ZETA FUNCTIONS IN HIGHER TEICHMÜLLER THEORY

MARK POLLICOTT AND RICHARD SHARP

Abstract. In this note we introduce zeta functions and L-functions for faithful representations of surface groups in $\mathrm{PSL}(d, \mathbb{R})$, for $d \geq 3$, which are natural generalizations of the well known classical Selberg zeta function and L-function for Fuchsian groups, corresponding to the case $d = 2$. We show that these complex functions have meromorphic extensions to the entire complex plane $\mathbb{C}$.

1. Introduction

1.1. Selberg and Ruelle zeta functions for $\mathrm{PSL}(2, \mathbb{R})$.

In 1956, Selberg introduced a zeta function associated to the fundamental group Γ of a compact oriented surface V of genus $g \geq 2$ or, more precisely, to representations of such groups as Fuchsian groups, i.e. discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$.

The surface V admits a family of hyperbolic metrics (metrics of constant curvature -1) which are parametrised by the Teichmüller space $\mathcal{T}(V)$. Let us consider a particularly convenient viewpoint using representations. We recall that $\mathcal{T}(V)$ can be identified with the connected component of

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$$

which consists of discrete and faithful representations $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$. (Here $\mathrm{PSL}(2, \mathbb{R})$ acts on $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ by conjugation.) If $\rho \in \mathcal{T}(V)$ then one recovers the hyperbolic metric by realising V as $\mathbb{H}^2/\rho(\Gamma)$, where $\mathbb{H}^2$ is the Poincaré upper half plane and where $\mathrm{PSL}(2, \mathbb{R})$ acts by Möbius transformations as orientation preserving isometries of $\mathbb{H}^2$. (We also note that, since Γ is torsion free, $\rho : \Gamma \to \mathrm{PSL}(2, \mathrm{R})$ can be lifted to a representation $\widetilde{\rho} : \Gamma \to \mathrm{SL}(2, \mathbb{R})$ [6]. To simplify notation, we will use $\rho(g)$ to denote the matrix in $\mathrm{SL}(2, \mathbb{R})$ in the hope that the context will make clear the usage.) Another connected component $\mathcal{T}'(V)$ corresponds to reversing the orientation.

The Selberg zeta function is a function of a single complex variable $s \in \mathbb{C}$, formally defined by

$$S(s, \rho) = \prod_{n=0}^{\infty} \prod_{[g] \in \mathcal{P}} \left(1 - e^{-(s+n)\ell_\rho(g)}\right), \tag{1.1}$$

where the second product is over the set $\mathcal{P}$ of all primitive conjugacy classes $[g]$ of elements $g \in \Gamma \setminus \{1\}$. (A conjugacy class is called primitive if it does not contain an element of the form g_0^n, for $g_0 \in \Gamma$ and $n > 1$.) The real number $\ell_\rho(g)$ is twice the logarithm of the largest eigenvalue for $\rho(g)$ (or, equivalently, the length of the unique closed geodesic on V in the free homotopy class corresponding to $[g]$). [1]

[1] The Selberg zeta function is more usually denoted by Z but we wish to reserve this for a related but slightly different function defined below.

The product converges to a non-zero analytic function for $\mathrm{Re}(s) > 1$ and Selberg showed the following fundamental result.

Theorem 1.1 (Selberg)**.** *The Selberg zeta function $S(s,\rho)$ extends to an entire function of order 2 and has a simple zero at $s = 1$.*

An account of this theorem is contained in Hejhal's book [11], where it appears as Theorem 4.11 and Theorem 4.25. The result was obtained independently by Randol [18].

One may also, following Ruelle (cf. [19]), consider the related zeta function

$$R(s,\rho) = \prod_{[g]\in\mathcal{P}} \left(1 - e^{-s\ell_\rho(g)}\right)^{-1}. \tag{1.2}$$

Since $R(s,\rho) = S(s+1,\rho)/S(s,\rho)$, one immediately sees from Theorem 1.1 the following result.

Theorem 1.2. *The Ruelle zeta function $R(s,\rho)$ has a meromorphic extension to $\mathbb{C}$ with a simple pole at $s = 1$. Moreover, $R(s,\rho)$ can be written as a ratio of two entire functions of order 2.*

In [19], Ruelle gave an alternative proof of Theorem 1.2 which will be the basis of our approach when we define and study analogues of these zeta functions in higher rank Teichmüller spaces.

We now want to consider a natural generalization of these definitions and results.

1.2. **Higher rank Teichmüller theory.** In recent years there has been considerable interest in generalising results in classical Teichmüller theory to what is now often referred to as higher Teichmüller theory, involving representations of surface groups (and more general groups) in higher rank Lie groups. This point of view, which had its origins in the work of Goldman [10] and [12], has received considerable attention, see, for example, the surveys [4], [13] and [23]. In this note we will address the natural problem of studying analogues of the Selberg and Ruelle zeta functions in the context of higher Teichmüller theory.

We begin with a few preliminaries. We wish to consider representations of Γ in the higher rank group $\mathrm{PSL}(d,\mathbb{R})$ (for $d \geq 3$). The most elementary class of such representations are the so-called Fuchsian representations, obtained directly from representations into $\mathrm{PSL}(2,\mathbb{R})$.

Example 1.3 (Fuchsian representations)**.** It is well known that there is an irreducible representation $\widetilde{\tau} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$, unique up to conjugation. This has an explicit construction, which we briefly recall. Let $\mathcal{S}_d$ denote the d-dimensional vector space of homogeneous polynomials in 2 variables of degree $d-1$. We can choose a basis for $\mathcal{S}_d$ of the form

$$x^{d-1}, x^{d-2}y, \cdots, x^{d-i-1}y^i, \cdots, xy^{d-2}, y^{d-1}.$$

For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

we can define $\widetilde{\tau}(g) \in \mathrm{SL}(d,\mathbb{R})$ by specifying its action on the basis elements as follows:

$$\widetilde{\tau}(g) : x^{d-i-1}y^i \mapsto (ax+cy)^{d-i-1}(bx+dy)^i.$$

(It is an easy calculation to check that $\det \widetilde{\tau}(g) = 1$.) Furthermore, $\widetilde{\tau}$ factors to an irreducible representation $\tau : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$. Representations $\rho : \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ of the special form $\rho = \tau \circ \rho_0$ where $\rho_0 \in \mathcal{T}(V)$ (or $\mathcal{T}'(V)$), are called *Fuchsian representations.*

We can now define the representations we wish to study.

Definition 1.4. The natural generalization of the Teichmüller space are connected components of

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(d,\mathbb{R}))/\mathrm{PSL}(d,\mathbb{R})$$

which contain Fuchsian representations. These are called *Hitchin components.* When d is odd there is a single Hitchin component but when d is even there are two Hitchin components (corresponding to $\mathcal{T}(V)$ and $\mathcal{T}'(V)$ having distinct images).

The study of representations in the Hitchin component has proved particularly fruitful. Hitchin originally described the properties of the Hitchin component [12] (see also [10]), but more recently the dynamics of the individual representations in the component have attracted attention (see, for example, the survey [13]).

We note that any $\rho : \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ in a Hitchin component may be lifted to a representation $\widetilde{\rho} : \Gamma \to \mathrm{SL}(d,\mathbb{R})$. This follows since a Fuchsian representation $\tau \circ \rho_0$ into $\mathrm{PSL}(d,\mathbb{R})$ lifts to a representation $\widetilde{\tau} \circ \widetilde{\rho}_0$ into $\mathrm{SL}(d,\mathbb{R})$ and, by Theorem 4.1 of [6], everything in the same connected component has the same lifting property. In what follows, we shall consider representations into $\mathrm{SL}(d,\mathbb{R})$, which shall denote by ρ (but remembering that this is the lift of a representation into $\mathrm{PSL}(d,\mathbb{R})$).

We now come to a major result due to Labourie, which is crucial in what follows. We first need to recall the following definition.

Definition 1.5. A matrix is called *proximal* if it has an algebraically simple eigenvalue which is real and has strictly larger modulus than any other eigenvalue.

We then have the following.

Proposition 1.6 (Labourie [14])**.** *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation in a Hitchin component. Then for each $g \in \Gamma \setminus \{1\}$ the matrix $\rho(g)$ is proximal.*

1.3. **Zeta functions for higher rank Teichmüller spaces.** We now come to the main object of study in this note. We can associate a natural weight to each $g \in \Gamma \setminus \{1\}$. Let

$$d_\rho(g) = \log |\lambda(g)|,$$

where $\lambda(g)$ is the maximal eigenvalue of the matrix $\rho(g)$, whose existence follows from Proposition 1.6 above. Since $\rho(g) \in \mathrm{SL}(d,\mathbb{R})$, $|\lambda(g)| > 1$ and hence $d_\rho(g) > 0$. Note that $d_\rho(g)$ only depends on the conjugacy class $[g]$. The purpose of this note is to generalise the definition of the Selberg and Ruelle zeta functions to these representations.

Definition 1.7. We can associate to each representation $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ in a Hitchin component a *generalized Ruelle zeta function* defined by

$$\zeta(s,\rho) = \prod_{[g] \in \mathcal{P}} \left(1 - e^{-s d_\rho(g)}\right)^{-1}, \tag{1.2}$$

and a *generalized Selberg zeta function* defined by

$$Z(s,\rho) = \prod_{n=0}^{\infty} \prod_{[g]\in\mathcal{P}} \left(1 - e^{-(s+n)d_\rho(g)}\right), \tag{1.3}$$

for $s \in \mathbb{C}$ wherever the products converge.

In particular, from (1.2) and (1.3) we have the identities

$$\zeta(s,\rho) = \frac{Z(s+1,\rho)}{Z(s,\rho)}. \tag{1.4}$$

Remark 1.8. Note that, in the special case $d = 2$, with $\rho \in \mathcal{T}(V)$ we have that $d_\rho(g) = \ell_\rho(g)/2$. Therefore, $\zeta(s,\rho) = R(s/2,\rho)$. However, the relationship between $Z(s,\rho)$ and $S(s,\rho)$ is more complicated with $Z(s,\rho) = S(s/2,\rho)S(s/2+1/2,\rho)$.

To describe the half-plane of convergence of these new zeta functions, we recall the following quantity. See [4], [5] or [17] for a further discussion, including the existence of the limit.

Definition 1.9. The *entropy* $h(\rho)$ of a representation $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ in a Hitchin component is defined to be the growth rate of the number of primitive conjugacy classes $[g] \in \mathcal{P}$ with $d_\rho(g)$ at most T as $T \to +\infty$, i.e.,

$$h(\rho) = \lim_{T\to+\infty} \frac{1}{T} \log\left(\#\{[g] \in \mathcal{P} : d_\rho(g) \le T\}\right). \tag{1.5}$$

(The value of $h(\rho)$ would be the same if we used all non-trivial conjugacy classes in the definition, rather than just the primitive ones.)

It is easy to see from the definitions that $Z(s,\rho)$ and $\zeta(s,\rho)$ converge to non-zero analytic functions provided $\mathrm{Re}(s) > h(\rho)$. In the original setting, with $d = 2$ and $\rho \in \mathcal{T}(V)$, the entropy is always equal to 2 and so does not need to be explicitly introduced. (We remark that, in this case, $h(\rho) = 2$ is twice the topological entropy of the geodesic flow over $\mathbb{H}^2/\rho(\Gamma)$ because $d_\rho(g) = \ell_\rho(g)/2$, where $\ell_\rho(g)$ is the length of the corresponding closed geodesic.)

1.4. **Meromorphic extensions of the zeta functions.** Our main results on the zeta functions for representations in a Hitchin component can be viewed as a generalization of Theorem 1.2 and Theorem 1.1, respectively. We begin with the result for $\zeta(s,\rho)$, since, with our approach, the proof of this naturally comes first.

Theorem 1.10. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation in the Hitchin component. Then $\zeta(s,\rho)$ extends to a meromorphic function in the entire complex plane and has a simple pole at $s = h(\rho)$. Furthermore, $\zeta(s,\rho)$ may be written as the quotient of two entire functions of order at most d.*

This will allow us to deduce the following result for $Z(s,\rho)$.

Theorem 1.11. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation in the Hitchin component. Then $Z(s,\rho)$ is a meromorphic function to the entire complex plane and has a simple zero at $s = h(\rho)$. Furthermore, $Z(s,\rho)$ may be written as the quotient of two entire functions of order at most $d+1$.*

We will prove these results in section 3. We will also consider the zeros of the zeta functions in section 4. This leads to estimates on the error terms in the counting functions for the weights.

Remark 1.12. An alternative way of weighting the matrices, and hence of producing zeta functions, would be to take the difference between the logarithms of largest and smallest eigenvalues. It is possible to prove similar results for the corresponding zeta functions by considering for $g \in \Gamma$ the diagonal action $\Gamma \ni g \mapsto \widehat{\rho}(g) \times \widehat{\rho}(g^{-1})$ on $\mathbb{R}P^{d-1} \times \mathbb{R}P^{d-1}$. We then observe that $\mathrm{Jac}(\widehat{\rho}(g) \times \widehat{\rho}(g^{-1})) = \mathrm{Jac}(\widehat{\rho}(g))\mathrm{Jac}(\widehat{\rho}(g^{-1}))$.

We end the introduction, we briefly outline the contents of the rest of the paper. In the next section, we discuss surface groups and their symbolic coding. We associate to this a dynamical system, namely a subshift of finite type and explain how the induced action on projective space allows us to introduce a a function defined on the shift that encodes the weightings $d_\rho(g)$. In section 3, we use an approach due to Ruelle [19], to obtain results on extending our zeta functions as meromorphic functions in the entire complex plane by writing them in terms of determinants of transfer operators. In section 4, we discuss the location of zeros and poles for the zeta functions and obtain an error estimate on the associated counting function. This is based on the ideas originally introduced by Dolgopyat citedolgopyat for estimating iterates of transfer operators. A final section considers the more general case of L-functions associated to unitary representations. Some of the technical results we use in the paper (Lemma 2.3, Lemma 2.7 and Lemma 4.3) originally appeared in the preprint [17] – we include their proofs for completeness.

2. Surface groups and symbolic dynamics

In this section we recall the ideas that are used to build a bridge between the geometry and the dynamics. We write $\partial\Gamma$ for the Gromov boundary of Γ, which is defined as follows. Choosing a finite generating set for Γ, consider the set of geodesic rays $(g_n)_{n=0}^\infty$, where each $g_n \in \Gamma$ and, for each $n \geq 0$, $g_0 g_1 \cdots g_{n-1}$ has word length n. Say that two such rays $(g_n)_{n=0}^\infty$ and $(h_n)_{n=0}^\infty$ are equivalent if there exists a constant $K > 0$ such that the distance between g_n and h_n in the word metric is bounded above by K for all $n \geq 0$. Then $\partial\Gamma$ is the set of equivalence classes of geodesic rays. It can be equipped with a natural "visual" metric, where two points are close if one can choose representative rays which agree for a large initial segment (see Chapitre 7 of [9]). In this case, $\partial\Gamma$ is homeomorphic to the unit circle (by a quasisymmetric homeomorphism) [9]. There is a natural action of Γ on $\partial\Gamma$ and each $g \in \Gamma \setminus \{1\}$ has a unique attracting fixed point $g^+ \in \partial\Gamma$.

2.1. Action on projective space.

The representation $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ induces an action of Γ on the projective space $\mathbb{R}P^{d-1}$. More explicitly, for $g \in \Gamma$, we define $\widehat{\rho}(g) : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ by

$$\widehat{\rho}(g)[v] = \frac{\rho(g)v}{\|\rho(g)v\|_2},$$

where $v \in \mathbb{R}^d \setminus \{0\}$ is a representative element.

Suppose $g \in \Gamma \setminus \{1\}$. Since, by Proposition 1.6, $\rho(g)$ has a real eigenvalue which is strictly maximal in modulus, the map $\widehat{\rho}(g)$ has a unique attracting fixed point; we will denote this by $\xi_g \in \mathbb{R}P^{d-1}$.

Proposition 2.1 (Labourie [14], Theorem 1.4)**.** *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation in the Hitchin component then there is an associated ρ-equivariant Hölder continuous map $\xi_0 : \partial\Gamma \to \mathbb{R}P^{d-1}$.*

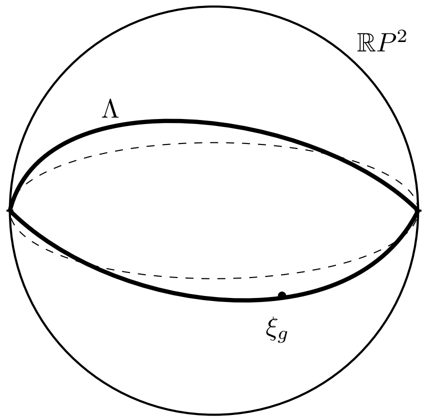

FIGURE 1. $\Lambda \subset \mathbb{R}P^2$ is a C^1 embedded curve. For $g \in \Gamma \setminus \{1\}$, the point ξ_g is an attracting fixed point for $\widehat{\rho}(g) : \mathbb{R}P^2 \to: \mathbb{R}P^2$.

As an immediate consequence of this result, for each $g \in \Gamma \setminus \{1\}$, we have $\xi_0(g^+) = \xi_g$, the fixed point for $\widehat{\rho}(g)$.

We now define a compact subset of projective space which will be useful in the sequel.

Definition 2.2. We denote by $\Lambda := \xi_0(\partial\Gamma) \subset \mathbb{R}P^{d-1}$ the image set. By a result of Labourie [14], Λ is a C^1 curve. (In the case $d = 3$, this follows from earlier work of Benoist [3].) We also have $\{\xi_g : g \in \Gamma \setminus \{e\}\} \subset \Lambda$.

We can use the following simple lemma to relate the weight $d_\rho(g)$ to the action of $\widehat{\rho}(g)$ on $\mathbb{R}P^{d-1}$.

Lemma 2.3. *If $g \in \Gamma \setminus \{1\}$ then*

$$d_\rho(g) = -\frac{1}{d} \log \det(D_{\xi_g} \widehat{\rho}(g)).$$

Proof. We can choose a basis for $\mathbb{R}^d$ which puts $\rho(g)$ into Jordan block form and, without loss of generality, assume that the first coordinate corresponds to an eigenvector v for the maximal positive eigenvalue $\lambda > 1$. In these coordinates we can write

$$\rho(g) = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$$

where B is a $(d-1) \times (d-1)$-matrix. In particular, $1 = \det \rho(g) = \lambda \det B$. We can consider a chart for $\mathbb{R}P^{d-1}$ corresponding to fixing $x_1 = 1$ then in these coordinates the derivative of the projective action at the fixed point $[v]$ can be written as $D_{[v]}\widehat{\rho}(g) = \frac{1}{\lambda} B$. Therefore the jacobian is $\det\left(\frac{1}{\lambda} B\right) = \lambda^{-(d-1)} \det(B) = \lambda^{-d}$. □

2.2. **Surface groups and coding.** We now want to introduce a natural coding for the group Γ which will allow us to analyse the projective action of Γ on Λ. This is motivated by the more familiar boundary coding associated to Fuchsian groups due to Bowen and Series [22]. We take the same basic approach to introducing a dynamical perspective to the study of representations in a Hitchin component based on subshifts of finite type that was used in [17].

Since Γ is the fundamental group of a compact surface V with genus $g \geq 2$, it has the presentation

$$\Gamma = \left\langle a_1, \dots, a_g, b_1, \dots, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

We write $\Gamma_0 = \{a_1^{\pm 1}, \cdots, a_g^{\pm 1}, b_1^{\pm 1}, \cdots, b_g^{\pm 1}\}$ for the symmetrized generating set and $|\cdot| : \Gamma \to \mathbb{Z}^+$ for the word length with respect to Γ_0.

Such surface groups are particular examples of Gromov hyperbolic groups and as such they are strongly Markov groups in the sense of Ghys and de la Harpe [9], i.e. they can be encoded using a directed graph and an edge labelling by elements in Γ_0. This approach allows allows us to relate the group to a dynamical system, namely a subshift of finite type, and thus to use the machinery of thermodynamic formalism to define a so called pressure function and, later, transfer operators, which may be used to analyse our zeta functions. In the particular case of surface groups, the coding follows directly from the work of Adler and Flatto [1] and Series [22] on coding the action on the boundary and the associated subshift of finite type is mixing.

We have the following result.

Lemma 2.4. *We can associate to* (Γ, Γ_0)

(i) *a directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with a distinguished vertex* $*$*; and*
(ii) *an edge labelling* $\omega : \mathcal{E} \to \Gamma_0$,

such that

(1) *no edge terminates at* $*$*;*
(2) *there is at most one directed edge joining each ordered pair of vertices;*
(3) *the map from the set finite paths in the graph starting at* $*$ *to* $\Gamma \setminus \{e\}$ *defined by*

$$(e_1, \dots, e_n) \mapsto \omega(e_1) \cdots \omega(e_n)$$

is a bijection and $|\omega(e_1) \cdots \omega(e_n)| = n$*;*
(4) *the map from closed paths in* $\mathcal{G}$ *to conjugacy classes to* Γ *induced by* ω *is a bijection and for such a closed path* $(e_1, \dots, e_n, e_1)$, n *is the minimum word length in the conjugacy class of* $\omega(e_1) \cdots \omega(e_n)$.

Furthermore, the subgraph obtained be deleting the vertex $*$ *has the aperiodicity property that there exists* $N \geq 1$ *such that, given any two* $v, v' \in \mathcal{V} \setminus \{*\}$, *there is a directed path of length* N *from* v *to* v'.

2.3. **Symbolic dynamics.** We can associate to the graph $\mathcal{G}$ a subshift of finite type where the states are labelled by the edges in the graph after deleting the edges that originate in the vertex $*$. In particular, if there are k such edges then we can define a $k \times k$ matrix A by $A(e, e') = 1$ if e' follows e in the directed graph and then define a space

$$\Sigma = \{x = (x_n) \in \{1, \cdots, k\}^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1, n \geq 0\}.$$

This is a compact space with respect to the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1 - \delta(x_n, y_n)}{2^n}.$$

The shift map is the local homeomorphism $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_n = x_{n+1}$. By the final statement in Lemma 2.4, A is aperiodic (i.e. there exists $N \geq 1$ such that A^N has all entries positive) and, equivalently, the shift $\sigma : \Sigma \to \Sigma$ is mixing (i.e. for all open non-empty $U, V \subset \Sigma$, there exists $N \geq 1$ such that $\sigma^{-n}(U) \cap V \neq \varnothing$ for all $n \geq N$).

There is a natural surjective Hölder continuous map $\pi : \Sigma \to \partial\Gamma$ defined by letting $\pi((x_n)_{n=0}^{\infty})$ be the equivalence classes of the the geodesic ray $(g_n)_{n=0}^{\infty}$ in $\partial\Gamma$, where $g_n = \omega(x_0)\omega(x_1)\cdots\omega(x_n)$.

Definition 2.5. We define a map $\xi : \Sigma \to \Lambda \subset \mathbb{R}P^{d-1}$ by $\xi = \xi_0 \circ \pi$.

However, the shift $\sigma : \Sigma \to \Sigma$ only encodes information about Γ as an abstract group. In order to keep track of the additional information given by the representation of Γ in $\mathrm{SL}(d, \mathbb{R})$ and its action on $\mathbb{R}P^{d-1}$ we need to introduce a Hölder continuous function $r : \Sigma \to \mathbb{R}$.

Definition 2.6. We can associate a map $r : \Sigma \to \mathbb{R}$ defined by

$$r(x) = -\frac{1}{d} \log \det(D_{\xi(x)} \widehat{\rho}(g_{x_0})),$$

(i.e., the Jacobian of the derivative of the projective action), where $g_{x_0} = \omega(x_0)$ is the generator corresponding to the first term in $x = (x_n)_{n=0}^{\infty} \in \Sigma$.

There is a natural correspondence between non-trivial conjugacy classes in Γ and periodic orbits for the shift σ, by part (4) of Lemma 2.4.

We now have the following simple but key result.

Lemma 2.7. *The function $r : \Sigma \to \mathbb{R}$ is Hölder continuous, and if $\sigma^n x = x$ is a periodic point corresponding to the conjugacy class $[g]$ of an element $g \in \Gamma$ then $r^n(x) = d_\rho(g)$.*

Proof. The Hölder continuity of r follows immediately from the Hölder continuity of ξ_0, which is given by Proposition 2.1. The second part of the lemma follows from the equivariance and the observation $\xi(\sigma x) = \rho(g_{x_0})\xi(x)$. Moreover, that the periodic point x has an image $\xi(x)(= \xi_g)$ which is fixed by $\widehat{\rho}(g)$ and the result follows from Lemma 2.3. □

2.4. **Semi-flows and entropy.** In this subsection we give an alternative definition of the entropy $h(\rho)$ of a representation, as defined in the introduction, which is useful when taking the symbolic viewpoint.

We can associate to $\sigma : \Sigma \to \Sigma$ and the positive function $r : \Sigma \to \mathbb{R}^+$ the space

$$\Sigma^r = \{(x, u) \in \Sigma \times \mathbb{R} : 0 \leq u \leq r(x)\}/\sim,$$

where $(x, r(x)) \sim (\sigma x, 0)$, and the semi-flow $\sigma_t^r : \Sigma^r \to \Sigma^r$ defined by $\sigma_t^r(x, u) = (x, u + t)$, for $t \geq 0$ (respecting the identifications).

The topological entropy $h(\sigma^r)$ for the semi-flow is defined to be the entropy of the time one map. It is a standard result that an alternative way of defining $h(\sigma^r)$ is as the growth rate

$$h(\sigma^r) = \lim_{T \to +\infty} \frac{1}{T} \log \#\{\tau \in \mathcal{P}(\sigma^r) : l(\tau) \leq T\},$$

where $\mathcal{P}(\sigma^r)$ denotes the set of prime periodic orbits of σ^r and $l(\tau)$ denotes the least period of the orbit τ [15].

Lemma 2.8. *If ρ is in a Hitchin component then the entropy of the representation is equal to the topological entropy of the semi-flow σ^f.*

Proof. We observe that there is a one-one correspondence between the periodic orbits for the semi-flow σ^r and for the shift σ. By Lemma 2.7 and Definition 1.9, we see that $h(\rho) = h(\sigma^r)$. □

3. Extensions of Zeta Functions

We now turn to the results on the meromorphic extensions of zeta functions. The proof of Theorem 1.10 is based on the approach of Ruelle in [19]. We then use Therem 1.10 to prove the corresponding result for $Z(s, \rho)$ in Theorem 1.11.

3.1. **Partitions.** By the construction in the previous section we have associated to the representation ρ a Hölder continuous map $\xi : \Sigma \to \mathbb{R}P^{d-1}$ which intertwines the shift map on Σ with the projective action, i.e. $\xi(\sigma x) = \widehat{\rho}(\omega(x_0))\xi(x)$. In particular, we use ξ to define a particular partition $P_i = \xi([i])$ of the limit set $\Lambda \subset \mathbb{R}P^{d-1}$ (which we recall is a C^1 curve), where

$$[i] = \{x \in \Sigma : x_0 = i\}, \quad 1 \leq i \leq k.$$

We then have that

(1) $\Lambda = \bigcup_{i=1}^k P_i$, and
(2) $P_i \cap P_j$ consists of at most one point, for distinct $i \neq j$.

The analytic extension of the zeta function $\zeta(s, \rho)$ is based on the study of a transfer operator defined on analytic functions. In particular, we can realise the symbolic maps $\Sigma \supset [j] \ni x \mapsto ix \in [i] \subset \Sigma$ (where $A(i, j) = 1$) as local real analytic contractions

$$\psi_i : \coprod_{A(i,j)=1} P_j \to P_i$$

on the disjoint union of the sets.

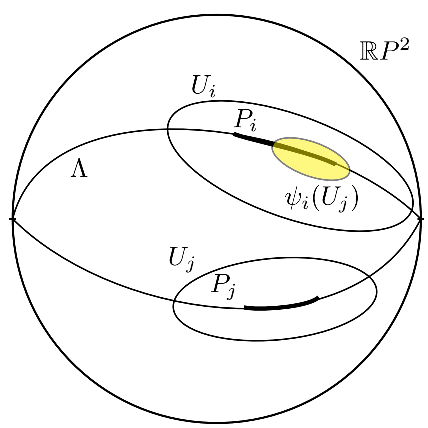

Figure 2. The contractions $\psi_i : P_j \to P_i$ satisfy $\psi_i(U_j) \subset U_i$ for the neighbourhoods of the complexifications (suggested in the figure).

3.2. **Transfer operators.** We can consider a Banach space of analytic functions on a neighbourhood of the disjoint union of the partition $\coprod_{i=1}^k P_i$. In particular, let us fix neighbourhoods $U_i \supset P_i$ in the complexification of $\mathbb{R}P^{d-1}$.

This allows us to define a convenient Banach space of analytic functions.

Definition 3.1. We define B_0 to be the space of bounded analytic functions $f : \coprod_{i=1}^k U_i \to \mathbb{C}$ with the supremum norm

$$\|f\| = \max_i \sup_{z \in U_i} |f(z)|,$$

i.e., $B_0 = \left\{ f : \coprod_{i=1}^k U_i \to \mathbb{C} \text{ analytic} : \|f\| < \infty \right\}$.

We can also define a natural family of bounded linear operators on this space.

Definition 3.2. For $s \in \mathbb{C}$, the *transfer operator* $\mathcal{L}_s : B_0 \to B_0$ is defined by

$$\mathcal{L}_s f(z) = \sum_{A(i,j)=1} \det(D\psi_i(z))^s f(\psi_i z), \text{ for } z \in U_j,$$

where $\det(D\psi_i(\cdot))^s$ is understood as the complexification of the real analytic function.

The advantage of being able to work in the Banach space of bounded analytic functions is that the operators have far better spectral properties than, say, Banach spaces of C^1 functions or Hölder continuous functions. More precisely, the transfer operators are compact operators and, furthermore, satisfy the stronger property of nuclearity. This following follows directly from the work of Ruelle, based on the ideas of Grothendieck.

Lemma 3.3 (Nuclearity: after Grothendieck and Ruelle)**.** *The transfer operators $\mathcal{L}_s : B_0 \to B_0$ are nuclear. More precisely, we can find*

(1) $v_m \in B_0$ *with* $\|v_m\|_{B_0} = 1$,
(2) $l_m \in B_0^*$ *with* $\|l_m\|_{B_0^*} = 1$, *and*
(3) $\lambda_m \in \mathbb{C}$ *with* $|\lambda_m| = O(e^{-cm})$, *for some* $c > 0$,

such that

$$\mathcal{L}_s f = \sum_{m=1}^{\infty} \lambda_m l_m(f) v_m.$$

Proof. This can be deduced from [19]. □

We can also consider the natural generalisation of these operators to the space of analytic j-forms on $\coprod_{i=1}^k U_i$ (for $j = 1, \ldots, d-1$). More precisely, let B_j denote the Banach space of analytic j-forms on $\coprod_{i=1}^k U_i$ whose coefficients are bounded. Then the definition of $\mathcal{L}_s$ naturally extends to define transfer operators $\mathcal{L}_{s,j} : B_j \to B_j$. The analogue of Lemma 3.3 holds for these operators.

3.3. **Traces and determinants.** A particular consequence of Lemma 3.3 is that the operator $\mathcal{L}_s$ (and its powers $\mathcal{L}_s^n$) are trace class, i.e. the eigenvalues are summable: $\sum_{m=1}^{\infty} |\lambda_m| < \infty$.

To relate the transfer operators to the zeta functions we need the following result.

Lemma 3.4. *The trace of* $\mathcal{L}_s^n : B_0 \to B_0$ *is given by*

$$\mathrm{tr}(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \frac{\det(D\psi_{\underline{i}}(z_{\underline{i}}))^s}{\det(I - D\psi_{\underline{i}}(z_{\underline{i}}))},$$

where:

1. *the summation is over allowed reduced strings* $\underline{i} = (i_1, \cdots, i_n)$ *of length* n *such that* $A(i_1, i_2) = \cdots = A(i_{n-1}, i_n) = A(i_n, i_1) = 1$*;*
2. $\psi_{\underline{i}}$ *denotes the composition* $\psi_{\underline{i}} = \psi_{i_n} \circ \cdots \circ \psi_{i_1}$*; and*
3. $z_{\underline{i}} \in P_{i_n}$ *is a fixed point for* $\psi_{\underline{i}}$*.*

Furthermore,

$$\mathrm{tr}(\mathcal{L}_s^n) + \sum_{j=1}^{d} (-1)^j \mathrm{tr}(\mathcal{L}_{s,j}^n) = \sum_{|\underline{i}|=n} \det(D\psi_{\underline{i}}(z_{\underline{i}}))^s. \tag{3.1}$$

Proof. This can be deduced from the general setting in [19]. □

We want to associate to the transfer operators a function of a single complex variable s. We can write

$$\det(I - \mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(\mathcal{L}_s^n)\right).$$

Lemma 3.4 explains the connection with the fixed points of the contractions. Similarly, we can associate to the transfer operators $\mathcal{L}_{s,j}$ the functions

$$\det(I - \mathcal{L}_{s,j}) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(\mathcal{L}_{s,j}^n)\right)$$

for $1 \leq j \leq d-1$.

The next lemma shows that these functions extend analytically to the entire complex plane. To simplify notation, we shall write $\mathcal{L}_{s,0} = \mathcal{L}_s$.

Lemma 3.5. *The functions* $s \mapsto \det(I - \mathcal{L}_{s,j})$ *(* $0 \leq j \leq d-1$*) extend to entire analytic functions of* $s \in \mathbb{C}$ *of order at most* d*.*

Proof. This can be (essentially) deduced from [19], although there is a minor correction required, as described in [8]. In particular, one can expand

$$\det(I - \mathcal{L}_{s,j}) = 1 + \sum_{m=1}^{\infty} c_m(s)$$

where

$$c_m(s) = O\left(e^{|s|Dm} e^{-Em^{\left(1+\frac{1}{d-1}\right)}}\right),$$

for some $E > 0$ and $D = \max_j \|D\phi_j\|$. In particular, by considering critical points of the real function $t \mapsto |s|Dt - Et^{\left(1+\frac{1}{d-1}\right)}$ we see that the upper bound on the terms occurs at

$$m = \left[\left(\frac{|s|D}{E(1+\frac{1}{d-1})}\right)^{d-1}\right] \text{ or } \left[\left(\frac{|s|D}{E(1+\frac{1}{d-1})}\right)^{d-1}\right] + 1.$$

giving an upper bound on the terms of $O\left(\exp\left(|s|^d D^d/(E(1+\frac{1}{d-1}))^{d-1}\right)\right)$. Moreover, it is easy to see that this gives an upper bound

$$|\det(I-\mathcal{L}_{s,j})| = O\left(e^{c|s|^d}\right),$$

for some $c > 0$. In particular, we see that the order is given by

$$\limsup_{R\to+\infty} \frac{\log\left(\max_{|s|\le R}\log|\det(I-\mathcal{L}_{s,j})|\right)}{\log R} \le d.$$

□

Proof of Theorem 1.10. To complete the proof of Theorem 1.10, we can then use the identity (3.1) to write

$$\zeta(s,\rho) = \frac{\prod_{j \text{ odd}} \det(I-\mathcal{L}_{s,j})}{\prod_{j \text{ even}} \det(I-\mathcal{L}_{s,j})},$$

from which we deduce that $\zeta(s,\rho)$ is a meromophic function which, by Lemma 3.5, may be written as the quotient of two entire functions of order $\le d$.

Proof of Theorem 1.11. Given $s_0 \in \mathbb{C}$, we can choose $m \in \mathbb{Z}$ with $\mathrm{Re}(s_0) \ge m$. Setting $n = |m| + [h(\rho)] + 1$ we can use (1.4) to write

$$Z(s,\rho) = \frac{Z(s+n,\rho)}{\zeta(s,\rho)\zeta(s+1,\rho)\cdots\zeta(s+n-1,\rho)}. \tag{3.2}$$

For s in a sufficiently small neighbourhood of s_0 we have that $\mathrm{Re}(s+n) > h(\rho)$ and therefore $Z(s+n,\rho)$ converges to an analytic function. Moreover, the denominator is meromorphic by Theorem 1.10 and thus using (3.2) we deduce that $Z(s,\rho)$ is meromorphic.

To obtain the order bound in Theorem 1.11 consider $|s| < m$, say, and set $n = m + [h(\rho)] + 2$. By the proofs of Theorem 1.10 and Lemma 3.5 we can write $\zeta(s,\rho) = \frac{f(s)}{g(s)}$, where for any $\epsilon > 0$ we can choose $C > 0$ such that $|f(s)|, |g(s)| \le \exp(C|s|^{d+\epsilon})$ and thus using (3.2) we can write

$$Z(s,\rho) = Z(s+n,\rho)\frac{G(s)}{F(s)} \tag{3.3}$$

where $F(s) = \prod_{k=0}^{n-1} f(s+k)$ and $G(s) = \prod_{k=0}^{n-1} g(s+k)$ are entire functions. In particular, $Z(s+n,\rho)$ is uniformly bounded and since

$$\log|F(s)| \le C\sum_{k=0}^{m+1} |s+k|^{d+\epsilon} \le C(m+2)(2m)^{d+\epsilon}$$

the order of $F(s)$ is bounded by

$$\lim_{m\to+\infty} \frac{\log(C(m+2)(2m)^{d+\epsilon})}{\log m} = d+1+\epsilon,$$

and similarly the order for $G(s)$ is bounded by $d+1+\epsilon$. Finally, since $\epsilon > 0$ can be arbitrarily small, we can deduce from (3.3) that $Z(s,\rho)$ is a ratio of two entire functions of order $d+1$.

Remark 3.6 (Equidistribution)**.** For each $t > \delta$ can define a probability measure on Λ by

$$\mu_t = \frac{\sum_{g \in \Gamma \setminus \{1\}} |\mathrm{Jac} D_{\xi_g} \widehat{\rho}(g)|^t \delta_{\xi_g}}{\sum_{g \in \Gamma \setminus \{1\}} |\mathrm{Jac} D_{\xi_g} \widehat{\rho}(g)|^t}$$

and then as t tends to δ this converges to a probability measure μ supported on Λ. This follows easily from the properties of the transfer operator $\mathcal{L}_s$, where μ is related to the maximal eigenprojection of the operator $\mathcal{L}_{h(\rho)}$.

4. Zeros and poles of the zeta functions

Having established that the zeta functions $\zeta(s,\rho)$ and $Z(s,\rho)$ have meromorphic extensions to $\mathbb{C}$ in Theorems 1.10 and 1.11, it is very natural to ask about the location of the zeros and poles of zeta function.

In the case of the classical Selberg zeta function for hyperbolic surfaces the following is well known. Recall from Theorem 1.1 that the Selberg zeta function for a hyperbolic surface is entire with a simple zero at $s = 0$. Furthermore, the following result is well known.

Proposition 4.1 ([11])**.** *There exists $\epsilon > 0$ such that $S(s,\rho)$ has no zeros in $\mathrm{Re}(s) > 1 - \epsilon$ other than $s = 1$.*

In this classical case it is even known that the zeros in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$ lie on $[0,1] \cup \left(\frac{1}{2} + i\mathbb{R}\right)$. This is a consequence of the way the zeta function can be extended using the trace formula and the interpretation of the zeros s_n in terms of the eigenvalues $\lambda_n = s_n(1 - s_n)$ of the Laplace-Beltrami operator.

Using the relation $R(s,\rho) = S(s+1,\rho)/S(s,\rho)$ we can deduce the corresponding result for $R(s,\rho)$.

Proposition 4.2. *There exists $\epsilon > 0$ such that $R(s,\rho)$ has no zeros or poles in $\mathrm{Re}(s) > 1 - \epsilon$ other than a simple pole $s = 1$.*

We can generalize both Proposition 4.1 and Proposition 4.2 to the zeta functions for representations in a Hitchin component. We will need the following "mixing condition" on the weights $d_\rho(g)$.

Lemma 4.3. *Let ρ be in a Hitchin component. Then there does not exist $a > 0$ such that $\{d_\rho(g) : g \in \Gamma \setminus \{1\}\} \subset a\mathbb{Z}$.*

Proof. Let $g, h \in \Gamma \setminus \{1\}$ be two distinct elements of the group. For any $N > 0$ we can consider $g^N, h^N \in \Gamma$. The linear maps on $\mathbb{R}^d$ for the associated matrices $\rho(g^N), \rho(h^N) \in \mathrm{SL}(d,\mathbb{R})$ can be written in the form $\lambda(g)^N \pi_g + U_{g^N}$ and $\lambda(h)^N \pi_h + U_{h^N}$, respectively, where $\lambda(g), \lambda(h)$ are the largest simple eigenvalues, $\pi_g, \pi_h : \mathbb{R}^d \to \mathbb{R}^d$ are the eigenprojections onto their one dimensional eigenspaces, $\limsup_{N \to +\infty} \|U_{g^N}\|^{1/N} < \lambda(g)$ and $\limsup_{N \to +\infty} \|U_{h^N}\|^{1/N} < \lambda(h)$.

Let us now consider $g^N h^N \in \Gamma$ and and associated matrix $\rho(g^N h^N)$. The associated linear map will be of the form $\lambda(g^N h^N) \pi_{g^N h^N} + U_{g^N h^N}$ where $\lambda(g^N h^N)$ is the largest simple eigenvalue, $\pi_{g^N h^N} : \mathbb{R}^d \to \mathbb{R}^d$ is the eigenprojection onto their one dimensional eigenspaces, and $\limsup_{N \to +\infty} \|U_{g^N h^N}\|^{1/N} < \lambda(g^N h^N)$. However, since we have the identity $\rho(g^N h^N) = \rho(g^N)\rho(h^N)$ for the matrix representations we can also write the corresponding relationship for the linear maps:

$$\lambda(g^N h^N) \pi_{g^N h^N} + U_{g^N h^N} = \left(\lambda(g^N) \pi_{g^N} + U_{g^N}\right) \left(\lambda(h^N) \pi_{h^N} + U_{h^N}\right). \tag{4.1}$$

In particular, we see that as N becomes larger

$$\lim_{N\to+\infty} \exp\left((d_\rho([g^N h^N]) - d_\rho([g^N]) - d_\rho([h^N])\right) = \lim_{N\to+\infty} \frac{\lambda(g^N h^N)}{\lambda(g^N)\lambda(h^N)} = \langle \pi_h, \pi_g \rangle$$

where $\langle \pi_h, \pi_g \rangle$ is simply the cosine of the angle between the eigenvectors associated to $\lambda(g)$ and $\lambda(h)$, respectively. However, if we assume for a contradiction that the conclusion of the lemma does not hold, then the right hand side of (4.1) must be of the form e^{na}, for some $n \in \mathbb{Z}$. However, the directions for the associated eigenprojections form an infinite set in $\mathbb{R}P^{d-1}$ and have an accumulation point. Thus for suitable choices of g, h we can arrange that $0 < \langle \pi_h, \pi_g \rangle < e^a$, leading to a contradiction. This completes the proof of the lemma. □

We will now obtain our results on the zeta functions. First we will consider the Ruelle-type zeta function $\zeta(s, \rho)$.

Theorem 4.4. *There exists $\epsilon > 0$ such that $\zeta(s, \rho)$ has no zero or poles in* $\mathrm{Re}(s) > h(\rho) - \epsilon$ *other than $h(\rho)$. Moreover, there exists $\alpha > 0$ such that we can bound* $\log|\zeta(s, \rho)| = O(|\mathrm{Im}(s)|^\alpha)$ *for s satisfying* $h(\rho) - \epsilon < \mathrm{Re}(s) < h(\rho)$ *and* $|\mathrm{Im}(s)| \geq 1$.

Proof. The absence of zeros in a half-plane $\mathrm{Re}(s) > h(\rho) - \epsilon$ is a standard result [15]. The pole free strip is a consequence of the method of Dolgopyat [7] and the fact that the limit set $\Lambda \subset \mathbb{R}P^{d-1}$ is a C^1 curve [14]. This method gives bounds on the iterates of the transfer operators $\mathcal{L}_s$ regarded as operators $\mathcal{L}_s : C^1\left(\coprod_{i=1}^k P_i, \mathbb{C}\right) \to C^1\left(\coprod_{i=1}^k P_i, \mathbb{C}\right)$. It is convenient to use the norm

$$\|h\|_{1,t} = \begin{cases} \max\left\{\|h\|_\infty, \frac{\|h'\|_\infty}{|t|}\right\} & \text{if } |t| \geq 1 \\ \max\left\{\|h\|_\infty, \|h'\|_\infty\right\} & \text{if } |t| < 1 \end{cases}$$

on $C^1\left(\coprod_{i=1}^k P_i, \mathbb{C}\right)$. Following Proposition 7.4 of [2], the key ingredients required for the proof are that:

(i) $r : \coprod_{i=1}^k P_i \to \mathbb{R}$ is a C^1 function; and
(ii) $r : \coprod_{i=1}^k P_i \to \mathbb{R}$ is not cohomologous to a constant, i.e. there is no C^1 function $u : \coprod_{i=1}^k P_i \to \mathbb{R}$ and constant $c \in \mathbb{R}$ such that $r = u \circ T - u + c$.

We see that (i) holds by construction and (ii) follows from Lemma 2.7 and Lemma 4.3.

We can apply the results from [7] and [16] to show that there exist constants $\sigma_0 < h$, $C > 0$ and $0 < \beta < 1$ such that whenever $s = \sigma + it$ and $n = p[\log|t|] + l$, where $p \geq 0$ and $0 \leq l \leq [\log|t|] - 1$, then

$$\|\mathcal{L}_{-sr}^n\|_{1,t} \leq C\beta^{p[\log|t|]} e^{lP(-\sigma r)}$$

with respect to the norm $\|h\|_{1,t}$.

By Lemma 2 of [16] (which, apart from the dependence on $|t|$ in the bound, appears in [20]) we have the estimate that for any $x_j \in P_j$, $j = 1, \ldots, k$, and any β_0 satisfying $\max\{\beta, \max_j \|D\psi_j\|_\infty\} < \beta_0 < 1$, there exists $C > 0$ such that

$$\left| \sum_{|\underline{i}|=n} \det(D\psi_{\underline{i}}(z_{\underline{i}}))^s - \sum_{j=1}^k \mathcal{L}_s \chi_{P_j}(x_j) \right| \leq C|t|n\beta_0^n, \quad \forall n \geq 1,$$

where χ_{P_j} is the indicator function for P_j and $\underline{i}$, $\psi_{\underline{i}}$ and $z_{\underline{i}}$ are as in Lemma 3.4. We can use this estimate to get a bound on the logarithm of the absolute value of the zeta function of the form $\log|\zeta(s,\rho)| = O\left(|t|^{\alpha}\right)$, for some $\alpha > 0$, as in (2.3) of [16]. □

In light of (1.4), and the observation that $Z(s+1,\rho)$ is uniformly bounded on the half-plane $\mathrm{Re}(s) > h(\rho) - \epsilon$, provided $\epsilon < 1$, this gives the following generalization of Proposition 4.1.

Theorem 4.5. *There exists $\epsilon > 0$ such that $Z(s,\rho)$ has no zero or poles in $\mathrm{Re}(s) > h(\rho) - \epsilon$ other than the zero at $s = h(\rho)$. Moreover, there exists $\alpha > 0$ such that we can bound $\log|Z(s,\rho)| = O(|\mathrm{Im}(s)|^{\alpha})$ for s satisfying $h(\rho) - \epsilon < \mathrm{Re}(s) < h(\rho)$ and $|\mathrm{Im}(s)| \geq 1$.*

Although Theorem 4.4 is weaker than the full result known for the Selberg zeta function in the case of hyperbolic surfaces it nonetheless has analogous consequences for error terms in counting functions. To conclude the section, we address how the positions of the zeros influence asymptotic formulae.

Definition 4.6. For $T > 0$, we define

$$\pi_\rho(T) = \#\{[g] \in \mathcal{P} : d_\rho([g]) \leq T\},$$

the number of primitive conjugacy classes with weight at most T.

It was shown in [21] (following the approach in [15]) that $\pi_\rho(T)$ satisfies the asymptotic formula

$$\pi_\rho(T) \sim \frac{e^{h(\rho)T}}{h(\rho)T}, \quad \text{as } T \to +\infty.$$

Using Theorem 4.4, we can deduce a stronger result with an error term provided we replace the leading term with a logarithmic integral. As usual, we will write

$$\mathrm{li}(x) := \int_2^x \frac{1}{\log u}\,du \sim \frac{x}{\log x}, \quad \text{as } x \to +\infty.$$

We then have the following result.

Theorem 4.7. *There exists $\epsilon > 0$ such that*

$$\pi_\rho(T) = \mathrm{li}(e^{h(\rho)T})\left(1 + O(e^{-\epsilon T})\right).$$

Proof. Given Theorem 4.4, we can apply the classical proof from number theory (see [16]). □

5. The L-function in higher rank

We can generalize the definition of the zeta functions to L-functions by incorporating additional information associated to a unitary representation $U : \Gamma \to \mathrm{U}(N)$. More precisely, we make the following definition.

Definition 5.1. We can associate to each representation $\rho : \Gamma \to \mathrm{PSL}(d,R)$ in a Hitchen component and to each unitary representation $R_\chi : \Gamma \to \mathrm{U}(N)$ (with character $\chi = \mathrm{Trace}(R_\chi)$), a Selberg type L-function formally defined by

$$Z(s,\rho,\chi) = \prod_{n=0}^{\infty} \prod_{[g]\in\mathcal{P}} \det\left(1 - e^{-(s+n)d_\rho(g)} R_\chi([g])\right), \quad s \in \mathbb{C},$$

and a Ruelle type L-function formally defined by

$$\zeta(s,\rho,\chi) = \prod_{[g]\in\mathcal{P}} \det\left(1 - e^{-sd_\rho(g)} R_\chi([g])\right)^{-1}, \quad s \in \mathbb{C},$$

where the products converge.

If R_{χ_0} is the trivial representation then we see that these L-functions reduce to the corresponding zeta functions, i.e. $Z(s,\rho,\chi_0) = Z(s,\rho)$ and $\zeta(s,\rho,\chi_0) = \zeta(s,\rho)$. In the particular case of Fuchsian representations then these definitions reduce to the familiar definitions of L-functions for Fuchsian groups [8].

It is easy to see that $Z(s,\rho,\chi)$ and $\zeta(s,\rho,\chi)$ converge for $\mathrm{Re}(s) > h(\rho)$. We can also observe directly from the definitions that

$$\zeta(s,\rho,\chi) = \frac{Z(s+1,\rho,\chi)}{Z(s,\rho,\chi)}. \tag{5.1}$$

The following result generalizes Theorem 1.10 and has a similar proof.

Theorem 5.2. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation in the Hitchin component and let $R_\chi : \Gamma \to \mathrm{U}(N)$ be a unitary representation. The Ruelle-type L-function $\zeta(s,\rho,\chi)$ has the following properties.*

(1) *$\zeta(s,\rho,\chi)$ has a meromorphic extension to the entire complex plane $\mathbb{C}$; and*
(2) *$\zeta(s,\rho,\chi)$ has a simple pole at $s = h(\rho)$ if and only if R_χ is trivial.*

The following result generalizes Theorem 1.11 and has a similar proof.

Theorem 5.3. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation in the Hitchin component and let $R_\chi : \Gamma \to \mathrm{U}(N)$ be a unitary representation. The generalized L-function $Z(s,\rho,\chi)$ has the following properties.*

(1) *$Z(s,\rho,\chi)$ has a meromorphic extension to the entire complex plane $\mathbb{C}$; and*
(2) *$Z(s,\rho,\chi)$ has a simple zero at $s = h(\rho)$ if and only if R_χ is trivial.*

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