Equidistribution of Periodic Orbits
Low-Dimensional Dynamics
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Abstract

A classical theorem of Bowen says that periodic orbits of hyperbolic diffeomorphisms and flows are on average equidistributed with respect to the measure of maximal entropy as the periods tends to infinity. Hannay–Ozorio de Almeida and Parry showed that, by introducing appropriate weightings, the SRB measure of an attractor may also be obtained in this way. Parry also showed that other Gibbs start could be obtained as a limit by varying the weighting. We will discuss different approaches to these results (dynamical zeta functions, large deviations) and more recent refinements for flows, where, for example, the window of which periods are averaged is allowed to shrink as the periods increase.

1 Hyperbolic flows

Let M be a smooth compact Riemannian manifold and the $\phi_t : M \to M$ be a C^1 flow generated by a vector field $\mathcal{X}_\phi$. We say a closed ϕ_t-invariant set $\Lambda \subset M$ is a *hyperbolic basic set* if

(i) there is a continuous splitting of $T_\Lambda M$ into three $D\phi_t$-invariant sub-bundles $T_\Lambda M = E^0 \oplus E^s \oplus E^u$, where E^0 is the one-dimensional bundle spanned by $\mathcal{X}_\phi$ and where there are constants $C > 0$, $\lambda > 0$ such that

  (a) $\|D\phi_t(v)\| \leq Ce^{-\lambda t}\|v\|$ for $t \geq 0$ and for all $v \in E^s$; and

  (b) $\|D\phi_{-t}(v)\| \leq Ce^{-\lambda t}\|v\|$ for $t \geq 0$ and for all $v \in E^u$,

(ii) $\phi_t : \Lambda \to \Lambda$ is topologically transitive,

(iii) the set of ϕ-periodic orbits in Λ is dense in Λ,

(iv) there is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t=-\infty}^{\infty} \phi_t(U)$.

If Λ is a hyperbolic basic set then we call $\phi_t : \Lambda \to \Lambda$ a *hyperbolic flow*. If the stronger property (iv') there is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t=0}^{\infty} \phi_t(U)$

then we call $\phi_t : \Lambda \to \Lambda$ an *attractor*. If property (i) holds for $\Lambda = M$ then $\phi_t : M \to M$ is called an *Anosov flow*. An Anosov flow is not necessarily topologically transitive; however property (iii) holds for M (Anosov Closing Lemma). Of course, property (iv') always holds for an Anosov flow.

We say that a hyperbolic flow is *topologically weak-mixing* if the equation

$$f \circ \phi_t = e^{iat} f$$

has no continuous solution $f : \Lambda \to \mathbb{C}$ for $a \in \mathbb{R}$.

Example 1.1. Let S be a compact surface with negative Gaussian curvature and let $M = T^1 S$, the unit tangent bundle. Then the geodesic flow $\phi_t : M \to M$ is a (topologically transitive) Anosov flow.

2 Periodic orbits and equidistribution

Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow. This flow has a countably infinite set of prime periodic orbits, which we denote by $\mathcal{P}$. (A periodic orbit is prime if is not a multiple of another periodic orbit.) For $\gamma \in \mathcal{P}$, we write $\ell(\gamma)$ for its (least) period and

$$\mathcal{P}(T) = \{\gamma \in \mathcal{P} : \ell(\gamma) \leq T\}$$

and, for $\Delta > 0$,

$$\mathcal{P}(T, \Delta) = \{\gamma \in \mathcal{P} : \ell(\gamma) \in [T - \Delta/2, T + \Delta/2]\}.$$

We note that the flow is topologically weak-mixing if and only if the set of periods $\{\ell(\gamma) : \gamma \in \mathcal{P}\}$ is *not* contained in a discrete subgroup of $\mathbb{R}$.

There is a unique ϕ_t-invariant probability measure μ_0 on Λ for which the measure-theoretic entropy of ϕ_t is maximised – we call this the *measure of maximal entropy* for ϕ_t. For $f \in \Lambda \to \mathbb{R}$, we write

$$\int_\gamma f = \int_0^{\ell(\gamma)} f(\phi_t x_\gamma)\, dt,$$

where x_γ is any point on γ.

A classical result of Bowen says that the periodic orbits become equidistributed with respect to the measure of maximal entropy.

Theorem 2.1 (Bowen [1]). *For all $f \in C(\Lambda, \mathbb{R})$, we have*

$$\lim_{T \to \infty} \frac{1}{\#\mathcal{P}(T, \Delta)} \sum_{\gamma \in \mathcal{P}(T,\Delta)} \frac{1}{\ell(\gamma)} \int_\gamma f = \int f\, d\mu_0.$$

An alternative formulation of this result is that, letting δ_γ denote the probability measure obtained by normalising one-dimensional Lebesgue measure around γ, the measures

$$\frac{1}{\#\mathcal{P}(T, \Delta)} \sum_{\gamma \in \mathcal{P}(T,\Delta)} \delta_\gamma$$

converge weak$*$ to μ_0, as $T \to \infty$.

We will discuss two approaches to obtaining this result:

(0) *Bowen's approach.* Bowen originally used a "bare hands" approach involving the growth of periodic orbits and specification properties of the flow. We will not discuss this further.

(1) *Large deviations.* The input required here are growth estimates for certain sums over periodic orbits. From these, it is possible to conclude that, in a precise sense, most long periodic orbits γ have $\int_\gamma f$ close to $\int f \, d\mu_0$, and hence that the result in Theorem 2.1 holds.

(2) *Zeta functions.* Suppose f is positive a Hölder continuous. Study of the domain of a zeta function
$$\zeta(s,z) = \prod_{\gamma \in \mathcal{P}} \left(1 - \exp\left(-s\ell(\gamma) + z \int_\gamma f\right)\right)^{-1}$$
gives an asymptotic for
$$\sum_{\ell(\gamma) \in [T-\Delta/2, T+\Delta/2]} \int_\gamma f,$$
as $T \to \infty$. Elementary arguments then give Theorem 2.1.

Roughly speaking, approach (1) is simpler but approach (2) is more flexible if we want to obtain more precise results (e.g. rates of convergence) We will discuss both of these methods. Before we go on to do this, we will state a result that complements Bowen's equidistribution result.

Let $\phi_t : \Lambda \to \Lambda$ be a $C^{1+\alpha}$ attractor. There is a ϕ_t-invariant probability measure m supported on Λ with the property that for almost every x in a neighbourhood U of Λ we have that, for every continuous function $f : U \to \mathbb{R}$,
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\phi_t x) \, dt = \int f \, dm$$
[2]. The measure m is called the SRB (Sinai–Ruelle–Bowen) measure. If ϕ_t preserved the Riemannian volume then m is the normalised volume. This holds, for example, for geodesic flows.

Write
$$\varphi^u(x) = \lim_{t \to 0} \frac{1}{t} \log |\mathrm{Jac}(D\phi_t | E^u_x)|$$
and, for $\gamma \in \mathcal{P}$,
$$\ell^u(\gamma) = \int_\gamma \varphi^u.$$
We have the following result of Parry, inspired by earlier heuristic results of Hannay and Ozorio de Almeida (for discrete maps) [4].

Theorem 2.2 (Parry [8]). *For all $f \in C(\Lambda, \mathbb{R})$, we have*

$$\lim_{T\to\infty} \left(\sum_{\gamma \in \mathcal{P}(T,\Delta)} e^{-\ell^u(\gamma)} \right)^{-1} \sum_{\gamma \in \mathcal{P}(T,\Delta)} e^{-\ell^u(\gamma)} \frac{1}{\ell(\gamma)} \int_\gamma f = \int f \, dm.$$

In terms of orbital measures, this says that the measures

$$\left(\sum_{\gamma \in \mathcal{P}(T,\Delta)} e^{-\ell^u(\gamma)} \right)^{-1} \sum_{\gamma \in \mathcal{P}(T,\Delta)} e^{-\ell^u(\gamma)} \delta_\gamma$$

converge weak$*$ to m, as $T \to \infty$.

It turns out that Theorem 2.1 and Theorem 2.2 are part of a family of equidistribution results for different weightings. We will discuss this further below, after we have made some more definitions.

3 Pressure

An important tool in this theory is the pressure function $P : C(\Lambda, \mathbb{R}) \to \mathbb{R}$. For $f \in C(\Lambda, \mathbb{R})$, its pressure $P(f)$ is defined as the exponential growth rate of sums:

$$P(f) = \lim_{\epsilon \to 0} \limsup_{T\to\infty} \frac{1}{T} \log \sum_x \exp \left(\int_0^T f(\phi_t x) \, dt \right),$$

where the sum is taken over points x is a (T, ϵ) spanning or separated sets. For our purposes, it is more important to note that

$$P(f) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{\gamma \in \mathcal{P}(T,\Delta)} \exp \int_\gamma f. \tag{1}$$

(This follows from the fact that $P(f)$ is the abscissa of convergence of the function

$$\sum_{\gamma \in \mathcal{P}} \ell(\gamma) \exp \left(-s\ell(\gamma) + \int_\gamma f \right) = \int_{\ell_0}^\infty e^{-st} d \left(\sum_{\gamma \in \mathcal{P}(T)} \ell(\gamma) \exp \int_\gamma f \right),$$

where $\ell_0 = \min_{\gamma \in \mathcal{P}} \ell(\gamma)$, which is the principal part of the logarithmic derivative of the zeta function $\zeta(s, 1)$ defined above.)

Pressure is also characterised by the variational principle:

$$P(f) = \sup_{\mu \in \mathcal{M}_\phi} \left(h_\mu(\phi) + \int f \, d\mu \right),$$

where $\mathcal{M}_\phi$ is the space of ϕ_t-invariant probability measures on Λ and $h_\mu(\phi)$ is the measure-theoretic entropy. If f is Hölder continuous then the supremum is attained at a unique

$\mu_f \in \mathcal{M}_\phi$, which we call the Gibbs measure of f. This measure is ergodic and fully supported on Λ. We also have the relation

$$h_\mu(\phi) = \inf_{f \in C(\Lambda,\mathbb{R})} \left(P(f) - \int f \, d\mu \right)$$

for each $\mu \in \mathcal{M}_\phi$. (This is a particular example of the duality of the Fenchel–Legendre transform in convex analysis.)

If $f = 0$ then $P(0)$ is equal to the topological entropy $h_{\text{top}}(\phi)$ and the Gibbs measure μ_0 is the measure of maximal entropy (so the notation we have chosen is consistent). If $f = -\varphi^u$ then $\mu_{-\varphi^u} = m$, the SRB measure.

An important property of pressure is its analyticity. If $C^\alpha(\Lambda, \mathbb{R})$ denotes the Banach space of Hölder continuous functions with exponent α then the map

$$C^\alpha(\Lambda, \mathbb{R}) \to \mathbb{R} : f \mapsto P(f)$$

is real analytic. Furthermore, for a neigbourhood of $C^\alpha(\Lambda, \mathbb{R})$ is $C^\alpha(\Lambda, \mathbb{C})$, it extends to give a complex analytic map $f \to P(f)$ (once we can define $P(f)$ have chosen a branch of the logarithm.) In particular, for $f, g \in C^\alpha(\Lambda, \mathbb{R})$, $t \mapsto P(tf + g)$ is real analytic for $t \in \mathbb{R}$ and, for each $t_0 \in \mathbb{R}$, $s \mapsto P(sf + g)$ is complex analytic for s in a complex neighbourhood of t_0. Furthermore,

$$\left. \frac{dP(tf + g)}{dt} \right|_{t=0} = \int f \, \mu_g.$$

The notion of Gibbs measure allows us to extend the theorems of Bowen and Parry above to more general weightings.

Theorem 3.1 (Parry [9], Pollicott [11]). *Let $g : \Lambda \to \mathbb{R}$ be Hölder continuous. For all $f \in C(\Lambda, \mathbb{R})$, we have*

$$\lim_{T \to \infty} \left(\sum_{\gamma \in \mathcal{P}(T,\Delta)} \exp \int_\gamma g \right)^{-1} \sum_{\gamma \in \mathcal{P}(T,\Delta)} \left(\exp \int_\gamma g \right) \frac{1}{\ell(\gamma)} \int_\gamma f = \int f \, d\mu_g.$$

In terms of orbital measures, this says that the measures

$$\left(\sum_{\gamma \in \mathcal{P}(T,\Delta)} \exp \int_\gamma g \right)^{-1} \sum_{\gamma \in \mathcal{P}(T,\Delta)} \left(\exp \int_\gamma g \right) \delta_\gamma$$

converge weak$*$ to μ_g, as $T \to \infty$. Setting $g = 0$ recovers Theorem 2.1 and $g = -\varphi^u$ recovers Theortem 2.2.

This theorem was originally proved by Parry, using zeta function techniques, for weights g with $P(g) \geq 0$ (this includes the case of the weight $-\varphi^u$). Pollicott gave a proof based on large deviations which avoids this condition.

4 Large deviations

Following Pollicott [11], itself inspired by ideas of Kifer [5, 6], we will prove a so-called Level II large deviations theorem from which we will deduce Theorem 3.1.

Theorem 4.1 (Pollicott [11]). *For any closed set $\mathcal{K} \subset \mathcal{M}_\phi$, we have that*

$$\limsup_{T\to\infty} \frac{1}{T} \log \left(\left(\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma g \right)^{-1} \sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\in\mathcal{K}}} \exp \int_\gamma g \right) \le -\rho(\mathcal{K}),$$

where

$$\rho(\mathcal{K}) = \inf_{\mu\in\mathcal{K}} \left(P(g) - \left(h_\mu(\phi) + \int g\, d\mu \right) \right).$$

Furthermore, $\rho(\mathcal{K}) > 0$ whenever $\mu_g \notin \mathcal{K}$.

Remark 4.2. There is also a corresponding lower bound involving lim inf for open sets $\mathcal{U} \subset \mathcal{M}_\phi$.

Before we give the proof of this theorem, we shall show that it implies Theorem 3.1.

Proof of Theorem 3.1. Let $f : \Lambda \to \mathbb{R}$ be continuous. For $\epsilon > 0$, let $\mathcal{U}$ be the open set

$$\mathcal{U} = \left\{ \mu \in \mathcal{M}_\phi : \left| \int f\, d\mu - \int f\, d\mu_g \right| < \epsilon \right\}.$$

Writing

$$\Sigma(T) = \sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma g,$$

we have that, using Theorem 4.1,

$$\begin{aligned} &\Sigma(T)^{-1} \sum_{\gamma\in\mathcal{P}(T,\Delta)} \left(\exp \int_\gamma g \right) \frac{1}{\ell(\gamma)} \int_\gamma f \\ &= \Sigma(T)^{-1} \sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\in\mathcal{U}}} \left(\exp \int_\gamma g \right) \frac{1}{\ell(\gamma)} \int_\gamma f + \Sigma(T)^{-1} \sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\notin\mathcal{U}}} \left(\exp \int_\gamma g \right) \frac{1}{\ell(\gamma)} \int_\gamma f \\ &= \Sigma(T)^{-1} \sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\in\mathcal{U}}} \left(\exp \int_\gamma g \right) \left(\int f\, d\mu_g + E(\gamma) \right) + O(e^{-\eta T}), \end{aligned}$$

where $|E(\gamma)| < \epsilon$ and $0 < \eta < \rho(\mathcal{M}_\phi \setminus U)$. Hence,

$$\limsup_{T\to\infty} \Sigma(T)^{-1} \sum_{\gamma\in\mathcal{P}(T,\Delta)} \left(\exp \int_\gamma g \right) \frac{1}{\ell(\gamma)} \int_\gamma f \le \int f\, d\mu_g + \epsilon$$

and

$$\liminf_{T\to\infty} \Sigma(T)^{-1} \sum_{\gamma\in\mathcal{P}(T,\Delta)} \left(\exp \int_\gamma g \right) \frac{1}{\ell(\gamma)} \int_\gamma f \le \int f\, d\mu_g - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this gives the result. □

We now turn our attention to the proof of Theorem 4.1.

We define $I : \mathcal{M}_\phi \to \mathbb{R}$ by

$$I(\mu) = \sup_{f \in C(\Lambda,\mathbb{R})} \left(\int f\, d\mu - P(f+g) + P(g) \right).$$

Lemma 4.3.

$$I(\mu) = P(g) - \int g\, d\mu - h_\mu(\phi).$$

so that

$$\rho(\mathcal{K}) = \inf_{\mu \in \mathcal{K}} I(\mu).$$

Proof. By the variational principle, we have

$$P(f+g) = \sup_{\mu \in \mathcal{M}_\phi} \left(h_\mu(\phi) + \int (f+g)\, d\mu \right)$$

By the definition of $I(\mu)$, we have

$$\begin{aligned} I(\mu) &= \sup_{f \in C(\Lambda,\mathbb{R})} \left(\int f\, d\mu - P(f+g) + P(g) \right) \\ &= \sup_{f \in C(\Lambda,\mathbb{R})} \left(\int (f+g)\, d\mu - P(f+g) \right) + P(g) - \int g\, d\mu \\ &= \sup_{f \in C(\Lambda,\mathbb{R})} \left(\int f\, d\mu - P(f) \right) + P(g) - \int g\, d\mu \\ &= -\inf_{f \in C(\Lambda,\mathbb{R})} \left(P(f) - \int f\, d\mu \right) + P(g) - \int g\, d\mu \\ &= -h_\mu(\phi) + P(g) - \int g\, d\mu. \end{aligned}$$

□

Proof of Theorem 4.1. We will prove the upper bound in Theorem 4.1. Let $\mathcal{K} \subset \mathcal{M}_\phi$ be closed and hence compact. For $\epsilon > 0$ and $f \in C(X,\mathbb{R})$, define open sets

$$\mathcal{U}(f,\epsilon) := \left\{ \mu \in \mathcal{M}_\phi :\ \int f\, d\mu - P(f+g) + P(g) > \rho(\mathcal{K}) - \epsilon \right\}.$$

From the definition of $I(\mu)$, it is clear that

$$\mathcal{K} \subset \{\mu \in \mathcal{M}_\phi :\ I(\mu) > \rho(\mathcal{K}) - \epsilon\} = \bigcup_{f \in C(X,\mathbb{R})} \mathcal{U}(f,\epsilon),$$

so $\{\mathcal{U}(f,\epsilon)\}_{f \in C(X,\mathbb{R})}$ is an open cover of $\mathcal{K}$. Since $\mathcal{K}$ is compact, we can find $f_1, \ldots, f_k \in C(X,\mathbb{R})$ such that

$$\mathcal{K} \subset \bigcup_{i=1}^{k} \mathcal{U}(f_i,\epsilon).$$

We then have

$$\begin{aligned}\Sigma(T)^{-1}\sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\in\mathcal{K}}}\exp\int_\gamma g &\le \Sigma(T)^{-1}\sum_{i=1}^{k}\sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\in\mathcal{U}(f_i,\epsilon)}}\exp\int_\gamma g\\ &= \Sigma(T)^{-1}\sum_{i=1}^{k}\sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \int f_i\,d\delta_\gamma>P(f_i+g)-P(g)+\rho(\mathcal{K})-\epsilon}}\exp\int_\gamma g\\ &\le \Sigma(T)^{-1}\sum_{i=1}^{k}\sum_{\gamma\in\mathcal{P}(T,\Delta)}e^{-\ell(\gamma)(P(f_i+g)-P(g)+\rho(\mathcal{K})-\epsilon)}\exp\int_\gamma(f_i+g)\end{aligned}$$

Taking logs, dividing by T and taking the lim sup, we get (using that each γ satisfies $\ell(\gamma)\in[T-\Delta/2,T+\Delta/2]$),

$$\limsup_{T\to\infty}\frac{1}{T}\log\left(\Sigma(T)^{-1}\sum_{\substack{\gamma\in\mathcal{P}(T,\Delta)\\ \delta_\gamma\in\mathcal{K}}}\exp\int_\gamma g\right)\le-\rho(\mathcal{K})+\epsilon.$$

Since $\epsilon >$ is arbitrary, the required bound follows.

Finally we show that if $\mu_g\notin\mathcal{K}$ then $\rho(\mathcal{K})>0$. If $\mu\in\mathcal{K}$ then $\mu\neq\mu_g$ and so, since the supremum in the variational principle is attained uniquely at μ_g, we have

$$I(\mu)=P(g)-\int g\,d\mu-h_\mu(\phi)>0.$$

Furthermore, the map $\mu\mapsto h_\mu(\phi)$ is upper semi-continuous (and $\mu\mapsto\int g\,d\mu$ is continuous), so $\mu\mapsto I(\mu)$ is upper semi-continuous. Therefore, $\rho(\mathcal{K})=\int_{\mu\in\mathcal{K}}I(\mu)>0$. □

By taking $g=0$ and, given $f\in C(\Lambda,\mathbb{R})$, setting

$$\mathcal{K}=\left\{\mu\in\mathcal{M}_\phi:\ \left|\int f d\mu-\int f\,d\mu_0\right|\ge 0\right\},$$

we have the following corollary.

Corollary 4.4. *For any $\epsilon>0$,*

$$\frac{\#\left\{\gamma\in\mathcal{P}(T,\Delta)\ :\ \left|\frac{1}{\ell(\gamma)}\int_\gamma f-\int f\,d\mu_0\right|<\epsilon\right\}}{\#\mathcal{P}(T,\Delta)}\to 1$$

exponentially fast, as $T\to\infty$.

In other words, with density one, $\int_\gamma f/\ell(\gamma)$ is close to $\int f\,d\mu_0$.

5 Zeta functions

Our second approach to equidistribution involves the zeta functions. Let us suppose for the moment that $g : \Lambda \to \mathbb{R}$ and $f : \Lambda \to \mathbb{R}$ are Hölder continuous, that $P(g) > 0$ and f is strictly positive. We consider the two variable zeta function

$$\begin{aligned}\zeta(s,z) &= \prod_{\gamma \in \mathcal{P}} \left(1 - \exp\left(-s\ell(\gamma) + z\int_\gamma f\right)\right)^{-1} \\ &= \exp \sum_{\gamma\in\mathcal{P}} \sum_{m=1}^{\infty} \frac{1}{m} \exp\left(-sm\ell(\gamma) + m\int_\gamma g + zm\int_\gamma f\right)\end{aligned}$$

defined for all $s \in \mathbb{C}$ and $z \in \mathbb{C}$ for which the product converges. In fact, we have convergence for $\mathrm{Re}(s) > P(g)$ and $|z|$ small (depending on s). The following is a standard extension result.

Theorem 5.1 ([10]). *Suppose that $\phi_t : \Lambda \to \Lambda$ is topologically weak-mixing. Then $\zeta(s,z)$ extends to a non-zero analytic function in a neighbourhood of $\mathrm{Re}(s) \geq P(g)$ and $|z|$ small apart from a singularity at $s = P(g)$.For (s,z) close to $(P(g),0)$, we have*

$$\zeta(s,z) = \frac{\exp a(s,z)}{s - P(g+zf)},$$

where $a(s,z)$ is an analytic function and $\lambda(s,z)$ is an analytic function satisfying $\lambda(P(g+zf),z) = 1$.

As a consequence, for s close to $P(g)$,

$$\begin{aligned}\eta(s) = \left.\frac{\partial_z\zeta(s,z)}{\zeta(s,z)}\right|_{z=0} &= \partial_z \log\zeta(s,z)\Big|_{z=0} \\ &= -\left.\frac{\partial_z(s - P(g+zf))}{s-P(g+zf)}\right|_{z=0} + b(s) \\ &= \frac{\int f\, d\mu_g}{s - P(g)} + b(s),\end{aligned}$$

where $b(s)$ is analytic. On the other hand, taking the logarithmic derivative with respect to z in the definition of the zeta function, we have

$$\eta(s) = \sum_{\gamma\in\mathcal{P}}\sum_{m=1}^{\infty}\left(\int_\gamma f\right)\exp\left(-sm\ell(\gamma) + m\int_\gamma g\right).$$

The sum over $m \geq 2$ will converge for $\mathrm{Re}(s) > P(g)/2$, so it suffices to consider

$$\eta_1(s) = \sum_{\gamma\in\mathcal{P}}\left(\int_\gamma f\right)\exp\left(-s\ell(\gamma) + \int_\gamma g\right).$$

Combining these calculations, we have that:

Lemma 5.2. *$\eta_1(s)$ is analytic in a neighbourhood of $\mathrm{Re}(s) \geq P(g)$, apart from a simple pole at $s = P(g)$ with residue $\int f \, d\mu_g$.*

We can now apply the following.

Theorem 5.3 (Wiener–Ikehara Tauberian Theorem [3])**.** *Suppose that $A(T)$ is a non-negative monotone nodecreasing function defined fir $T \geq 0$ and such that the Stieltjes integral*

$$\omega(s) = \int_1^\infty e^{-st} dA(t)$$

is analytic for $\mathrm{Re}(s) > 1$ and, for some $c > 0$,

$$\omega(s) - \frac{c}{s-1}$$

has an extension to a continuous function for $\mathrm{Re}(s) \geq 1$. Then

$$A(T) \sim ce^T, \quad \text{as } T \to \infty.$$

We apply this to the function

$$\psi_f(T) := \sum_{\gamma \in \mathcal{P}(T)} \left(\int_\gamma f \right) \exp \int_\gamma g.$$

With the appropriate rescaling, we obtain

$$\psi_f(T) \sim \int f \, d\mu_g \frac{e^{P(g)T}}{P(g)}, \quad \text{as } T \to \infty.$$

Hence

$$\begin{aligned} \sum_{\gamma \in \mathcal{P}(T,\Delta)} \left(\int_\gamma f \right) \exp \int_\gamma g &= \psi_f(T + \Delta/2) - \psi_f(T - \Delta/2) \\ &\sim (e^{P(g)\Delta/2} - e^{-P(g)\Delta/2}) \int f \, d\mu_g \frac{e^{P(g)T}}{P(g)}, \end{aligned}$$

as $T \to \infty$. Comparing with the case $f = 1$, we get the following.

Proposition 5.4.

$$\lim_{T \to \infty} \frac{\sum_{\gamma \in \mathcal{P}(T,\Delta)} \left(\int_\gamma f \right) \exp \int_\gamma g}{\sum_{\gamma \in \mathcal{P}(T,\Delta)} \ell(\gamma) \exp \int_\gamma g} = \int f \, \mu_g.$$

Write

$$\pi_f(T) := \sum_{\gamma \in \mathcal{P}(T,)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma g.$$

We have

$$\psi_f(T) \leq (T + \Delta/2) \pi_f(T).$$

Therefore

$$\liminf_{T\to\infty} \pi_f(T)\frac{T}{e^{P(g)T}} \geq \lim_{T\to\infty} \psi_f(T)e^{-P(g)T} = \frac{1}{P(g)}\int f\, d\mu_g.$$

Now we want an asymptotic inequality in the other direction. For $\sigma > P(g)$,

$$\eta(\sigma) \geq \sum_{\gamma\in\mathcal{P}(T)} \left(\int_\gamma f\right) e^{-\sigma\ell(\gamma)+\int_\gamma g} \geq \sum_{\gamma\in\mathcal{P}(T)} \frac{\ell_0 \int_\gamma f}{\ell(\gamma)} e^{-\sigma T+\int_\gamma g} = \pi_f(T)\ell_0 e^{-\sigma T},$$

so that $\lim_{T\to\infty} e^{-\sigma T}\pi_f(T)$ fior all $\sigma > P(g)$.

Now let $\tau > 1$ and set $y = x/\tau$, then

$$\begin{aligned}\pi_f(x) - \pi_f(y) &= \sum_{y<\ell(\gamma)\leq x} \frac{\int_\gamma f}{\ell(\gamma)} e^{\int_\gamma g} \\ &\leq \sum_{\ell(\gamma)\leq x} \frac{\int_\gamma f}{y} e^{\int_\gamma g} \leq \psi_f(x)/y.\end{aligned}$$

Hence

$$\pi_f(x)\frac{x}{e^{P(g)x}} \leq \frac{\pi_f(y)\tau y}{e^{P(g)\tau y}} + \frac{\psi_f(x)}{y}\frac{\tau y}{e^{P(g)x}}$$

and so

$$\limsup_{x\to\infty} \pi_f(x)\frac{x}{e^{P(g)x}} \leq \tau \limsup_{x\to\infty} \frac{\psi_f(x)}{e^{P(g)x}} = \frac{\tau}{P(g)}\int f\, d\mu_g.$$

Since $\tau > 1$ is arbitrary, we conclude that

$$\limsup_{x\to\infty} \pi_f(x)\frac{x}{e^{P(g)x}} \leq \frac{1}{P(g)}\int f\, d\mu_g.$$

Combing the two inequalities,

$$\pi_f(T) \sim \int f\, d\mu_g \frac{e^{P(g)T}}{P(g)T}.$$

Therefore we obtain

$$\sum_{\gamma\in\mathcal{P}(T,\Delta)} e^{\int_\gamma g}\frac{\int_\gamma f}{\ell(\gamma)} \sim (e^{P(g)\Delta/2} - e^{-P(g)\Delta/2})\int f\, d\mu_g \frac{e^{P(g)T}}{P(g)T},$$

as $T\to\infty$. Comparing with the case $f = 1$, we get the following.

Proposition 5.5.

$$\lim_{T\to\infty} \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp\int_\gamma g}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp\int_\gamma g} = \int f\, \mu_g.$$

Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Then $f^+ + 1$ and $f^- + 1$ are strictly positive and

$$f = (f^+ + 1) - (f^- + 1).$$

This decomposition allows us to transfer the convergence result from positive Hölder continuous functions to general Hölder continuous functions. Since Hölder continuous functions are uniformly dense in the continuous functions, a simple approximation argument extends the result to continuous functions.

If the flow is not weak-mixing then $\zeta(s, z)$ will have poles at $(s, z) = (P(g) + ina, 0)$, fior some $a > 0$ and $n \in \mathbb{Z}$. This leads to slightly different asymptotic behaviour but the equidistribution theorem still holds.

The case $P(g) = 0$. To complete our discussion, suppose now that $P(g) = 0$. Let $\epsilon > 0$. Then $P(g + \epsilon) = P(g) + \epsilon > 0$ and, furthermore, we have $\mu_{g+\epsilon} = \mu_g$. Therefore, for f continuous, we have

$$\lim_{T\to\infty} \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma (g+\epsilon)}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma (g+\epsilon)} = \int f \, \mu_g.$$

Now

$$\begin{aligned} e^{-\epsilon\Delta} \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma (g+\epsilon)}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma (g+\epsilon)} &\le \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma g}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma g} \\ &\le e^{\epsilon\Delta} \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma (g+\epsilon)}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma (g+\epsilon)}, \end{aligned}$$

so that

$$e^{-\epsilon\Delta} \int f \, d\mu_g \le \liminf_{T\to\infty} \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma g}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma g} \le \limsup_{T\to\infty} \frac{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \frac{\int_\gamma f}{\ell(\gamma)} \exp \int_\gamma g}{\sum_{\gamma\in\mathcal{P}(T,\Delta)} \exp \int_\gamma g} \le e^{\epsilon\Delta} \int f \, \mu_g.$$

Since $\epsilon > 0$ is arbitrary, we get Theorem 3.1 for g with $P(g) = 0$.

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