phi'_w|^t \, dm;$$

and

(ii) for every $w = (e_1, \ldots, e_n), w' = (e'_1, \ldots, e'_m) \in W^*$, such that $n \leq m$ and $(e_1, \ldots, e_n) \neq (e'_1, \ldots, e'_n)$, we have

$$m(\phi_w(X_{\mathfrak{t}(w)}) \cap X_{\mathfrak{t}(w')})) = 0.$$

**Proposition 2.8.** *Let $\mathcal{S}$ be a CGDMS satisfying (C1)-(C5), with limit set $\Lambda$, and let $\delta$ denote the Hausdorff dimension of $\Lambda$. Then $\Lambda$ supports a unique $t$-conformal probability measure $\mu$ and, furthermore, $t = \delta$.*

*Proof.* This is essentially Lemma 4.2.10 of [23]. $\square$

**Example 2.9.** Let $M$ be a compact connected smooth Riemannian manifold and suppose that $U \subset M$ is open. Let $T : U \to M$ be a conformal expanding map, i.e. a conformal map such that $\inf_{x\in U} \|DT_x\| > 1$. Then $T$ admits a finite Markov partition $\{X_1, \ldots, X_k\}$ such that if $T(\mathrm{int}(X_i)) \cap \mathrm{int}(X_j) \neq \varnothing$ then $T(X_i) \supset X_j$. The set $\Lambda = \bigcap_{n=0}^{\infty} T^{-n}U$ is the limit set of a CGDMS, where the set of vertices $V = \{1, \ldots, k\}$, the set of edges $E$ is defined by $E = \{(i,j) \in V \times V : f(X_i) \supset X_j\}$ and $\{\phi_e : e \in E\}$ consists of the local inverses of $T$ between appropriate pairs of $X_i$'s. Provided $T$ is topologically mixing, the conditions (C1)-(C5) are satisfied.

**Example 2.10.** Generalizing the previous example, we can consider maps $T : \coprod_{i=1}^{k} X_i \to \coprod_{i=1}^{k} X_i$ such that $T$ is conformal and expanding on a neighbourhood of each $X_i$ and such that $T$ satisfies the Markov property: if $T(\mathrm{int}(X_i)) \cap \mathrm{int}(X_j) \neq \varnothing$ then $T(X_i) \supset X_j$. The CGDMS is defined as in Example 2.9 and, provided $T$ is topologically mixing, the conditions (C1)-(C5) are satisfied.

**Remark 2.11.** An arbitrary piecewise expanding map need not be Markov. For example, $T : [0,1] \to [0,1]$ defined by $T(x) = \beta x \pmod 1$, $1 < \beta < 2$, is only Markov when $\beta^2 = \beta + 1$, i.e when $\beta = (1+\sqrt{5})/2$. Thus, the Markov property needs to be assumed in Example 2.10.

**Example 2.12.** The limit sets of convex co-compact Fuchsian groups, quasi-Fuchsian groups and Kleinian groups with the even angle property are examples of limit sets of CGDMS as above. The case of Fuchsian groups is well-known [1, 33] and quasi-Fuchsian groups have symbolic dynamics of the same type. For even angle Kleinian groups, we follow [6]. Let $R$ be a polyhedron as specified by the even angle property. Label the faces of $R$ by $\{R_1, \ldots, R_m\}$ and let $g_i \in \Gamma$ denote the isometry for which $g_i R \cap R = R_i$. For each $i = 1, \ldots, m$, $R_i$ extends to a co-dimension one hyperbolic hyperplane, which divides $\mathbb{H}^d \cup S^{d-1}$ into two half- spaces. Let $H_i$ denote the half-space which does not contain $R$ and let $U_i = H_i \cap S^{d-1}$. In general, the $U_i$'s will overlap; to obtain a partition we let $\mathcal{X} = \{X_1, \ldots, X_k\}$ denote the sets formed by taking the closure of all possible intersections of the interiors of the $U_i$'s.

Choose an arbitrary ordering $\prec$ on $\mathcal{S} = \{g_1, \ldots, g_m\}$. The map $T : \coprod_{i=1}^{k} X_i \to S^{d-1}$ defined by $T|_{X_i}(x) = a_i^{-1}x$, where $\mathrm{int}(X_i) = \mathrm{int}(U_{j_1}) \cap \cdots \cap \mathrm{int}(U_{j_l})$, and where $a_i$ is the $\prec$-smallest element of $\{g_{j_1}, \ldots, g_{j_l}\}$, is conformal and expanding. If necessary refining a finite number of times by considering intersections of sets in $\mathcal{X}, T^{-1}\mathcal{X}, \ldots, T^{-n}\mathcal{X}$, for some $n \geq 0$, $T$ will satisfy the Markov property: if $T(\mathrm{int}(X_i)) \cap \mathrm{int}(X_j) \neq \varnothing$ then $T(X_i) \supset X_j$. The CGDMS is defined as in Example 2.9 and satisfies (C1)-(C5). The limit set $\Lambda_\Gamma$ is the limit set of the CGDMS and the Patterson-Sullivan measure is the conformal measure.

2.3. **Spectral triples.** We will use $\ell^2(W^*)$ to denote the $\ell^2$ space

$$\ell^2(W^*) = \left\{ \xi : W^* \to \mathbb{C} : \sum_{w \in W^*} |\xi(w)|^2 < +\infty \right\}.$$

Our Hilbert space will be

$$H_{\mathcal{S}} = \ell^2(W^*) \oplus \ell^2(W^*) \subset \bigoplus_{w \in W^*} \mathbb{C} \oplus \mathbb{C},$$

where we write a typical element as

$$\xi = \bigoplus_{w \in W^*} \begin{pmatrix} \xi_1(w) \\ \xi_2(w) \end{pmatrix}$$

and our C$^*$-algebra will be $A_{\mathcal{S}} = C(\Lambda, \mathbb{C})$ with the uniform norm.

The action of $A_{\mathcal{S}}$ on $H_{\mathcal{S}}$ and the definition of the Dirac operator will depend on the choice of a finite number of points in $\coprod_{v \in V} X_v$. More precisely, for each $v \in V$, choose $x^v, y^v \in X_v \cap \Lambda$ with $x^v \neq y^v$. We define a $*$-representation $\pi : A_{\mathcal{S}} \to B(H_{\mathcal{S}})$ by setting $\pi(a)$ to be the multiplication operator

$$\pi(a)\left(\bigoplus_{w\in W^*}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}\right) = \bigoplus_{w\in W^*}\begin{pmatrix}a\left(\phi_w(x^{\mathrm{t}(w)})\right)\xi_1(w)\\ a\left(\phi_w(y^{\mathrm{t}(w)})\right)\xi_2(w)\end{pmatrix}.$$

We define a Dirac operator $D_{\mathcal{S}} : H_{\mathcal{S}} \to H_{\mathcal{S}}$ by

$$D_{\mathcal{S}}\left(\bigoplus_{w\in W^*}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}\right) = \bigoplus_{w\in W^*}\frac{1}{|\phi_w'(x^{\mathrm{t}(w)})|}\begin{pmatrix}\xi_2(w)\\ \xi_1(w)\end{pmatrix}.$$

We have the following theorem, which will be proved in section 5.

**Theorem 2.13.** *$(A_{\mathcal{S}}, H_{\mathcal{S}}, D_{\mathcal{S}})$ is a spectral triple.*

## 3. Singular traces and noncommutative measures

In order to state our main result, we need to briefly discuss the theory of singular traces of compact operators. For more details, see [2] or [17]. Let $B(H)$ denote the algebra of bounded linear operators on a separable Hilbert space $H$ and let $K(H)$ denote the ideal of compact operators. A *singular trace* on a two-sided ideal $I \subset K(H)$ is a positive linear functional $\mathrm{Tr} : I \to \mathbb{R}$ such that Tr is unitary invariant (the trace property) and vanishes on finite rank operators.

The most important singular traces are the so-called *Dixmier traces* [14]. These are defined on an ideal $I = \mathcal{L}^{1,\infty}(H)$, the *Dixmier ideal*, given by

$$\mathcal{L}^{1,\infty}(H) = \left\{A \in K(H) : \limsup_{n\to+\infty}\frac{1}{\log n}\sum_{k=1}^{n} a_k < +\infty\right\},$$

where $\{a_n\}_{n=1}^{\infty}$ denote the eigenvalues of $|A| := \sqrt{A^*A}$, written in decreasing order. Then a Dixmier trace is a singular trace $\mathrm{Tr}_\omega$ on $\mathcal{L}^{1,\infty}(H)$ defined, for a positive operator $A$, by

$$\mathrm{Tr}_\omega(A) = \omega\text{-}\lim\frac{1}{\log n}\sum_{k=1}^{n} a_k,$$

where this is a generalized limit corresponding to a state $\omega$ on $l^\infty$, and extended to $\mathcal{L}^{1,\infty}(H)$ by linearity. If the limit

$$\lim_{n\to+\infty}\frac{1}{\log n}\sum_{k=1}^{n} a_k$$

exists then we say that $A$ is *measurable* and call the value of the limit the *noncommutative integral* of $A$. (There are more general definitions of the Dixmier trace – see, for example, Chapter IV, §2.$\beta$ of [11], [22] or Chapter 5 of [39]. Correspondingly, there are more general definitions of measurability. It is shown in [22] that these are equivalent to the definition given here.)

We now return to the spectral triple $(A_{\mathcal{S}}, H_{\mathcal{S}}, D_{\mathcal{S}})$ defined in the preceding section. For $a \in A_{\mathcal{S}}$, the operator $\pi(a)|D_{\mathcal{S}}|^{-s}$ is given by the formula

$$\pi(a)|D_{\mathcal{S}}|^{-s}\left(\bigoplus_{w\in W^*}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}\right) = \bigoplus_{w\in W^*}|\phi_w'(x^{\mathrm{t}(w)})|^s\begin{pmatrix}a(\phi_w(x^{\mathrm{t}(w)}))\xi_1(w)\\ a(\phi_w(y^{\mathrm{t}(w)}))\xi_2(w)\end{pmatrix}.$$

The following is our main result. In particular, it tells us that the conformal measure $\mu$ may be recovered from the spectral triple $(A_{\mathcal{S}}, H_{\mathcal{S}}, D_{\mathcal{S}})$. The proof will be given in section 6.

**Theorem 3.1.** *Let $\Lambda$ be the limit set of a CGDMS $\mathcal{S}$ on $\mathbb{R}^d$ satisfying conditions (C1)-(C5), and let $\delta > 0$ denote the Hausdorff dimension of $\Lambda$. For any $a \in C(\Lambda, \mathbb{C})$, $\pi(a)|D_{\mathcal{S}}|^{-\delta} \in \mathcal{L}^{1,\infty}(H)$ and, furthermore, $\pi(a)|D_{\mathcal{S}}|^{-\delta}$ is measurable and*

$$\mathrm{Tr}_\omega(\pi(a)|D_{\mathcal{S}}|^{-\delta}) = c \int a \, d\mu,$$

*where $\mu$ is the $\delta$-conformal measure for $\mathcal{S}$ and where $c > 0$ is independent of $a$.*

In fact, the constant $c$ takes the form

$$c = \frac{2}{-\delta \int f \, dm_{\delta f}} \left( \sum_{v \in V} h_{\delta f}(\overline{x^v}) \right),$$

where $f : \Sigma_A^+ \to \mathbb{R}$ is defined by

$$f(\underline{e}) = \log |\phi'_{e_1}(\Pi(\sigma \underline{e}))|,$$

$h_{-\delta f} : \Sigma_A^+ \to \mathbb{R}$ and $\nu_{\delta f}$ are the unique function and probaility measure satisfying $L_{\delta f} h_{\delta f} = h_{\delta f}$, $L_{\delta f}^* \nu_{\delta f} = \nu_{\delta f}$ and $\int h_{\delta f} \, d\nu_{\delta f} = 1$, where $L_{\delta f}$ is the operator

$$(L_{\delta f} a)(\underline{e}) = \sum_{\sigma \underline{e}' = \underline{e}} e^{\delta f(\underline{e}')} a(\underline{e}'),$$

$m_{\delta f} = h_{\delta f} \nu_{\delta f}$, and $\overline{x^v} = \Pi(x^v)$.

We state the special case of limit sets of Kleinian groups and Patterson-Sullivan measure as a separate result.

**Corollary 3.2.** *Let $\Gamma$ be a non-elementary Kleinian group which is either*

(a) *a convex co-compact Fuchsian group acting on $\mathbb{H}^2$;*
(b) *a quasi-Fuchsian group acting on $\mathbb{H}^3$; or*
(c) *a convex co-compact Kleinian group acting on $\mathbb{H}^d$, $d \geq 3$ with the even corners property,*

*let $\Lambda_\Gamma \subset S^{d-1}$ denote its limit set and let $\delta$ be its critical exponent. Let $(A_{\mathcal{S}}, H_{\mathcal{S}}, D_{\mathcal{S}})$ be the spectral triple defined above (depending on the representation of $\Lambda_\Gamma$ as the limit set of a CGDMS). For any $a \in C(\Lambda_\Gamma, \mathbb{C})$, $\pi(a)|D_{\mathcal{S}}|^{-\delta} \in \mathcal{L}^{1,\infty}(H)$ and, furthermore, $\pi(a)|D_{\mathcal{S}}|^{-\delta}$ is measurable and*

$$\mathrm{Tr}_\omega(\pi(a)|D_{\mathcal{S}}|^{-\delta}) = c \int a \, d\mu,$$

*where $\mu$ is the Patterson-Sullivan measure for $\Gamma$ and where $c > 0$ is independent of $a$.*

The next result recovers a special case of Theorem 2.10 of [18]. Let $\phi_i(x) = a_i x + b_i$, $i = 1, \ldots, k$, with $0 < a_i < 1$ and $b_i \in \mathbb{R}^d$, be a finite family of contracting similarities on $\mathbb{R}^d$, satisfying the Open Set Condition. Define $\delta > 0$ by $\sum_{i=1}^k a_i^\delta = 1$. Then $\delta$ is equal to the Hausdorff dimension of the self-similar limit set $\Lambda$. Furthermore, the conformal measure $\mu$ is the solution of

$$\sum_{i=1}^k a_i^\delta T_{i*} \mu = \mu$$

and is equal to the normalized $\delta$-dimensional Hausdorff measure.

**Corollary 3.3.** *Suppose $\phi_1, \ldots, \phi_k$ be a finite collection of contracting similarities on $\mathbb{R}^d$ satisfying the Open Set Condition. Let $\Lambda$ denote the associated self-similar set and let $\delta$ denote its Hausdorff dimension. For any $a \in C(\Lambda, \mathbb{C})$, $\pi(a)|D_{\mathcal{S}}|^{-\delta} \in \mathcal{L}^{1,\infty}(H)$ and, furthermore, $\pi(a)|D_{\mathcal{S}}|^{-\delta}$ is measurable and*

$$\mathrm{Tr}_\omega(\pi(a)|D_{\mathcal{S}}|^{-\delta}) = c \int a \, d\mu,$$

*where $\mu$ is the normalized $\delta$-dimensional Hausdorff measure on $\Lambda$ and where $c > 0$ is independent of $a$.*

In this special case, one can calculate that $f(\underline{e}) = \log a_{e_1}$, $h_{\delta f} = 1$ and $m_{\delta f}$ is the $(a_1^\delta, \ldots, a_k^\delta)$-Bernoulli measure, so that

$$c = \frac{2}{-\delta \sum_{i=1}^{k} (\log a_i) a_i^\delta} = \frac{2}{h(m_{\delta f})},$$

where $h(m_{\delta f})$ is the measure theoretic entropy of the shift map $\sigma$ with respect to $m_{\delta f}$.

## 4. Transfer Operators

In this section we shall discuss some of the ergodic theory associated to the subshift of finite type $(\Sigma_A^+, \sigma)$, where $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is the shift map defined by $(\sigma \underline{e})_n = e_{n+1}$, and hence to the CGDMS $\mathcal{S}$. The main references are [8, 25] and, for CGDMSs, [23]. We shall write $\mathcal{M}_\sigma$ for the space of $\sigma$-invariant probability measures on $\Sigma_A^+$. Given $m \in \mathcal{M}_\sigma$, we write $h_\sigma(m) \geq 0$ for the entropy of $\sigma$ as a measure preserving transformation of $(\Sigma_A^+, m)$ (see [40] for the definition). For a continuous function $f : \Sigma_A^+ \to \mathbb{R}$, we define its *pressure* $P(f)$ by

$$P(f) = \sup_{m \in \mathcal{M}_\sigma} \left( h_\sigma(m) + \int f \, dm \right).$$

If $f$ is Hölder continuous, then there is a unique probability measure $m_f$, called the *equilibrium measure* for $f$, for which this supremum is realized [4, 8, 25].

Let $f \in C(\Sigma_A^+, \mathbb{R})$. We define the *transfer operator* $L_f : C(\Sigma_A^+, \mathbb{C}) \to C(\Sigma_A^+, \mathbb{C})$ by

$$L_f a(\underline{e}) = \sum_{\sigma \underline{e}' = \underline{e}} e^{f(\underline{e}')} a(\underline{e}').$$

A key element of our approach will be to relate the eigenvalue asymptotics of our Dirac operators to the spectral properties of transfer operators. For this approach to work, we need a space on which $L_f$ acts quasi-compactly.

If $f \in C^\alpha(\Sigma_A^+, \mathbb{R})$ then $L_f$ restricts to a bounded linear operator $L_f : C^\alpha(\Sigma_A^+, \mathbb{C}) \to C^\alpha(\Sigma_A^+, \mathbb{C})$, where $C^\alpha(\Sigma_A^+, \mathbb{C})$ is the Banach space of complex valued Hölder continuous functions of exponent $\alpha$ with the norm

$$\|a\|_\alpha = \|a\|_\infty + \sup_{\underline{e} \neq \underline{e}'} \frac{|a(\underline{e}) - a(\underline{e}')|}{d(\underline{e}, \underline{e}')^\alpha}.$$

The basic spectral properties of $L_f$ on this space are contained in the following result, which is Ruelle's generalization of the classical Perron-Frobenius Theorem for non-negative matrices.

**Proposition 4.1.** [4, 8, 25, 31] *If $f \in C^\alpha(\Sigma_A^+, \mathbb{R})$ then $L_f : C^\alpha(\Sigma_A^+, \mathbb{C}) \to C^\alpha(\Sigma_A^+, \mathbb{C})$ has a simple eigenvalue equal to $e^{P(f)}$ with the rest of the spectrum contained in a disk $\{z \in \mathbb{C} : |z| \leq \theta e^{P(f)}\}$, for some $0 < \theta < 1$. Furthermore, there exist*

(i) *a strictly positive eigenfunction $h_f \in C^\alpha(\Sigma_A^+, \mathbb{R})$ such that $L_f h_f = e^{P(f)} h_f$; and*
(ii) *an eigenmeasure $\nu_f \in C(\Sigma_A^+, \mathbb{R})^*$ such that $L_f^* \nu_f = e^{P(f)} \nu_f$.*

*If $\nu_f$ is chosen to be a probability measure and the eigenfunction $h_f$ is chosen so that $\int h_f \, d\nu_f = 1$ then $m_f = h_f \nu_f$ is the equilibrium measure for $f$.*

This result has two consequences, stated below as corollaries, which will be important for our analysis.

**Corollary 4.2.** *There exists $\lambda_f < e^{P(f)}$ such that, for any $a \in C^\alpha(\Sigma_A^+, \mathbb{C})$, we have*

$$L_f^n a = \left( \int a \, d\nu_f \right) h_f e^{nP(f)} + O(\lambda_f^n).$$

*Proof.* We recall the following basic fact from spectral theory (see, for example, [5] or [19]). Let $L : B \to B$ be a bounded linear operator on a Banach space $B$ with spectrum $\mathrm{spec}(L) = \Sigma \subset \mathbb{C}$. If $\Sigma$ can be decomposed into two disjoint non-empty sets $\Sigma_1$ and $\Sigma_2$ and if $\gamma$ is a simple closed curve which is disjoint from $\Sigma$ and which has $\Sigma_1$ in its interior and $\Sigma_2$ in its exterior then $\mathcal{P} : B \to B$ defined by

$$\mathcal{P} = \frac{1}{2\pi i} \int_\gamma (z - L)^{-1} \, dz,$$

is a projection (i.e. $\|\mathcal{P}\| = 1$ and $\mathcal{P}^2 = \mathcal{P}$). Moreover, $B = B_1 \oplus B_2$, where $B_1 = \mathcal{P}(B)$ and $B_2 = (I - \mathcal{P})(B)$ are closed and $L$-invariant subspaces with $\mathrm{spec}(L|B_1) = \Sigma_1$ and $\mathrm{spec}(L|B_2) = \Sigma_2$.

Now consider the operator $L_f : C^\alpha(\Sigma_A^+, \mathbb{C}) \to C^\alpha(\Sigma_A^+, \mathbb{C})$. By Theorem 4.1, we may decompose its spectrum into $\Sigma_1 = \{e^{P(f)}\}$ and a disjoint set $\Sigma_2$. Thus, we may decompose the operator $L_f$ as a sum

$$L_f^n = L_f^n \mathcal{P} + L_f^n (I - \mathcal{P}) = e^{nP(f)} \nu_f(\cdot) h_f + L_f^n (I - \mathcal{P}),$$

where $\mathcal{P} = \nu_f(\cdot) h_f$ is the projection onto the eigenspace spanned by $e^{P(f)}$. Furthermore, since $e^{P(f)}$ is strictly maximal in modulus, we have

$$\lim_{n \to +\infty} \|L_f^n (I - \mathcal{P})\|^{1/n} = \sup\{|z| \, : \, z \in \Sigma_2\} < e^{P(f)}.$$

Choosing $\lambda_f$ such that

$$\lim_{n \to +\infty} \|L_f^n (I - \mathcal{P})\|^{1/n} < \lambda_f < e^{P(f)}$$

completes the proof. □

**Corollary 4.3.** *The quantities $e^{P(f)}$, $h_f$ and $\nu_f$ in Proposition 4.1 all depend analytically on $f$.*

*Proof.* First we note that $L_f$ depends analytically on $f$. The result is then a standard consequence of the fact that $e^{P(f)}$ is an isolated simple eigenvalue for $L_f$ [5, 19]. □

In order to use the transfer operator formalism to study CGDMSs and, in particular, their conformal measures, we need to introduce a one-parameter family of functions of $\Sigma_A$ which capture information related to the set of contractions $\{\phi_e\}_{e \in E}$. We define a function $f : \Sigma_A^+ \to \mathbb{R}$ by the formula

$$f(\underline{e}) = \log |\phi'_{e_1}(\Pi(\sigma \underline{e}))|.$$

Clearly, $f$ is strictly negative. By Lemma 2.4 and the Hölder continuity of $\Pi$, $f$ is Hölder continuous. Then the $\delta$-conformal measure for $\mathcal{S}$ is $\mu = \Pi_* \nu_{\delta f}$.

To prove Theorem 3.1, we shall need to consider a family of transfer operators $L_{tf}$, for $t \in \mathbb{R}$. By Proposition 4.1, these will have a maximal eigenvalue equal to $e^{P(tf)}$. We end the section with a result on the regularity and derivative of the function $t \mapsto P(tf)$.

**Lemma 4.4.** *The function $t \mapsto P(tf)$ is real-analytic and strictly decreasing. Furthermore,*

$$\left. \frac{dP(tf)}{dt} \right|_{t=\delta} = \int f \, dm_{\delta f}.$$

*Proof.* The analyticity of $P(tf)$ follows from Corollary 4.3. From Proposition 4.10 of [25] we have that

$$P'(tf) = \int f dm_{tf}.$$

Since $f$ is strictly negative, it follows that $P(tf)$ is strictly decreasing and the formula is obtained by evaluating at $t = \delta$. □

## 5. Proof of Theorem 2.13

In this section we prove that the the triple $(A_{\mathcal{S}}, H_{\mathcal{S}}, D_{\mathcal{S}})$ constructed above is a spectral triple. The key point is that the Lipschitz functions give a dense subalgebra of $A_{\mathcal{S}} = C(\Lambda, \mathbb{C})$ on which $\|[D_{\mathcal{S}}, \pi(\cdot)]\|$ is finite.

*Proof of Theorem 2.13.* Suppose that $a_1, a_2 \in C(\Lambda, \mathbb{C})$ and that $\pi(a_1) = \pi(a_2)$. Then, in particular, by definition, for each $w \in W^*$, $a_1(\phi_w(x^{\mathrm{t}(w)})) = a_2(\phi_w(x^{\mathrm{t}(w)}))$. The set $\{\phi_w(x^{\mathrm{t}(w)}) : w \in W^*\}$ is dense in $\Lambda$. Hence $a_1 = a_2$ and $\pi : C(\Lambda, \mathbb{C}) \to B(H_{\mathcal{S}})$ is faithful.

Next we prove that $D_{\mathcal{S}}$ is self-adjoint. Note that $D_{\mathcal{S}}^{-1}$ is defined by

$$D_{\mathcal{S}}^{-1}\left(\bigoplus_{w\in W^*}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}\right) = \bigoplus_{w\in W^*}|\phi_w'(x^{\mathrm{t}(w)})|\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}$$

and is a bounded operator. Let $\xi, \eta \in H_{\mathcal{S}}$. Then

$$\langle D_{\mathcal{S}}^{-1}\xi, \eta\rangle_{H_{\mathcal{S}}} = \sum_{w\in W^*}|\phi_w'(x^{\mathrm{t}(w)})|\left\langle \begin{pmatrix}\xi_2(w)\\ \xi_1(w)\end{pmatrix}, \begin{pmatrix}\eta_1(w)\\ \eta_2(w)\end{pmatrix}\right\rangle_{\mathbb{C}^2} = \langle \xi, D_{\mathcal{S}}^{-1}\eta\rangle_{H_{\mathcal{S}}},$$

so that $D_{\mathcal{S}}^{-1}$ is symmetric and, since it is bounded, self-adjoint. By Proposition 2.4 of chapter X of [13], if $D_{\mathcal{S}}^{-1}$ is injective then $D_{\mathcal{S}}$ is self-adjoint. However, it is clear that $D_{\mathcal{S}}^{-1}\xi = 0$ if and only if $\xi = 0$, so self-adjointness holds.

The eigenvalues of $D_{\mathcal{S}}$ are the numbers

$$\left\{\frac{1}{|\phi_w'(x^{\mathrm{t}(w)})|} : w \in W^*\right\}$$

(counted with the appropriate multiplicity). In particular, 0 is not an eigenvalue. Thus, the resolvent of $D_{\mathcal{S}}$ is compact provided $D_{\mathcal{S}}^{-1}$ is compact and it is clear that $D_{\mathcal{S}}^{-1}$ is a compact operator.

For $a \in C(\Lambda, \mathbb{C})$,

$$[D_{\mathcal{S}}, \pi(a)]\left(\bigoplus_{w\in W^*}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}\right) = \bigoplus_{w\in W^*}\frac{a(\phi_w(x^{\mathrm{t}(w)})) - a(\phi_w(y^{\mathrm{t}(w)}))}{|\phi_w'(x^{\mathrm{t}(w)})|}\begin{pmatrix}-\xi_2(w)\\ \xi_1(w)\end{pmatrix}.$$

Let $A_0$ denote the subalgebra of Lipschitz functions on $\Lambda$, i.e.

$$A_0 = C_{\mathrm{Lip}}(\Lambda, \mathbb{C}) = \{a \in C(\Lambda, \mathbb{C}) : |a|_{\mathrm{Lip}} < +\infty\},$$

where $|a|_{\mathrm{Lip}}$ is the seminorm

$$|a|_{\mathrm{Lip}} = \sup_{x\neq y}\frac{|a(x) - a(y)|}{\|x - y\|}.$$

Then, by Lemma 2.6, for $a \in A_0$,

$$\left\|[D_{\mathcal{S}}, \pi(a)]\left(\bigoplus_{w\in W^*}\begin{pmatrix}\xi_1(w)\\ \xi_2(w)\end{pmatrix}\right)\right\|_2^2$$

$$\leq C^2\sum_{w\in W^*}\left(\frac{a(\phi_w(x^{\mathrm{t}(w)})) - a(\phi_w(y^{\mathrm{t}(w)}))}{\|\phi_w(x^{\mathrm{t}(w)}) - \phi_w(y^{\mathrm{t}(w)})\|}\right)^2((-\xi_2(w))^2 + \xi_1(w)^2) \leq C^2|a|_{\mathrm{Lip}}^2\|\xi\|_2^2.$$

Hence $\|[D_{\mathcal{S}}, \pi(a)]\| < +\infty$. □

## 6. Proof of Theorem 3.1

We will use the following version of the Hardy-Littlewood Tauberian Theorem, adapted from Proposition 4 in chapter IV, §2.$\beta$ of [11].

**Lemma 6.1.** *Suppose that*

$$\zeta(t) = \sum_{n=1}^{\infty} a_n \lambda_n^t,$$

*where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers decreasing to 0 and $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers, has abscissa of convergence $\delta > 0$. Then*

$$\lim_{t \to \delta+} (t-\delta)\zeta(t) = \delta L,$$

*if and only if*

$$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} a_k \lambda_k^{\delta} = L.$$

We will suppose that $a \in C_{\mathrm{Lip}}(\Lambda, \mathbb{R})$ and that $a \geq 0$, so that $\pi(a)|D_{\mathcal{S}}|^{-\delta}$ is a positive operator. The eigenvalues of $\pi(a)|D_{\mathcal{S}}|^{-\delta}$ are the numbers

$$\left\{ a(\phi_w(x^{\mathfrak{t}(w)}))|\phi_w'(x^{\mathfrak{t}(w)})|^{\delta}, a(\phi_w(y^{\mathfrak{t}(w)}))|\phi_w'(x^{\mathfrak{t}(w)})|^{\delta} \,:\, w \in W^* \right\}$$

(counted with multiplicity). We define a spectral zeta function

$$\zeta_a(t) = \sum_{w \in W^*} \left( a(\phi_w(x^{\mathfrak{t}(w)})) + a(\phi_w(y^{\mathfrak{t}(w)})) \right) |\phi_w'(x^{\mathfrak{t}(w)})|^t$$

and we also write

$$\zeta_{a,+}(t) = \sum_{w \in W^*} a(\phi_w(x^{\mathfrak{t}(w)}))|\phi_w'(x^{\mathfrak{t}(w)})|^t, \ \zeta_{a,-}(t) = \sum_{w \in W^*} a(\phi_w(y^{\mathfrak{t}(w)}))|\phi_w'(x^{\mathfrak{t}(w)})|^t.$$

We will use the results of section 4 to analyse $\zeta_{a,+}(t)$ (and hence $\zeta_{a,-}(t)$ and $\zeta_a(t)$) as $t$ approaches $\delta$.

**Lemma 6.2.** *For $a \in C_{\mathrm{Lip}}(\Lambda, \mathbb{R})$ with $a > 0$, $\zeta_a(t)$ converges for $t > \delta$ and*

$$\lim_{t \to \delta+} (t-\delta)\zeta_a(t) = \frac{2\int a\, d\mu}{-\int f\, dm_{\delta f}} \left( \sum_{v \in V} h_{\delta f}(\overline{x^v}) \right),$$

*where, for each $v \in V$, $\overline{x^v} \in \Sigma_A^+$ is chosen so that $\Pi(\overline{x^v}) = x^v$.*

*Proof.* First, we shall obtain the corresponding result for $\zeta_{a,+}(t)$. Write $\overline{a} = a \circ \Pi$. Then $\overline{a} \in C^{\alpha}(\Sigma^+, \mathbb{R})$, for some $\alpha > 0$. Provided $\zeta_{a,+}(t)$ converges, we may use the definitions of $x^{\mathfrak{t}(w)}$ and $L_{tf}$ to write

$$\zeta_{a,+}(t) = \sum_{n=1}^{\infty} \sum_{v \in V} (L_{tf}^n \overline{a})(\overline{x^v}).$$

By Theorem 4.1 and Lemma 4.4, for $t > \delta$, $L_{tf}$ has spectral radius $e^{P(tf)} < 1$. Thus, using the spectral radius formula, it is easy to see that $\zeta_{a,+}(t)$ converges. Furthermore, by Corollary 4.2, we have

$$\begin{aligned} \zeta_{a,+}(t) &= \sum_{n=0}^{\infty} \left( \int \overline{a}\, d\nu_{tf} \right) e^{nP(tf)} \sum_{v \in V} h_{tf}(\overline{x^v}) + \sum_{n=0}^{\infty} q_n(t) \\ &= \left( \int \overline{a}\, d\nu_{tf} \right) \sum_{v \in V} \frac{h_{tf}(\overline{x^v})}{1 - e^{P(tf)}} + \sum_{n=1}^{\infty} q_n(t), \end{aligned}$$

where $h_t$ and $\nu_t$ are the eigenfunction and eigenmeasure for $L_{tf}$ given by Theorem 4.1 and where $q_n(t) = O(\lambda_{tf}^n)$ (with $\lambda_{tf} < e^{P(tf)}$). Since

(i) $t \mapsto e^{P(tf)}$, $t \mapsto h_{tf}$ and $t \mapsto \nu_{tf}$ are all analytic;
(ii) $e^{P(\delta f)} = 1$ and $\Pi_* \nu_{\delta f} = \mu$; and
(iii)
$$\left.\frac{de^{P(tf)}}{dt}\right|_{t=\delta} = \int f \, dm_{\delta f};$$

we see that
$$\zeta_{a,+}(t) = \frac{\int a \, d\mu}{-\int f \, dm_{\delta f}} \left( \sum_{v \in V} h_{\delta f}(\overline{x^v}) \right) \frac{1}{t-\delta} + \alpha(t),$$
where $\alpha(t)$ is finite for $t \geq \delta$. Thus,
$$\lim_{t \to \delta+} (t-\delta)\zeta_{a,+}(t) = \frac{\int a \, d\mu}{-\int f \, dm_{\delta f}} \left( \sum_{v \in V} h_{\delta f}(\overline{x^v}) \right).$$

We shall complete the proof by showing that $\zeta_{a,-}(t)$ also converges for $t > \delta$ and that $\lim_{t \to \delta+}(t-\delta)\zeta_{a,-}(t) = \lim_{t \to \delta+}(t-\delta)\zeta_{a,+}(t)$. Note that, for $t > \delta$,
$$|\zeta_{a,+}(t) - \zeta_{a,-}(t)| \leq \sum_{w \in W^*} |a(\phi_w(x^{\mathfrak{t}(w)})) - a(\phi_w(y^{\mathfrak{t}(w)}))||\phi_w'(x^{\mathfrak{t}(w)})|^t.$$

Let $\{w_m\}_{m=1}^{\infty}$ be any enumeration of $W^*$. Then
$$\lim_{m \to +\infty} \|\phi_{w_m}(x^{\mathfrak{t}(w_m)}) - \phi_{w_m}(y^{\mathfrak{t}(w_m)})\| = 0,$$
so that, since $a$ is continuous,
$$\lim_{m \to +\infty} a(\phi_{w_m}(x^{\mathfrak{t}(w_m)})) - a(\phi_{w_m}(y^{\mathfrak{t}(w_m)})) = 0.$$
Hence, given $\epsilon > 0$, there exists $N \geq 1$ such that
$$|a(\phi_{w_m}(x^{\mathfrak{t}(w_m)})) - a(\phi_{w_m}(y^{\mathfrak{t}(w_m)}))| < \epsilon$$
whenever $m \geq N$. Thus,
$$\begin{aligned} |\zeta_{a,+}(t) - \zeta_{a,-}(t)| &\leq 2\|a\|_\infty \sum_{m=1}^{N-1} |\phi_{w_m}'(x^{\mathfrak{t}(w)})|^t + \epsilon \left( \sum_{m=N}^{\infty} |\phi_{w_m}'(x^{\mathfrak{t}(w_m)})|^t \right) \\ &\leq 2\|a\|_\infty (N-1) + \epsilon \zeta_{1,+}(t). \end{aligned}$$
Hence, $\zeta_{a,-}(t)$ converges provided $\zeta_{a,+}(t)$ converges and we have the estimate
$$\lim_{t \to \delta+} (t-\delta)(\zeta_{a,+}(t) - \zeta_{a,-}(t)) \leq \epsilon \lim_{t \to \delta+} (t-\delta)\zeta_{1,+}(t) = \frac{\epsilon}{-\int f \, dm_{\delta f}}.$$
Since $\epsilon > 0$ is arbitrary, this shows that
$$\lim_{t \to 1+} (t-1)\zeta_{a,+}(t) = \lim_{t \to 1+} (t-1)\zeta_{a,-}(t)$$
and hence that
$$\lim_{t \to \delta+} (t-\delta)\zeta_a(t) = \frac{2\int a \, d\mu}{-\int f \, dm_{\delta f}} \left( \sum_{v \in V} h_{\delta f}(\overline{x^v}) \right),$$
as required. □

**Remark 6.3.** In fact, one can show (using the type of methods described in [25, 34]) that, considered as a function of a complex variable $s$, $\zeta_a(s)$ is analytic for $\mathrm{Re}(s) > \delta$, has a simple pole at $s = \delta$ and $\zeta_a(s)$ has a meromorphic extension to a neighbourhood of $\mathrm{Re}(s) \geq \delta$. Furthermore, poles can only occur at values of $s$ for which $L_{sf}$ has 1 as an eigenvalue. For the analysis of similar functions, see, for example, [28, 29, 30].

*Proof of Theorem 3.1.* We need to show that, whenever $a \in C(\Lambda, \mathbb{R})$ with $a \geq 0$, we have

$$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} \beta_k(a) = c \int a \, d\mu,$$

where $\{\beta_k(a)\}_{k=1}^{\infty}$ are the eigenvalues of $\pi(a)|D_{\mathcal{S}}|^{-\delta}$, counted with multiplicity and written in decreasing order, and where

$$c = \frac{2}{-\delta \int f \, dm_{\delta f}} \left( \sum_{v \in V} h_{\delta f}(\overline{x^v}) \right).$$

First, suppose that $a \in C_{\mathrm{Lip}}(\Lambda, \mathbb{R})$ and that $a \geq 0$. Lemma 6.2 shows that $\zeta_a(t)$ converges for $t > \delta$ and diverges for $t = \delta$. Thus $\pi(a)|D_{\mathcal{S}}|^{-\delta} \in \mathcal{L}^{1,\infty}(H)$. It follows immediately from Lemma 6.1 and Lemma 6.2 that

$$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} \beta_k(a) = \delta^{-1} \lim_{t \to \delta+} (t - \delta)\zeta_a(t) = c \int a \, d\mu.$$

Now suppose $a \in C(\Lambda, \mathbb{R})$ and $a \geq 0$. Given $\epsilon > 0$, we may choose $a_1, a_2 \in C_{\mathrm{Lip}}(\Lambda, \mathbb{C})$ such that $0 \leq a_1 \leq a \leq a_2$ and

$$\int a \, d\mu - \epsilon \leq \int a_1 \, d\mu \leq \int a_2 \, d\mu \leq \int a \, d\mu + \epsilon.$$

Then we have

$$\begin{aligned}
c\left(\int a \, d\mu - \epsilon\right) &\leq c \int a_1 \, d\mu = \lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} \beta_k(a_1) \\
&\leq \liminf_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} \beta_k(a) \leq \limsup_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} \beta_k(a) \\
&\leq \lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} \beta_k(a_2) = c \int a_2 \, d\mu \\
&\leq c\left(\int a \, d\mu - \epsilon\right).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the required convergence result holds for $a$. □

## References


[1] (MR1085823) R. Adler and L. Flatto, *Geodesic flows, interval maps, and symbolic dynamics*, Bull. Amer. Math. Soc. 25 (1991), 229–334.

[2] (MR1387512) S. Albeverio, D. Guido, A. Ponosov, and S. Scarlatti, *Singular traces and compact operators*, J. Funct. Anal. 137 (1996), 281–302.

[3] N. Benakli, *Polyèdres hyperboliques passage du local au global*, Thesis, Paris Sud, 1992.

[4] (MR1793194) V. Baladi, *Positive transfer operators and decay of correlations*, Advanced Series in Nonlinear Dynamics, 16, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[5] (MR0506158) R. Bhatia and K. Parthasarathy, *Lectures on functional analysis. Part I. Perturbation by bounded operators*, ISI Lecture Notes, 3, Macmillan Co. of India, Ltd., New Delhi, 1978.

[6] M. Bourdon, *Actions quasi-convexes d'un groupe hyperbolique, flot géodésique*, Thesis, Paris Sud, 1993.

[7] (MR0556580) R. Bowen, *The Hausdorff dimension of quasi-circles*, Publ. Math. IHES 50 (1979), 11–25.

[8] (MR2423393) R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, second revised edition, with a preface by David Ruelle, edited by Jean-René Chazottes, Lecture Notes in Mathematics, 470, Springer-Verlag, Berlin, 2008.

[9] (MR0556585) R. Bowen and C. Series, *Markov maps associated with Fuchsian groups*, Publ. Math. IHES 50 (1979), 153–170.

[10] (MR2261610) E. Christensen and C. Ivan, *Spectral triples for AF $C^*$-algebras and metrics on the Cantor set*, J. Operator Theory 56 (2006), 17–46.

[11] (MR1303779) A. Connes, *Noncommutative Geometry*, Academic Press, New York, 1994.

[12] (MR1345552) A. Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. 34 (1995), 203–238.

[13] (MR1070713) J. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics 96, Springer-Verlag, New York, 1990.

[14] (MR0196508) J. Dixmier, *Existence de traces non normales*, C. R. Acad. Sci. Paris Sér. A-B 262 (1966) 1107–1108.

[15] (MR2776380) K. Falconer and T. Samuel, *Dixmier traces and coarse multifractal analysis*, Ergodic Theory Dynam. Systems 31 (2011), 369–381.

[16] (MR1867554) D. Guido and T. Isola, *Fractals in non-commutative geometry*, Mathematical Physics in Mathematics and Physics, Sienna 2000, Field Institute Communications, 30, American Mathematical Society, Providence, RI, 2001.

[17] (MR2003353) D. Guido and T. Isola, *Dimensions and singular traces for spectral triples, with applications to fractals*, J. Funct. Anal. 203 (2003), 362–400.

[18] (MR2238285) D. Guido and T. Isola, *Dimensions and spectral triples for fractals in $\mathbb{R}^N$*, Advances in operator algebras and mathematical physics, 89–108, Theta Ser. Adv. Math., 5, Theta, Bucharest, 2005.

[19] (MR1335452) T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[20] (MR3084488) M. Kesseböhmer and T. Samuel, *Spectral metric spaces for Gibbs measures*, J. Funct. Anal. 265 (2013), 1801–1828.

[21] (MR1189091) M. Lapidus and C. Pomerance, *The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums*, Proc. London Math. Soc. 66 (1993), 41–69.

[22] (MR2139105) S. Lord, A. Sedaev and F. Sukochev, *Dixmier traces as singular symmetric functionals and applications to measurable operators*, J. Funct. Anal. 224 (2005), 72–106.

[23] (MR2003772) R. D. Mauldin and M. Urbanski, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Tracts in Mathematics 148, Cambridge University Press, Cambridge, 2003.

[24] I. Palmer, *Riemannian Geometry of Compact Metric Spaces*, PhD Thesis, Georgia Tech, 2010.

[25] (MR1085356) W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque 187–188 (1990), 1-268.

[26] (MR0450547) S.J. Patterson, *The limit set of a Fuchsian group*, Acta Math. 136 (1976), 241–273.

[27] (MR2511637) J. Pearson and J. Bellissard, *Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets*, J. Noncommut. Geom. 3 (2009), 447–480.

[28] (MR1323676) M. Pollicott, *A symbolic proof of a theorem of Margulis on geodesic arcs on negatively curved manifolds*, Amer. J. Math. 117 (1995), 289–305.

[29] (MR1376553) M. Pollicott and R. Sharp, *Comparison theorems and orbit counting in hyperbolic geometry*, Trans. Amer. Math. Soc. 350 (1998), 473–499.

[30] (MR1819191) M. Pollicott and R. Sharp, *Poincaré series and comparison theorems for variable negative curvature*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, 202, Amer. Math. Soc., Providence, RI, 2001, pp. 229–240.

[31] (MR2129258) D. Ruelle, *Thermodynamic Formalism*, second edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004.

[32] T. Samuel, *A Commutative Noncommutative Fractal Geometry*, PhD Thesis, St Andrews University, 2010.

[33] (MR0873435) C. Series, *Geometrical Markov coding of geodesics on surfaces of constant negative curvature*, Ergod. Th. and Dynam. Sys. 6 (1986), 601–615.

[34] (MR2035655) R. Sharp, *Periodic orbits of hyperbolic flows*, in G. Margulis, On some aspects of the theory of Anosov systems, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

[35] (MR2990125) R. Sharp, *Spectral triples and Gibbs measures for expanding maps on Cantor sets*, J. Noncommut. Geom. 6 (2012), 801–817.

[36] (MR0556586) D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. IHES 50 (1979), 171–202.

[37] (MR0766265) D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. 153 (1984), 259–277.

[38] (MR0736215) P. Tukia, *The Hausdorff dimension of the limit set of a geometrically finite Kleinian group*, Acta Math. 152 (1984), 127–140.

[39] (MR2239597) J. Várilly, *An introduction to noncommutative geometry*, EMS Series of Lectures in Mathematics, European Mathematical Society, Zürich, 2006.
[40] (MR0648108) P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, 79, Springer-Verlag, New York-Berlin, 1982

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England
*E-mail address*: R.J.Sharp@warwick.ac.uk