RELATIVE GROWTH SERIES IN SOME HYPERBOLIC GROUPS

Richard Sharp

University of Manchester

Abstract. We study certain groups of isometries of hyperbolic space (including certain hyperbolic Coxeter groups) and normal subgroups with quotient $\mathbb{Z}^\nu$, $\nu \geq 1$. We show that the associated relative growth series are not rational. This extends results obtained by Grigorchuk and de la Harpe for free groups and Pollicott and the author for surface groups.

0. Introduction

Given a finitely generated group G, a finite generating set S for G and a (not necessarily finitely generated) subgroup G_0, we define the *relative* growth series $\xi_0(z) = \xi_0(G, G_0, S, z)$ by $\xi_0(z) = \sum_{g \in G_0} z^{|g|}$, where $|\cdot|$ denotes word length with respect to the generators S.

This notion was discussed in, for example, [12], where it was shown (using results from [10]) that if G is the free group on two generators and G_0 is the commutator subgroup, then $\xi_0(z)$ is *not* a rational function. In [18], the corresponding result was obtained in the case where G is a the fundamental group of a compact surface of genus at least 2 (with the standard one-relator presentation) and G_0 is the commutator subgroup. (In fact, the results shown in both these papers are a little stronger: $\xi_0(z)$ is not algebraic.)

Observe that in these examples, G may be realized as a convex co-compact group of isometries of the hyperbolic plane. (Recall that a group of isometries of $\mathbb{H}^n$ is called convex co-compact if the convex hull of its limit set has compact quotient.) In this paper, we shall extend these results to some groups of isometries of hyperbolic space $\mathbb{H}^n$, $n \geq 3$ (equiped with geometrically natural generating sets). More precisely, we have the following definition.

Definition. We say that a group G of isometries of $\mathbb{H}^n$ together with a finite generating set S satisfies Property (*) if G admits a fundamental domain R such that

(i) R is a finite sided polyhedron (with finite or infinite volume);
(ii) $S = \{g \in G : gR \cap R \neq \emptyset\}$; and
(iii) the set $\cup_{g \in G} g\partial R$ is a union of hyperplanes.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

Theorem 1. *Let G be a convex co-compact group of isometries of $\mathbb{H}^n$ and let S be a finite generating set such that (G,S) satisfies Property (*). Suppose that $G_0 \lhd G$ such that $G/G_0 \cong \mathbb{Z}^\nu$, $\nu \geq 1$. Then the relative growth series $\xi_0(z)$ is not a rational function.*

Remark. Property (*) was introduced for Fuchsian groups by Bowen and Series [2] and studied in higher dimensions by Bourdon [1]. The usual definition is that G satisfies (i) and (iii) and does not make reference to the generating set. However, we need to fix a suitable set of generators and, for convenience, we have made this part of the definition.

We state two classes of examples where Property (*) is satisfied.

Example 1. An important class of examples satisfying (*) is is the following class of Coxeter groups. Let R be a polyhedron in $\mathbb{H}^n$ with a finite number of faces and with interior dihedral angles all equal to π/k, $k \in \mathbb{N}$, $k \geq 2$. Let S be the set of inversions in the faces of R and let G be the Coxeter group it generates. Then (G,S) satisfies (*) [1].

Example 2. A second example is given by the following. Let Σ be a compact surface of genus $\mathfrak{g} \geq 2$ and let $f : \Sigma \to \Sigma$ be a pseudo-Anosov homeomorphism [4]. Let $G_0 = \pi_1\Sigma$ and let $f_* : G_0 \to G_0$ be the induced automorphism. Now define G to be the semi-direct product $G = G_0 \ltimes_{f_*} \mathbb{Z}$, where the $\mathbb{Z}$-action is induced by f_*. Then G may be represented as a co-compact group of isometries of $\mathbb{H}^3$ [5], [14]. The fundamental domain R for the action on $\mathbb{H}^3$ can be chosen so that it is a finite sided polyhedron each face of which is a $4\mathfrak{g}$-gon. Each face extends to a hyperplane P_k meeting the ideal boundary $\hat{\mathbb{C}}$ in a circle C_k in such a way that C_k and $C_{k'}$, $k \neq k'$, are either orthogonal or disjoint [14]. If S denotes the set of inversions in the faces of R then (G,S) satisfies Property (*). To see this notice that condition (iii) of Property (*) amounts to the requirement that $\cup_{g\in G} g\partial R = \cup_{g \in G} \cup_k gP_k$. This holds if P_k and $P_{k'}$, $k \neq k'$, either meet orthogonally or are disjoint; however this follows from the corresponding condition on the circles C_k.

Since $G_0 \lhd G$ and $G/G_0 \cong \mathbb{Z}$ the hypotheses of Theorem 1 are satisfied. In this example, unlike those in [12] and [18], the subgroup G_0 is finitely generated.

Remark. In [11], Grigorchuk proved that for a finitely generated subgroup of a free group, the relative growth series is rational. Example 2 shows that this result does not extend to hyperbolic groups.

The proof will be based on estimating the growth of the coefficients in $\xi_0(z)$. If $\sum_{n=0}^\infty b_n z^n$ is the series of a rational function then there exist (not necessarily distinct) complex numbers $\alpha_1, \dots, \alpha_k$ and $c_1, \dots, c_k$ and non-negative integers $m_1, \dots, m_k$ such that $b_n \sim \sum_{i=1}^k c_i n^{m_i} \alpha_i^n$, as $n \to \infty$ [9, p.327]. The theorem will follow from the fact that this condition fails along some subsequence. More precisley, we shall show that there exists $D > 0$ and $\lambda > 1$ such that $b_{Dn} \sim \lambda^{Dn}/n^{\nu/2}$.

Remark. The referee raised the interesting problem of whether or not $\xi_0(z)$ is algebraic. If $\xi_0(z)$ is algebraic then $b_n = \sum_{i=1}^k c_i n^p \omega_i^n \lambda^n + O(n^q \lambda^n)$, where $|\omega_1| = \cdots = |\omega_k| = 1$, $p \in \mathbb{Q} - \{-1,-2,-3,\dots\}$ and $q < p$ [6]. Thus our result implies that $\xi_0(z)$ is not algebraic if ν is even.

The author would like to thank Marc Bourdon and Mark Pollicott for helpful discussions about hyperbolic groups and growth series.

1. Growth series

We begin by recalling the notion of growth series for finitely generated groups. Let G be a finitely generated group and let S be a finite generating set. For simplicity, we shall assume that $S^{-1} = S$. For an element $g \in G$, we define its word length $|g| = |g|_S$ by

$$|g| = \inf\{n : g = g_1 \cdots g_n, g_i \in S, i = 1, \dots, n\}.$$

(By convention, we set $|e| = 0$, where e is the identity element in G.) We now define the growth series $\xi(z) = \xi(G, S, z)$ by

$$\xi(z) = \sum_{g \in G} z^{|g|} = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \#\{g \in G : |g| = n\}$. The radius of convergence of $\xi(z)$ is given by $1/\lambda$, where

$$\lambda = \limsup_{n \to \infty} a_n^{1/n}.$$

The number $\lambda = \lambda(G, S)$ is called the exponential growth rate of the pair (G, S). For the groups we consider, $\lambda > 1$.

The groups of isometries that we consider are examples of hyperbolic groups in the sense of Gromov [8],[13]. By a celebrated result of Cannon [3], if G is a hyperbolic group then, for any finite generating set S, $\xi(z)$ is (the series of) a rational function.

Now suppose that we have a subgroup G_0 of G. We define the *relative* growth series $\xi_0(z) = \xi_0(G, G_0, S, z)$ by

$$\xi_0(z) = \sum_{g \in G_0} z^{|g|} = \sum_{n=0}^{\infty} b_n z^n,$$

where $|\cdot|$ is the restriction of $|\cdot| : G \to \mathbb{Z}$ to G_0 and where $b_n = \#\{g \in G_0 : |g| = n\}$.

We shall consider a family of auxiliary growth sequences. Let $\psi : G \to G/G_0 \cong \mathbb{Z}^\nu$ denote the natural projection. For $\alpha \in \mathbb{Z}^\nu$, we define

$$b_n(\alpha) = \#\{g \in G : |g| = n \text{ and } \psi(g) = \alpha\}$$

and $B_n(\alpha) = \sum_{m=0}^{n} b_m(\alpha)$. In particular, since $g \in G_0$ if and only if $\psi(g) = 0$, $b_n = b_n(0)$.

If we write $n_\alpha = \inf\{|g| : \psi(g) = \alpha\}$ then it is easy to see that

$$B_n(0) \le B_{n+n_\alpha}(\alpha) \le B_{n+2n_\alpha}(0)$$

and so

$$\lambda_0 := \limsup_{n \to \infty} B_n(\alpha)^{1/n}$$

is independent of α.

Clearly $\lambda_0 \le \lambda$. Let $A_n = \{\alpha \in \mathbb{Z}^\nu : n_\alpha \le n\}$. Since $a_n = \sum_{\alpha \in A_n} b_n(\alpha)$, we have

$$\lambda \le \left(\limsup_{n \to \infty} (\#A_n)^{1/n} \right) \left(\limsup_{n \to \infty} (\sup_{\alpha \in \mathbb{Z}^\nu} b_n(\alpha))^{1/n} \right).$$

One easily shows that $\limsup_{n \to \infty} (\#A_n)^{1/n} = 1$, so that $\lambda_0 = \lambda$. Since $\lambda > 0$, we see that we also have

$$\limsup_{n \to \infty} b_n(\alpha)^{1/n} = \lambda, \tag{1.1}$$

for every $\alpha \in \mathbb{Z}^\nu$.

2. Strongly Markov groups

An important property of the groups we consider which will prove crucial to our analysis is that they are strongly Markov in the sense of [8].

Definition. A finitely generated group G is called strongly Markov if for every finite (symmetric) generating set S, there exists

(i) a finite directed graph consisting of vertices V and edges $E \subset V \times V$;
(ii) a distinguished vertex $* \in V$, with no edges terminating at $*$;
(iii) a labeling map $\rho : E \to S$;

such that

(a) there is a bijection between finite paths in the graph starting at $*$ and passing through the consecutive edges $e_1, \dots, e_n$, say and elements $g \in G$ given by the correspondence $g = \rho(e_1) \cdots \rho(e_n)$ (where the empty path corresponds to the identity element);
(b) the word length $|g|$ is equal to the path length n.

Proposition 1 ([3],[7])**.** *If G is a convex co-compact group of isometries of $\mathbb{H}^n$ then G is strongly Markov.*

Let A denote the incidence matrix of the graph (V, E), i.e., A is a $|V| \times |V|$ matrix with entries $A(i,j) = 1$ if $(i,j) \in E$ and 0 otherwise. For future use, we shall write $k + 1 = |V|$. From the definition it is easy to see that we have $a_n = \sum_{j \in V} A^n(*, j)$ and that A has spectral radius equal to the growth rate λ. Furthermore, one can easily deduce that $\xi(z)$ is a rational function [7, Chapitre 9].

Let B denote the submatrix of A obtained by deleting the index corresponding to the vertex *. It is easy to see that A and B have the same non-zero eigenvalues. We have the following.

Proposition 2 ([1])**.** *If (G, S) satisfies Property (*) then B is irreducible.*

Recall that that the matrix B, indexed by $\mathcal{I}$, is said to be *irreducible* if for each pair of indices (i,j), there exists $n > 0$ such that $B^n(i,j) > 0$ and is said to be *aperiodic* if there exists $n > 0$ such that $B^n(i,j) > 0$ for each pair of indices $(i,j) \in \mathcal{I} \times \mathcal{I}$. The integer $d = \mathrm{hcf}\{n : B^n(i,i) > 0, \text{ for every } i\}$ is called the period of B and B is aperiodic precisely when $d = 1$. The indexing set $\mathcal{I}$ can be uniquely partitioned into sets $\mathcal{I}_1, \dots, \mathcal{I}_d$ such that B^d restricted to $\mathcal{I}_l \times \mathcal{I}_l$ is aperiodic, $l = 1, \dots, d$. Furthermore, we can arrange the indexing so that if $B(i,j) = 1$ with $i \in \mathcal{I}_l$ then $j \in \mathcal{I}_{l+1}$ (mod d). In view of this decomposition we may, after an appropriate recoding, suppose that B is aperiodic. More precisely, we introduce a new index set

$$\mathcal{I}^{(d)} = \{(i_1, \dots, i_d) \in \mathcal{I}_1 \times \cdots \times \mathcal{I}_d : B(i_l, i_{l+1}) = 1, l = 1, \dots, d\}$$

and replace B by $\tilde{B}$, indexed by $\mathcal{I}^{(d)}$ and defined by

$$\tilde{B}((i_1, \dots, i_d), (j_1, \dots, j_d)) = B^d(i_1, j_1).$$

This amounts to replacing the sequence b_n with the sequence b_{dn}.

The spectrum of an aperiodic matrix is partially described by the next result.

Lemma 1 (Perron-Frobenius Theorem [7]). *If B is an aperiodic non-negative square matrix then B has a simple positive eigenvalue λ equal to its spectral radius with a strictly positive associated eigenvector. The remaining eigenvalues of B all have modulus strictly less that λ.*

3. The calculation

In order to analyse their growth we shall express the coefficients b_n as integrals over the torus $\mathbb{R}^\nu/\mathbb{Z}^\nu$. Define a function $f : V \times V \to \mathbb{Z}^\nu$ by $f(i,j) = \psi \circ \rho(i,j)$ if $(i,j) \in E$ and $f(i,j) = 0$ otherwise. Let $g \in G$ correspond to the path $(*, i_1), \ldots, (i_{|g|-1}, i_{|g|})$ in (V, E). Then $g \in G_0$ if and only if

$$f(*, i_1) + f(i_1, i_2) + \cdots + f(i_{|g|-1}, i_{|g|}) = 0.$$

For $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, define a matrix $A(t)$ by

$$A(t)(i,j) = A(i,j)e^{2\pi i t \cdot f(i,j)},$$

for $i, j \in V$. (Here, the dot denotes scalar product.) In particular, $A(0) = A$. Recalling the orthogonality identity

$$\int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} e^{2\pi i t \cdot y} dt = \begin{cases} 1 & \text{for } y = 0 \\ 0 & \text{for } y \in \mathbb{Z}^\nu - \{0\}, \end{cases}$$

we then see that

$$b_n = \sum_{|g|=n} \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} e^{2\pi i t \cdot \psi(g)} dt = \sum_{j \in V} \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} A(t)^n(*, j) dt. \tag{3.1}$$

We also define submatrices $B(t)$, as before, by deleting the index corresponding to the vertex $*$. Once again $A(t)$ and $B(t)$ have the same non-zero eigenvalues, which we shall denote by $\lambda_1(t), \ldots, \lambda_k(t)$, arranged so that $|\lambda_1(t)| \geq |\lambda_2(t)| \geq \cdots \geq |\lambda_k(t)|$. We shall be interested in those values of t for which $|\lambda_1(t)| = \lambda$. These are characterized by the following lemma.

Lemma 2 (Wielandt's Theorem [7, p.57]). *The matrix $B(t)$ has maximal eigenvalue $e^{2\pi i\theta}\lambda$ if and only if $B(t) = e^{2\pi i\theta}\Delta B \Delta^{-1}$, where Δ is a diagonal matrix with diagonal entries of modulus one. In this case the eigenvalue is simple and the rest of the spectrum of $B(t)$ is contained in a disk of radius strictly less than λ.*

In order to continue our analysis, we recall some objects familiar from the coding theory of subshifts of finite type which were employed in [17] and [18].

We call a sequence $\gamma = (x_0, x_1, \ldots, x_n)$ such that $x_0 = x_n$ and $B(x_i, x_{i+1}) = 1$, $i = 0, 1, \ldots, n$, a cycle and denote its length by $l(\gamma) = n$. We define the f-weight $w_f(\gamma)$ of γ by $w_f(\gamma) = (f(x_0, x_1) + \cdots + f(x_{n-1}, x_n))$. We shall use $\mathcal{C}$ to denote the set of all cycles.

Following [16], we let Γ_f denote the subgroup of $\mathbb{Z}^\nu$ generated by the set $\{w_f(\gamma) : \gamma \in \mathcal{C}\}$ and let Δ_f be the subgroup of Γ_f defined by

$$\Delta_f = \{w_f(\gamma) - w_f(\gamma') : \gamma, \gamma' \in \mathcal{C}, l(\gamma) = l(\gamma')\}.$$

We also choose two cycles γ, γ' such that $l(\gamma) = l(\gamma') + 1$ and set $c_f = w_f(\gamma) - w_f(\gamma')$. Although this choice is arbitrary, it was shown in [16] that the coset $\Delta_f + c_f$ is well-defined and that Γ_f/Δ_f is the cyclic group generated by $\Delta_f + c_f$.

We say that two functions $g, g' : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ are cohomologous if there exists a function $u : \mathcal{I} \to \mathbb{R}$ such that $g'(x, y) = g(x, y) + u(y) - u(x)$. It is well-known that g and g' are cohomologous if and only if $w_g(\gamma) = w_{g'}(\gamma)$ for all $\gamma \in \mathcal{C}$. The functions that we need to consider are the inner products $t \cdot f : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$. The following was shown in [15].

Lemma 3 [15]. *If $t \cdot f$ is not cohomologous to a constant for any $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu - \{0\}$ then Γ_f/Δ_f has finite order, D_0 say.*

We now use the growth estimates on $b_n(\alpha)$ which we obtained earlier to show that the hypothesis of Lemma 3 is satisfied.

Proposition 3. *The inner product $t \cdot f$ is not cohomologous to a constant for any $t \in \mathbb{R}^\nu - \{0\}$.*

Proof. We shall assume that $t \cdot f$ is cohomologous to a constant c and obtain a contradiction. Suppose first that $c > 0$. Then there exists $n_0 \geq 1$ such that for all $n \geq n_0$ we have

$$t \cdot f(*, i_1) + t \cdot f(i_1, i_2) + \cdots t \cdot f(i_{n-1}, i_n) > 0$$

so that if $|g| \geq n_0$ then $t \cdot \psi(g) > 0$. However, this gives $b_n(0) = 0$ for $n \geq n_0$, contradicting (1.1). The same argument deals with the case $c < 0$. Now suppose that $c = 0$. Then, by similar reasoning to the above, $|t \cdot \psi(g)| \leq Q$, for some $Q \geq 0$ and all $g \in G$. Since $t \neq 0$, we can find $\alpha \in \mathbb{Z}^\nu$ such that $t \cdot \alpha > Q$. Then the above bound gives $b_n(\alpha) = 0$ for all $n \geq 0$, again a contradiction.

The above proposition also shows that $\mathrm{rank}_{\mathbb{Z}}(\Gamma_f) = \nu$, so that $\mathbb{Z}^\nu/\Gamma_f$ is finite. Let us write $D_1 = |\mathbb{Z}^\nu/\Gamma_f|$.

The importance of these results is due to the following.

Proposition 4 ([17]). *For $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, define $\chi_t \in \widehat{\mathbb{Z}^\nu}$ by $\chi_t(x) = e^{2\pi i t \cdot x}$. Then we have the following identities*

$$\{\chi_t : |\lambda_1(t)| = \lambda\} = \Delta_f^\perp;$$

$$\{\lambda_1(t) : \chi_t \in \Delta_f^\perp\} = \{\lambda^d e^{2\pi i r/D_0} : r = 0, 1, \ldots, d_0 - 1\}$$

where $\Delta_f^\perp = \{\chi \in \widehat{\mathbb{Z}^\nu} : \chi(\Delta_f) = 1\}$.

In fact, this result is a slight modification of the one in [17], due to that fact that here we do not assume $\Gamma_f = \mathbb{Z}^\nu$. Observe that $\Delta_f^\perp$ has cardinality $D = D_0 D_1$. We shall denote the values of t for which $|\lambda_1(t)|$ is maximal by by $t_0 = 0, t_1, \ldots, t_{D-1}$.

Choose a neighbourhood U_0 of 0 and write $U_r = U_0 + t_r$, $r = 1, \ldots, D-1$. Provided U_0 is sufficiently small, we can suppose that $\lambda_1(t)$ is a simple eigenvalue for $t \in \cup_{r=0}^{D-1} U_r$; in particular, it depends analytically on t. Furthermore, if we choose $\epsilon > 0$ sufficiently small, then we can suppose that

(1) $|\lambda_1(t)| \leq \lambda - \epsilon$, $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu - \cup_{r=0}^{D-1} U_r$;
(2) $|\lambda_j(t)| \leq \lambda - \epsilon$, $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, $j = 2, \ldots, k$.

Substituting this information into (3.1), we obtain

$$b_n(0) = \sum_{r=0}^{D-1} \int_{U_r} \lambda_1(t)^n \chi w_t dt + O\left((\lambda-\epsilon)^n\right),$$

where $\chi = (1, 0, \ldots, 0)$, with the 1 occuring in position *, and where w_t is the right eigenvector for $A(t)$ associated to $\lambda_1(t)$. Since $B(t-t_r)^d = B(t)^D$ for $r = 0, 1, \ldots, D-1$, we see that there exist $c_0, c_1, \ldots, c_{D-1}$ such that

$$\begin{aligned} b_n &= \sum_{r=0}^{D-1} e^{2\pi i n r/D} c_r \int_{U_r} \lambda_1(t)^n \left(1 + O(||t-t_r||)\right) dt + O\left((\lambda-\epsilon)^n\right) \\ &= \left(\sum_{r=0}^{D-1} e^{2\pi i n r/D} c_r\right) \int_{U_0} \lambda_1(t)^n \left(1 + O(||t||)\right) dt + O\left((\lambda-\epsilon)^n\right). \end{aligned}$$

Now this last integral is of the form considered in [17]. Given the growth estimate (1.1) the results of that paper give us that

$$\int_{U_0} \lambda_1(t)^n \left(1 + O(||t||)\right) dt \sim C \frac{\lambda^n}{n^{\nu/2}}, \text{ as } n \to \infty,$$

for some constant $C > 0$. Passing to the subsequence of multiples of D, we thus obtain that

$$b_{Dn} \sim \left(\sum_{r=0}^{D-1} c_r\right) \frac{C}{D^{\nu/2}} \frac{\lambda^{Dn}}{n^{\nu/2}}, \text{ as } n \to \infty,$$

which was the asymptotic formula we claimed in the introduction. This proves Theorem 1.

References

1. M. Bourdon, *Actions quasi-convexes d'un groupe hyperbolique, flot géodésique*, Thesis, Orsay (1993).
2. R. Bowen and C. Series, *Markov maps associated with Fuchsian groups*, Publ. Math. IHES **50** (1979), 153-170.
3. J. Cannon, *The combinatorial structure of co-compact discrete hyperbolic groups*, Geom. Dedicata **16** (1984), 123-148.
4. A. Casson and S. Bleiler, *Automorphisms of Surfaces after Nielsen and Thurston*, Cambridge University Press, Cambridge, 1988.
5. C. Epstein, *The spectral theory of geometrically periodic hyperbolic 3-manifolds*, Memoirs Amer. Math. Soc. **335** (1985), 1-161.
6. P. Flajolet, *Analytic models and ambiguity of context-free languages*, Theoretical Computer Science **49** (1987), 283-309.
7. F. R. Gantmacher, *The Theory of Matrices, Vol II*, Chelsea, New York, 1974.
8. E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Birkhauser, Basel, 1990.
9. R. Graham, D. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, Mass., 1989.
10. R. Grigorchuk, *Banach invariant means on homogeneous spaces and random walks*, Thesis (1978).

11. R. Grigorchuk, *Symmetrical random walks on discrete groups*, Multicomponent random systems (R. Dobrushin, Ya. Sinai and D. Griffeath, ed.), Advances in Probabilty and Related Topics 6, Dekker, 1980, pp. 285-325.
12. R. Grigorchuk and P. de la Harpe, *On problems related to growth, entropy and spectrum in group theory*, Journal of Dynamical and Control Systems **3** (1997), 51-89.
13. M. Gromov, *Hyperbolic groups*, Essays in Group Theory (S. Gersten, ed.), MSRI Publications, 1987.
14. T. Jorgensen, *Compact 3-manifolds of constant negative curvature fibering over the circle*, Annals of Math. **106** (1977), 61-72.
15. B. Marcus and S. Tuncel, *The weight-per-symbol polytope and scaffolds of invariants associated with Markov chains*, Ergodic Theory Dynam. Syst. **11** (1991), 129-180.
16. W. Parry and K. Schmidt, *Natural coefficients and invariants for Markov-shifts*, Invent. math. **76** (1984), 15-32.
17. M. Pollicott and R. Sharp, *Rates of recurrence for $\mathbb{Z}^q$ and $\mathbb{R}^q$ extensions of subshifts of finite type*, J. London Math. Soc. **49** (1994), 401-416.
18. M. Pollicott and R. Sharp, *Growth series for the commutator subgroup*, Proc. Amer. Math. Soc. **124** (1996), 1329-1335.

Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, U.K.