# CORRIGENDUM TO "ANOSOV FLOWS, GROWTH RATES ON COVERS AND GROUP EXTENSIONS OF SUBSHIFTS"

RHIANNON DOUGALL AND RICHARD SHARP

In the paper *Anosov flows, growth rates on covers and group extensions of subshifts* [1] the authors mistakenly claim in Lemma 5.2.(i) that

*There exists $C > 0$ so that for any $(\eta, g)$ and $(\xi, h)$ with $T_\psi^k(\eta, g) = (\xi, h)$ we have*

$$\nu_{\eta,g}(v) \geq C^k \nu_{\xi,h}(v).$$

This is in error as the proof actually shows the opposite inequality. Most significantly, the inequality was then used in the proof Lemma 5.3 (which is crucial to the main result Theorem 5.1). Less significantly, the proof of Theorem 5.1 is written with $\xi, \eta$ not restricted to the same cylinder, and the inequality in this case was not justified. Both of these errors are corrected by using transitivity of the system. In the beginning of the paper the authors set up the notation for a skew product with $\psi$ but then proceed to write all the proofs (and the definition of the transfer operator) with respect to a skew product with $\psi^{-1}$ – this easily seen to not change the validity of any of the statements. We give a revision of Lemma 5.2 to take into account the case where $\xi, \eta$ belong to different cylinders. We give a revision of Lemma 5.3. that does not depend on the mistaken claim. In addition show that the original statement claimed in Lemma 5.2(i) does in fact hold under transitivity (although it is no longer required for the proof of Theorem 5.1).

We remind the reader that we define the skew product $T_\psi$ with right multiplication (i.e. $T_\psi(\eta, g) = (\sigma\eta, g\psi(\eta))$), and so for the skew product with $\psi$ we have $T_\psi^k(\eta, g) = (\xi, h)$ if and only if $\sigma^k\eta = \xi$ and $g\psi(\eta)\psi(\sigma\eta)\cdots\psi(\sigma^k\eta) = h$. In this way for any letters $j_1, j_2$ we have $T_\psi^p([j_1] \times \{e\}) \cap [j_2] \times \{h\} \neq \varnothing$ if and only if $T_\psi^p([j_1] \times \{g\}) \cap [j_2] \times \{gh\} \neq \varnothing$. Or more precisely $T_\psi^k(\eta, e) = (\xi, h)$ if and only if $T_\psi^k(\eta, g) = (\xi, gh)$.

**Lemma 5.2.** (Corrected version)**.** *We have the following:*

*(i) There exists $C > 0$ so that for any $(\eta, g)$ and $(\xi, h)$ with $T_\psi^k(\eta, g) = (\xi, h)$ we have*

$$\nu_{\eta,g}(v) \leq C^k \nu_{\xi,h}(v).$$

*(ii) For any $\xi, \eta$ in the same length 1 cylinder we have*

$$\nu_{\eta,g}(v) \leq C_f \nu_{\xi,g}(v).$$

*Assume in addition that $T_\psi$ is transitive. Then we also have the following:*

*(iii) There exists $D > 0$ so that for any $\xi, \eta$ we have*

$$\nu_{\xi,g}(v) \leq D \nu_{\eta,g}(v).$$

*Proof.* Parts (i) and (ii) are the statements proved in [1] but with corrected inequality in the statement of (i).

We show (iii). Using transitivity of $T_\psi$, there is $r \in \mathbb{N}$ so that for any letters $j_1, j_2$ there is $p \leq r$ with $T_\psi^p([j_1] \times \{e\}) \cap [j_2] \times \{e\} \neq \varnothing$, whence for any $g \in G$ we have $T_\psi^p([j_1] \times \{g\}) \cap [j_2] \times \{g\} \neq \varnothing$.

Let $\eta \in [a]$ and $\xi \in [b]$. Let $(\zeta, e) \in [a] \times \{g\}$ with $T_\psi^p(\zeta, g) \in [b] \times \{g\}$ and set $T_\psi^p(\zeta, g) = (\zeta', g)$. Then $\zeta, \eta$ are in the same length 1 cylinder and $\zeta', \xi$ are in the same length 1 cylinder. Part (i) tells us that $\nu_{\zeta,g}(v) \geq C^p \nu_{\zeta',g}(v)$; and then part (ii) gives

$$\frac{\nu_{\eta,g}(v)}{\nu_{\xi,g}(v)} \geq C_f^{-2} \frac{\nu_{\zeta,g}(v)}{\nu_{\zeta',g}(v)} \geq C_f^{-2} C^p.$$

Noting that $p$ is bounded by $r$, which is independent of $g$, gives the result. □

We now give a correct proof of Lemma 5.3. and include the mistaken claim as a consequence.

**Lemma 5.3.** (Corrected version)**.** *Assume $T_\psi$ is transitive. We have the following:*

*(i) For any $a \in G$ there is a constant $M_a < \infty$ so that*

$$\sup_{g \in G} \frac{\nu_{o,ga}(v)}{\nu_{o,g}(v)} = M_a.$$

*(ii) There exists $L > 0$ so that for any $(\eta, g)$ and $(\xi, h)$ with $T_\psi^k(\eta, g) = (\xi, h)$ we have*

$$\nu_{\eta,g}(v) \geq L^k \nu_{\xi,h}(v).$$

*Proof.* We begin with part (i). Let $a \in G$ and let $o \in \Sigma^+$. Denote $b$ the first letter of $o$. so $o \in [b]$. Since $T_\psi$ is transitive there is $(\eta, a) \in [b] \times \{a\}$ and $k$ with $T_\psi^k(\eta, a) \in [b] \times \{e\}$. Then for any $g \in G$ we have $(\eta, ga) \in [b] \times \{ga\}$ and $T_\psi^k(\eta, ga) \in [b] \times \{g\}$. Set $(\xi, g) = T_\psi^k(\eta, ga)$. We use Lemma 5.2.(i) to say that

$$\frac{\nu_{\eta,ga}(v)}{\nu_{\xi,g}(v)} \leq C^k$$

and we use Lemma 5.2.(ii) to say that

$$\frac{\nu_{o,ga}(v)}{\nu_{o,g}(v)} = \frac{\nu_{o,ga}(v)}{\nu_{o,g}(v)} \frac{\nu_{\eta,ga}(v)}{\nu_{o,ga}(v)} \frac{\nu_{o,g}(v)}{\nu_{\xi,g}(v)} \frac{\nu_{\eta,ga}(v)}{\nu_{\xi,g}(v)} \leq C_f^2 C^k.$$

We have deduced part (i).

We now show part (ii). For brevity we will consider $\psi$ as being defined on letters.

Let $(\eta, g)$ and $(\xi, h)$ with $T_\psi^k(\eta, g) = (\xi, h)$. Let $b_0 \cdots b_k$ be the initial $k$ letters of $\eta$. Since $\eta \in [b_0 \cdots b_k]$ we have

$$T_\psi^k(\eta, g) = (\sigma^k \eta, g\psi(b_0)\psi(b_1) \cdots \psi(b_{k-1}))$$

and by hypothesis

$$(\sigma^k \eta, g\psi(b_0)\psi(b_1) \cdots \psi(b_{k-1})) = (\xi, h).$$

Therefore $\xi \in [b_k]$ and $g\psi(b_0)\psi(b_1) \cdots \psi(b_{k-1}) = h$. Upon setting $s_i = \psi(b_i)$ we may write $g = hs_0 \cdots s_{k-1}$. We use Lemma 5.2(iii) to say that $\nu_{\xi,h}(v) \leq D\nu_{\eta,h}(v)$ and then use Lemma 5.2.(ii) to change to some fixed $o$ belonging to the same cylinder as $\eta$, giving

$$\frac{\nu_{\eta,g}(v)}{\nu_{\xi,h}(v)} \geq D^{-1} \frac{\nu_{\eta,g}(v)}{\nu_{\eta,h}(v)} \geq C_f^{-2} D^{-1} \frac{\nu_{o,g}(v)}{\nu_{o,h}(v)}.$$

Hence it remains to find a lower bound for $\frac{\nu_{o,g}(v)}{\nu_{o,h}(v)}$. Now we have

$$\frac{\nu_{o,g}(v)}{\nu_{o,h}(v)} = \frac{\nu_{o,g}(v)}{\nu_{o,gs_0 \cdots s_{k-1}}(v)} = \frac{\nu_{o,g}(v)}{\nu_{o,gs_0}(v)} \frac{\nu_{o,gs_0}(v)}{\nu_{o,gs_0 s_1}(v)} \cdots \frac{\nu_{o,gs_0 \cdots s_{k-2}}(v)}{\nu_{o,gs_0 \cdots s_{k-1}}(v)} \geq \frac{1}{M_{s_0} M_{s_1} \cdots M_{s_{k-1}}}.$$

As $s_i$ belong to the bounded set of generators $S = \{\psi(B) : |B| = 1\}$ the result follows by setting

$$C_o = \min \left\{ C_f^{-2} D^{-1} \frac{1}{M_s} : s \in S \right\}$$

and taking the minimum over the finitely many choices of $o$ ($M_s$ depends on $o$). □

**Acknowledgement.** The authors are grateful to Zijie Lin for pointing out the errors in the original version of the paper.

## References

[1] R. Dougall and R. Sharp. Anosov flows, growth rates on covers and group extensions of subshifts *Inventiones Mathematicae*, 223, 445–483, 2021.

Department of Mathematical Sciences, Durham University, Upper Mountjoy, Durham DH1 3LE, UK

*Email address*: rhiannon.dougall@durham.ac.uk

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

*Email address*: R.J.Sharp@warwick.ac.uk