ZETA FUNCTIONS IN HIGHER TEICHMÜLLER THEORY

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Abstract. In this note we introduce zeta functions and L-functions for discrete and faithful representations of surface groups in $\mathrm{PSL}(d,\mathbb{R})$, for $d \geq 3$. These are natural generalizations of the well known classical Selberg zeta function and L-function for Fuchsian groups, corresponding to the case $d = 2$. We show that these complex functions have meromorphic extensions to the entire complex plane $\mathbb{C}$.

1. Introduction

Zeta functions and L-functions play a central role in number theory, geometry and dynamical systems. An important example is the Selberg zeta function associated to a compact hyperbolic surface. A hyperbolic metric on a compact surface is determined by a representation of its fundamental group into the rank-1 Lie group $\mathrm{PSL}(2,\mathbb{R})$. In this article, we address the natural question of how to extend the above approach to zeta functions $Z(s,\rho)$ naturally associated to appropriate representations ρ into higher rank Lie groups, which are the basis of the relatively new area of higher Teichmüller theory. A suitable class of representations to study are those known as projective Anosov (see section 2).

A simplified form of one of our main results is the following.

Theorem 1.1. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ $(d \geq 3)$ be a projective Anosov representation. Then $Z(s,\rho)$ converges for $Re(s)$ sufficiently large, and extends to a meromorphic function in the entire complex plane.*

This appears in a more precise form as Theorem 2.15, which is the analogue of the classical result that Selberg zeta functions have meromorphic extensions to the entire complex plane.

The original Selberg zeta function may also be considered dynamically as a zeta function associated to the geodesic flow over the surface. In this context, it common to also consider the closely related Ruelle zeta function. Both these functions may be studied via a body of tools and techniques known as Thermodynamic Formalism. In particular, this leads to results about extension beyond their half-plane of convergence.

We now briefly outline the contents of the rest of the paper. In the next section, we recall the Selberg and Ruelle zeta functions and introduce analgous zeta functions associated to projective Anosov representation. In section 3, we discuss surface groups and their symbolic coding. We associate to this a dynamical system, namely a subshift of finite type and explain how the induced action on projective space allows us to introduce a a function defined on the shift that encodes the weightings $d_\rho(g)$. In section 4, we use an approach due to Ruelle [26], to obtain results on extending our zeta functions as meromorphic functions in the entire complex plane by writing them in terms of determinants of transfer operators. In section 5, we discuss the location of zeros and poles for the zeta functions and obtain an error estimate on the associated counting function. This is based on the ideas originally introduced by Dolgopyat [10] for estimating iterates of transfer operators. A final section considers the more general case of L-functions associated to unitary representations. Some of the technical results we require in the paper (Lemma 3.3, Lemma 3.8 and Lemma 5.3) originally appeared in the preprint [24], and we include their proofs for completeness.

2. Representations and zeta functions

2.1. **Selberg and Ruelle zeta functions for** $\mathrm{PSL}(2,\mathbb{R})$**.** In 1956, Selberg introduced a zeta function associated to the fundamental group Γ of a compact oriented surface V of genus $g \geq 2$ or, more precisely, to representations of such groups as Fuchsian groups, i.e. discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

The surface V admits a family of hyperbolic metrics (metrics of constant curvature -1) which are parametrised by the Teichmüller space $\mathcal{T}(V)$. Let us consider a particularly convenient viewpoint using representations. We recall that $\mathcal{T}(V)$ can be identified with the connected component of

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

which consists of discrete and faithful representations $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$. (Here $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ by conjugation.) If $\rho \in \mathcal{T}(V)$ then one recovers the hyperbolic metric by realising V as $\mathbb{H}^2/\rho(\Gamma)$, where $\mathbb{H}^2$ is the Poincaré upper half plane and where $\mathrm{PSL}(2,\mathbb{R})$ acts by Möbius transformations as orientation preserving isometries of $\mathbb{H}^2$. (We also note that, since Γ is torsion free, $\rho : \Gamma \to \mathrm{PSL}(2,\mathrm{R})$ can be lifted to a representation $\widetilde{\rho} : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ [9]. To simplify notation, we will use $\rho(g)$ to denote the matrix in $\mathrm{SL}(2,\mathbb{R})$ in the hope that the context will make the usage clear.) Another connected component $\mathcal{T}'(V)$ corresponds to reversing the orientation.

The Selberg zeta function is a function of a single complex variable $s \in \mathbb{C}$, formally defined by

$$S(s,\rho) = \prod_{n=0}^{\infty} \prod_{[g]\in\mathcal{P}} \left(1 - e^{-(s+n)\ell_\rho(g)}\right), \tag{1.1}$$

where the second product is over the set $\mathcal{P}$ of all primitive conjugacy classes $[g]$ of elements $g \in \Gamma \setminus \{1\}$.[1] A conjugacy class is called primitive if it does not contain an element of the form g_0^n, for $g_0 \in \Gamma$ and $n > 1$. The real number $\ell_\rho(g)$ is twice the logarithm of the largest eigenvalue for $\rho(g)$ (or, equivalently, $\ell_\rho(g)$ is the length of the unique closed geodesic on V in the free homotopy class corresponding to $[g]$). The product converges to a non-zero analytic function for $\mathrm{Re}(s) > 1$ and Selberg showed the following fundamental result.

Theorem 2.1 (Selberg)**.** *The Selberg zeta function $S(s,\rho)$ extends to an entire function of order 2 and has a simple zero at $s = 1$.*

[1]The Selberg zeta function is more usually denoted by Z but we wish to reserve this for a related but slightly different function defined below.

An account of this theorem is contained in Hejhal's book [15], where it appears as Theorem 4.11 and Theorem 4.25. The result was obtained independently by Randol [25].

One may also, following Ruelle (cf. [26]), consider the related zeta function

$$R(s,\rho) = \prod_{[g]\in\mathcal{P}} \left(1 - e^{-s\ell_\rho(g)}\right)^{-1}. \tag{1.2}$$

Since $R(s,\rho) = S(s+1,\rho)/S(s,\rho)$, one immediately obtains the following result from Theorem 2.1.

Theorem 2.2. *The Ruelle zeta function $R(s,\rho)$ has a meromorphic extension to $\mathbb{C}$ with a simple pole at $s=1$. Moreover, $R(s,\rho)$ can be written as a ratio of two entire functions of order 2.*

In [26], Ruelle gave an alternative proof of Theorem 2.2 which will be the basis of our approach when we define and study analogues of these zeta functions in higher rank Teichmüller spaces.

We now want to consider a natural generalization of these definitions and results.

2.2. **Higher rank Teichmüller theory.** In recent years there has been considerable interest in generalising results in classical Teichmüller theory to what is now often referred to as higher Teichmüller theory, involving representations of surface groups (and more general groups) in higher rank Lie groups. This point of view, which had its origins in the work of Goldman [13] and [16], has received considerable attention, see, for example, the surveys [5], [19] and [31]. In this note we will address the natural problem of studying analogues of the Selberg and Ruelle zeta functions in the context of higher Teichmüller theory.

We wish to consider representations of Γ in the higher rank group $\mathrm{PSL}(d,\mathbb{R})$ (for $d \geq 3$). The most elementary class of such representations are the so-called Fuchsian representations, obtained directly from representations into $\mathrm{PSL}(2,\mathbb{R})$.

Example 2.3 (Fuchsian representations)**.** It is well known that there is an irreducible representation $\widetilde{\tau} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$, unique up to conjugation. This has an explicit construction, which we briefly recall. Let $\mathcal{S}_d$ denote the d-dimensional vector space of homogeneous polynomials in 2 variables of degree $\leq d-1$. We can choose a basis for $\mathcal{S}_d$ of the form

$$x^{d-1}, x^{d-2}y, \cdots, x^{d-i-1}y^i, \cdots, xy^{d-2}, y^{d-1}.$$

For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

we can define $\widetilde{\tau}(g) \in \mathrm{SL}(d,\mathbb{R})$ by specifying its action on the basis elements as follows:

$$\widetilde{\tau}(g) : x^{d-i-1}y^i \mapsto (ax+cy)^{d-i-1}(bx+dy)^i.$$

(It is an easy calculation to check that $\det\widetilde{\tau}(g) = 1$.) Furthermore, $\widetilde{\tau}$ factors to an irreducible representation $\tau : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$. Representations $\rho : \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ of the special form $\rho = \tau \circ \rho_0$ where $\rho_0 \in \mathcal{T}(V)$ (or $\mathcal{T}'(V)$), are called *Fuchsian representations*.

We can now define the representations we wish to study.

Definition 2.4. The natural generalization of the Teichmüller space are connected components of
$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(d,\mathbb{R}))/\mathrm{PSL}(d,\mathbb{R})$$
which contain Fuchsian representations. These are called *Hitchin components*. When d is odd there is a single Hitchin component but when d is even there are two Hitchin components (corresponding to $\mathcal{T}(V)$ and $\mathcal{T}'(V)$ having distinct images).

The study of representations in the Hitchin component has proved particularly fruitful. Hitchin originally described the properties of the Hitchin component [16] (see also [13]), but more recently the dynamics of the individual representations in the component have attracted attention (see, for example, the survey [19]).

We note that any $\rho : \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ in a Hitchin component may be lifted to a representation $\widetilde{\rho} : \Gamma \to \mathrm{SL}(d,\mathbb{R})$. This follows since a Fuchsian representation $\tau \circ \rho_0$ into $\mathrm{PSL}(d,\mathbb{R})$ lifts to a representation $\widetilde{\tau} \circ \widetilde{\rho}_0$ into $\mathrm{SL}(d,\mathbb{R})$ and, by Theorem 4.1 of [9], everything in the same connected component has the same lifting property. In what follows, we shall consider representations into $\mathrm{SL}(d,\mathbb{R})$, which shall denote by ρ (but remembering that this is the lift of a representation into $\mathrm{PSL}(d,\mathbb{R})$).

2.3. **Anosov representations.** It is now convenient to consider a slightly more general setting. Consider the case of discrete finitely generated torsion-free (word) hyperbolic group Γ. Let Γ_0 be a finite symmetric generating set and let $|\cdot| : \Gamma \to \mathbb{Z}^+$ be the word length with respect to Γ_0.

Definition 2.5. A discrete and faithful representation $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ is called *k-Anosov* (where $1 \leq k \leq d/2$) if there exists $C, \mu > 0$ such that for each $g \in \Gamma \setminus \{1\}$ the eigenvalues $\lambda_i(\rho(g))$, $1 \leq i \leq d$, for $\rho(g)$ ordered by modulus (i.e., $|\lambda_1(\rho(g))| \geq |\lambda_2(\rho(g))| \geq \cdots \geq |\lambda_d(\rho(g))|$) satisfy
$$\frac{\lambda_k(\rho(g))}{\lambda_{k+1}(\rho(g))} \geq Ce^{|g|}.$$

This particularly short definition is due to Kassel and Potrie [20] (see also [19]) but corresponds to earlier definitions, beginning with Labourie [21] and including Guichard and Wienhard [14].

A consequence of this definition is that there is a well behaved map from the Gromov boundary of Γ to $\mathcal{G}_k(\mathbb{R}^d)$, the Grassmannian of k-planes in $\mathbb{R}^k$. Here, the Gromov boundary is defined as follows. A sequence $(g_n)_{n=0}^{\infty} \in \Gamma^{\mathbb{Z}^+}$ is a geodesic ray (in the Cayley graph of Γ) if $|g_n| = n$ and $g_{n+1}g_n^{-1} \in \Gamma_0$. (In particular, $g_0 = 1$.) Then the Gromov boundary of Γ, which we denote by $\partial_\infty\Gamma$, is the space of equivalence classes of geodesic rays which remain a bounded distance apart. It can be equipped with a natural "visual" metric, where two points are close if one can choose representative rays which agree for a large initial segment (see Chapitre 7 of [12]). In our case, where Γ is the fundamental group of a compact surface of genus at least two, $\partial_\infty\Gamma$ is homeomorphic to the unit circle (by a quasisymmetric homeomorphism) [12]. There is a natural action of Γ on $\partial_\infty\Gamma$ and each $g \in \Gamma \setminus \{1\}$ has a unique attracting fixed point $g^+ \in \partial_\infty\Gamma$ and $\{g^+ : g \in \Gamma \setminus \{1\}\}$ is dense in $\partial_\infty\Gamma$.

We now have the following result.

Lemma 2.6. *If a representation $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ is P_k-Anosov then there exist Hölder continuous maps*
$$\xi^{(k)} : \partial_\infty\Gamma \to \mathcal{G}_k(\mathbb{R}^d) \text{ and } \theta^{(k)} : \partial_\infty\Gamma \to \mathcal{G}_{d-k}(\mathbb{R}^d)$$

such that for $\eta, \eta' \in \partial\Gamma_\infty$:

(1) *for $\eta \neq \eta'$ we have that $\xi^{(k)}(\eta) \oplus \theta^{(k)}(\eta') = \mathbb{R}^d$ (transversality);*
(2) *$\xi^{(k)}(g\eta) = \rho(g)\theta^{(k)}(\eta)$ for all $g \in \Gamma \setminus \{1\}$ (equivariance); and*
(3) *there exist $a > 0$ and $c > 0$ such that for every geodesic ray $(g_n)_{n=0}^\infty$ corresponding to $\eta \in \partial_\infty\Gamma$ we have that*
$$\frac{\sigma_k(\rho(g_n))}{\sigma_{k+1}(\rho(g_n))} \geq ce^{an},$$
where σ_i denotes the ith singular value (k-dominated).

Parts (1) and (2) are standard and part (3) may be found in [4].

In this paper, we will deal with representations which are P_1-Anosov. These representations are also called projective Anosov. Noting that $\mathcal{G}_k(\mathbb{R}^d)$ is the projective space $\mathbb{R}P^{d-1}$, in this case we will only need the map $\xi : \partial_\infty\Gamma \to \mathbb{R}P^{d-1}$ corresponding to $\xi^{(1)}$ in the lemma above. In this case, we have the following contraction property, which, though not part of the original definitions in [21], [14], comes from the equivalent definition due to Kapovich, Leeb and Porti ([17], Definition 6.45) and [18], or from part (3) of Lemma 2.6.

Lemma 2.7. *There exist $a > 0$ and $c > 0$ such that for every geodesic ray $(g_n)_{n=0}^\infty$ corresponding to $\eta \in \partial_\infty\Gamma$ we have that the associated action $\widehat{\rho}(g_n) : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ satisfies $\|D_{\xi(\eta)}\widehat{\rho}(g_n)\| \leq ce^{-na}$, for $n \geq 0$.*

Projective Anosov representations were originally introduced by Labourie [21]. Examples include the representations in the Hitchin component introduced above and also the Benoist representations introduced in [3].

Remark 2.8. (i) By consider the kth-exterior power, a P_k-Anosov representation into $\mathrm{SL}(d, \mathbb{R})$ induces a projective Anosov representation into $\mathrm{SL}\left(\binom{d}{k}, \mathbb{R}\right)$.
(ii) There may be additional restrictions on Anosov representations of hyperbolic groups. For example, if Γ is a torsion-free hyperbolic group which has an Anosov representation into $\mathrm{SL}(3, \mathbb{R})$ then Γ is either a free group or a surface group ([7], Theorem 1.1).

2.4. **Zeta functions for projective Anosov representations.** We now come to the main object of study in this note. Let Γ be the fundamental group of a compact oriented surface of genus $g \geq 2$ and let $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{R})$ be a projective Anosov representation. We recall the following definition.

Definition 2.9. A matrix is called *proximal* if it has a unique eigenvalue which is real and strictly maximal in modulus.

We then have the following, which is a consequence of the definition.

Lemma 2.10. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then for each $g \in \Gamma \setminus \{1\}$ the matrix $\rho(g)$ is proximal.*

We use this to associate a natural weight to each $g \in \Gamma \setminus \{1\}$. Let
$$d_\rho(g) = \log |\lambda(g)|,$$
where $\lambda(g)$ is the maximal eigenvalue of the matrix $\rho(g)$, guaranteed by Lemma 2.10. Since $\rho(g) \in \mathrm{SL}(d, \mathbb{R})$, $|\lambda(g)| > 1$ and hence $d_\rho(g) > 0$. Note that $d_\rho(g)$ only depends on the conjugacy class $[g]$. We will generalize the definition of the Selberg and Ruelle zeta functions to these representations.

Definition 2.11. Given a projective Anosov representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$, we can associate a *generalized Ruelle zeta function* defined by

$$\zeta(s,\rho) = \prod_{[g]\in\mathcal{P}} \left(1 - e^{-sd_\rho(g)}\right)^{-1}, \tag{1.2}$$

and a *generalized Selberg zeta function* defined by

$$Z(s,\rho) = \prod_{n=0}^{\infty} \prod_{[g]\in\mathcal{P}} \left(1 - e^{-(s+n)d_\rho(g)}\right), \tag{1.3}$$

for $s \in \mathbb{C}$ wherever the products converge.

In particular, from (1.2) and (1.3) we have the identities

$$\zeta(s,\rho) = \frac{Z(s+1,\rho)}{Z(s,\rho)}. \tag{1.4}$$

Remark 2.12. Note that, in the special case $d = 2$, with $\rho \in \mathcal{T}(V)$ we have that $d_\rho(g) = \ell_\rho(g)/2$. Therefore, $\zeta(s,\rho) = R(s/2,\rho)$. However, the relationship between $Z(s,\rho)$ and $S(s,\rho)$ is more complicated with

$$Z(s,\rho) = S(s/2,\rho)S(s/2 + 1/2, \rho).$$

To describe the half-plane of convergence of these new zeta functions, we recall the following definition. See [5], [6] or [24] for a further discussion, including the existence of the limit.

Definition 2.13. The *entropy* $h(\rho)$ of a projective Anosov representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is defined to be the growth rate of the number of primitive conjugacy classes $[g] \in \mathcal{P}$ with $d_\rho(g)$ at most T as $T \to +\infty$, i.e.,

$$h(\rho) = \lim_{T\to+\infty} \frac{1}{T} \log\left(\#\{[g] \in \mathcal{P} : d_\rho(g) \le T\}\right). \tag{1.5}$$

(The value of $h(\rho)$ would be the same if we used all non-trivial conjugacy classes in the definition, rather than just the primitive ones.)

It is easy to see from the definitions that $Z(s,\rho)$ and $\zeta(s,\rho)$ converge to non-zero analytic functions provided $\mathrm{Re}(s) > h(\rho)$. In the original setting, with $d = 2$ and $\rho \in \mathcal{T}(V)$, the entropy is always equal to 2 and so does not need to be explicitly introduced. (We remark that, in this case, $h(\rho) = 2$ is twice the topological entropy of the geodesic flow over $\mathbb{H}^2/\rho(\Gamma)$ because $d_\rho(g) = \ell_\rho(g)/2$, where $\ell_\rho(g)$ is the length of the corresponding closed geodesic.)

2.5. **Meromorphic extensions of the zeta functions.** Our main results on the zeta functions for projective Anosov representations can be viewed as generalizations of Theorem 2.2 and Theorem 2.1, respectively. We begin with the result for $\zeta(s,\rho)$, since, with our approach, the proof of this naturally comes first.

Theorem 2.14. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then $\zeta(s,\rho)$ extends to a meromorphic function in the entire complex plane and has a simple pole at $s = h(\rho)$. Furthermore, $\zeta(s,\rho)$ may be written as the quotient of two entire functions of order at most d.*

This will allow us to deduce the following result for $Z(s,\rho)$.

Theorem 2.15. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation. Then $Z(s, \rho)$ is a meromorphic function to the entire complex plane and has a simple zero at $s = h(\rho)$. Furthermore, $Z(s, \rho)$ may be written as the quotient of two entire functions of order at most $d + 1$.*

We will prove these results in section 4. We will also consider the zeros of the zeta functions in section 5. This leads to estimates on the error terms in the counting functions for the weights.

Remark 2.16. An alternative way of weighting the matrices, and hence of producing zeta functions, would be to take the difference between the logarithms of largest and smallest eigenvalues. It is possible to prove similar results for the corresponding zeta functions by considering for $g \in \Gamma$ the diagonal action $\Gamma \ni g \mapsto \rho(g) \times \rho(g^{-1})$ on $\mathbb{R}P^{d-1} \times \mathbb{R}P^{d-1}$. If $\rho(g)$ has eigenvalues $\lambda_1 > \cdots > \lambda_d$ and fixed point $\xi_g \in \mathbb{R}P^{d-1}$ then $\rho(g) \times \rho(g^{-1})$ has eigenvalues $\{\lambda_1^{\pm 1}, \ldots, \lambda_d^{\pm 1}\}$ and fixed point $(\xi_g, \xi_{g^{-1}})$. We then observe that for the action

$$\widehat{\rho}(g) \times \widehat{\rho}(g^{-1}) : \mathbb{R}P^{d-1} \times \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1} \times \mathbb{R}P^{d-1}$$

we have

$$\mathrm{Jac}_{(\xi_g, \xi_{g^{-1}})}(\widehat{\rho}(g) \times \widehat{\rho}(g^{-1})) = \mathrm{Jac}_{\xi_g}(\widehat{\rho}(g))\, \mathrm{Jac}_{\xi_{g^{-1}}}(\widehat{\rho}(g^{-1})).$$

Therefore most of our conclusions hold with the new weight function

$$\bar{d}_\rho(g) = \log \left|\lambda(g)/\lambda(g^{-1})\right| = \log\left(\lambda_d/\lambda_1\right).$$

3. Surface groups and symbolic dynamics

In this section we recall the ideas that are used to build a bridge between the geometry and the dynamics.

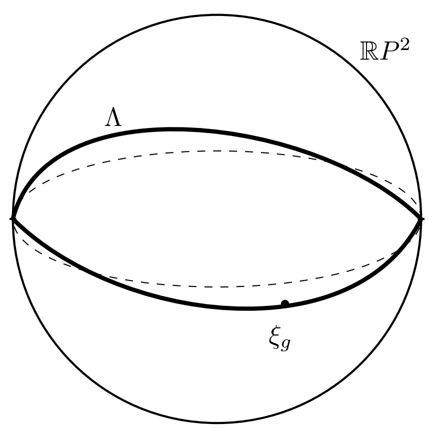

Figure 1. $\Lambda \subset \mathbb{R}P^2$ is a C^1 embedded curve. For $g \in \Gamma \setminus \{1\}$, the point ξ_g is an attracting fixed point for $\widehat{\rho}(g) : \mathbb{R}P^2 \to: \mathbb{R}P^2$.

3.1. **Action on projective space.** As discussed above, the representation $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ induces an action of Γ on the projective space $\mathbb{R}P^{d-1}$. More explicitly, for $g \in \Gamma$, we define $\widehat{\rho}(g) : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ by

$$\widehat{\rho}(g)[v] = \frac{\rho(g)v}{\|\rho(g)v\|_2},$$

where $v \in \mathbb{R}^d \setminus \{0\}$ is a representative element.

Suppose $g \in \Gamma \setminus \{1\}$. Since, by Lemma 2.10, $\rho(g)$ has a real eigenvalue which is strictly maximal in modulus, the map $\widehat{\rho}(g)$ has a unique attracting fixed point; we will denote this by $\xi_g \in \mathbb{R}P^{d-1}$. Let $\xi : \partial_\infty \Gamma \to \mathbb{R}P^{d-1}$ be the map introduced in the previous section. As an immediate consequence of Lemma 2.7, for each $g \in \Gamma \setminus \{1\}$, we have $\xi(g^+) = \xi_g$, the fixed point for $\widehat{\rho}(g)$.

We now define a compact subset of projective space which will be useful in the sequel.

Definition 3.1. We define the *limit set*

$$\Lambda := \overline{\{\xi_g \,:\, g \in \Gamma \setminus \{1\}\}} \subset \mathbb{R}P^{d-1}$$

to be the closure of the fixed points.

Since $\{g^+ \,:\, g \in \Gamma \setminus \{1\}\}$ is dense in $\partial_\infty \Gamma$, immediately follows that $\Lambda = \xi(\partial_\infty \Gamma)$. Moreover, by [3], Λ is a C^1 curve.

Example 3.2 (Case $d = 3$)**.** In this case the limit set $\Lambda \subset \mathbb{R}P^2$ is diffeomorphic to a circle bounding a strictly convex region Ω when ρ a representation in the Hitchin component. In particular, in Lemma 2.6 we have that for $\eta \in \partial_\infty \Gamma$ the point $\xi^{(1)}(\eta) \in \Lambda$ and $\theta^{(1)}(\eta)$ is tangent to the curve at that point. This is described in the earlier work of Choi and Goldman [8] (see also [21]) and applies to projective Anosov representations via the equivalence of the definitions shown in [20].

We can use the following simple lemma to relate the weight $d_\rho(g)$ to the action of $\widehat{\rho}(g)$ on $\mathbb{R}P^{d-1}$.

Lemma 3.3. *If $g \in \Gamma \setminus \{1\}$ then*

$$d_\rho(g) = -\frac{1}{d} \log \det(D_{\xi_g} \widehat{\rho}(g)).$$

Proof. We can consider the linear action of $\rho(g)$ on $\mathbb{R}^d$, then the fixed point corresponds to an eigenvector v and the result follows from a simple calculation using that the linear action of $\rho(g) \in \mathrm{SL}(d, \mathbb{R})$ preserves area in $\mathbb{R}^d$. More precisely, ξ_g corresponds to an eigenvector v for the maximal eigenvalue $\lambda(g)$, with $|\lambda(g)| > 1$, for the matrix $\rho(g)$. We can assume without loss of generality that $\|v\| = 1$ and then for arbitrarily small $\delta > 0$ we can consider a δ-neighbourhood of v which is the product of a $(d-1)$-dimensional neighbourhood in $\mathbb{R}P^{d-1}$ and a δ-neighbourhood in the radial direction. The effect of the linear action of $\rho(g)$ is to replace v by $\lambda(g)v$, and thus stretch the neighbourhood in the radial direction by a factor of $|\lambda(g)|$. Since $\rho(g)$ has determinant one, the volume of the $(d-1)$-dimensional neighbourhood contracts by $|\lambda(g)|^{-1}$. To calculate the effect of the projective action $\widehat{\rho}(g)$, we need to rescale $\lambda(g)v$ to have norm one, which corresponds to multiplication by the diagonal matrix $\mathrm{diag}(|\lambda(g)|^{-1}, \ldots, |\lambda(g)|^{-1})$. In particular, the $(d-1)$-dimensional neighbourhood in $\mathbb{R}P^{d-1}$ shrinks by a factor of approximately $|\lambda(g)|^{-d}$, giving the result. $\square$

3.2. **Surface groups and coding.** We now want to introduce a natural coding for the group Γ which will allow us to analyse the projective action of Γ on Λ. This is motivated by the more familiar boundary coding associated to Fuchsian groups due to Bowen and Series [30]. We take the same basic approach to introducing a dynamical perspective to the study of projective Anosov representations based on subshifts of finite type that was used in [24] (for representations in a Hitchin component).

Since Γ is the fundamental group of a compact surface V with genus $g \geq 2$, it has the presentation

$$\Gamma = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

We write $\Gamma_0 = \{a_1^{\pm 1}, \cdots, a_g^{\pm 1}, b_1^{\pm 1}, \cdots, b_g^{\pm 1}\}$ for the symmetrized generating set and $| \cdot | : \Gamma \to \mathbb{Z}^+$ for the word length with respect to Γ_0.

Such surface groups are particular examples of Gromov hyperbolic groups and as such they are strongly Markov groups in the sense of Ghys and de la Harpe [12], i.e. they can be encoded using a directed graph and an edge labelling by elements in Γ_0. This approach allows allows us to relate the group to a dynamical system, namely a subshift of finite type, and thus to use the machinery of thermodynamic formalism to define a so called pressure function and, later, transfer operators, which may be used to analyse our zeta functions. In the particular case of surface groups, the coding follows directly from the work of Adler and Flatto [1] and Series [30] on coding the action on the boundary and the associated subshift of finite type is mixing.

We have the following result.

Lemma 3.4. *We can associate to* (Γ, Γ_0)

(i) *a directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with a distinguished vertex* $*$*; and*
(ii) *an edge labelling* $\omega : \mathcal{E} \to \Gamma_0$*,*

such that

(1) *no edge terminates at* $*$*;*
(2) *there is at most one directed edge joining each ordered pair of vertices;*
(3) *the map from the set finite paths in the graph starting at* $*$ *to* $\Gamma \setminus \{1\}$ *defined by*

$$(e_1, \ldots, e_n) \mapsto \omega(e_1) \cdots \omega(e_n)$$

is a bijection and $|\omega(e_1) \cdots \omega(e_n)| = n$*;*

(4) *the map from closed paths in* $\mathcal{G}$ *to conjugacy classes to* Γ *induced by* ω *is a bijection and for such a closed path* $(e_1, \ldots, e_n, e_1)$*,* n *is the minimum word length in the conjugacy class of* $\omega(e_1) \cdots \omega(e_n)$*.*

Furthermore, the subgraph obtained be deleting the vertex $*$ *has the aperiodicity property that there exists* $N \geq 1$ *such that, given any two* $v, v' \in \mathcal{V} \setminus \{*\}$*, there is a directed path of length* N *from* v *to* v'*.*

Remark 3.5. Part (4) of Lemma 3.4 plays a subtle, but important, role in allowing us to relate the dynamical theory of zeta functions to conjugacy classes in Γ. In order to study other groups Γ than surface groups one would need to find a replacement for this property.

3.3. **Symbolic dynamics.** We now introduce the symbolic coding for the limit set Λ. This is particularly useful in introducing a partition of Λ which gives an effective way to reduce the group action to the orbits of a single transformation.

We can associate to the graph $\mathcal{G}$ a subshift of finite type where the states are labelled by the edges in the graph after deleting the edges that originate in the vertex $*$. In particular, if there are k such edges then we can define a $k \times k$ matrix A by $A(e, e') = 1$ if e' follows e in the directed graph and then define a space

$$\Sigma = \{x = (x_n) \in \{1, \cdots, k\}^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1, n \geq 0\}.$$

This is a compact space with respect to the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1 - \delta(x_n, y_n)}{2^n}.$$

The shift map is the local homeomorphism $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_n = x_{n+1}$. By the final statement in Lemma 3.4, A is aperiodic (i.e. there exists $N \geq 1$ such that A^N has all entries positive) and, equivalently, the shift $\sigma : \Sigma \to \Sigma$ is mixing (i.e. for all open non-empty $U, V \subset \Sigma$, there exists $N \geq 1$ such that $\sigma^{-n}(U) \cap V \neq \varnothing$ for all $n \geq N$).

There is a natural surjective Hölder continuous map $\pi : \Sigma \to \partial_\infty \Gamma$ defined by letting $\pi((x_n)_{n=0}^\infty)$ be the equivalence classes of the geodesic ray $(g_n)_{n=0}^\infty$ in $\partial_\infty \Gamma$, where $g_n = \omega(x_0)\omega(x_1) \cdots \omega(x_n)$.

Definition 3.6. We define a map $\kappa : \Sigma \to \Lambda \subset \mathbb{R}P^{d-1}$ by $\kappa = \xi \circ \pi$.

However, the shift $\sigma : \Sigma \to \Sigma$ only encodes information about Γ as an abstract group. In order to keep track of the additional information given by the representation of Γ in $\mathrm{SL}(d, \mathbb{R})$ and its action on $\mathbb{R}P^{d-1}$ we need to introduce a Hölder continuous function $r : \Sigma \to \mathbb{R}$.

Definition 3.7. We can associate a map $r : \Sigma \to \mathbb{R}$ defined by

$$r(x) = -\frac{1}{d} \log \det(D_{\kappa(x)} \widehat{\rho}(g_{x_0}))$$

(i.e. the Jacobian of the derivative of the projective action), where $g_{x_0} = \omega(x_0)$ is the generator corresponding to the first term in $x = (x_n)_{n=0}^\infty \in \Sigma$.

There is a natural one-to-one correspondence between non-trivial conjugacy classes in Γ and periodic orbits for the shift σ, by part (4) of Lemma 3.4.

We now have the following simple but key result.

Lemma 3.8. *The function $r : \Sigma \to \mathbb{R}$ is Hölder continuous, and if $\sigma^n x = x$ is a periodic point corresponding to the conjugacy class $[g]$ of an element $g \in \Gamma$ then $r^n(x) = d_\rho(g)$.*

Proof. The Hölder continuity of r follows immediately from the Hölder continuity of ξ, which is given by Lemma 2.7. The second statement follows from the equivariance and the observation $\kappa(\sigma x) = \rho(g_{x_0})\kappa(x)$. Moreover, that the periodic point x has an image $\kappa(x)(= \xi_g)$ which is fixed by $\widehat{\rho}(g)$ and the result follows from Lemma 3.3. □

3.4. **Semi-flows and entropy.** In this subsection we give an alternative definition of the entropy $h(\rho)$ of a representation, as defined in the introduction, which is useful when taking the symbolic viewpoint.

We can associate to $\sigma : \Sigma \to \Sigma$ and the positive function $r : \Sigma \to \mathbb{R}^+$ the space

$$\Sigma^r = \{(x, u) \in \Sigma \times \mathbb{R} : 0 \leq u \leq r(x)\}/\sim,$$

where $(x, r(x)) \sim (\sigma x, 0)$, and the semi-flow $\sigma_t^r : \Sigma^r \to \Sigma^r$ defined by $\sigma_t^r(x, u) = (x, u + t)$, for $t \geq 0$ (respecting the identifications).

The topological entropy $h(\sigma^r)$ for the semi-flow is defined to be the entropy of the time one map. It is a standard result that an alternative way of defining $h(\sigma^r)$ is as the growth rate

$$h(\sigma^r) = \lim_{T \to +\infty} \frac{1}{T} \log \#\{\tau \in \mathcal{P}(\sigma^r) : l(\tau) \leq T\},$$

where $\mathcal{P}(\sigma^r)$ denotes the set of prime periodic orbits of σ^r and $l(\tau)$ denotes the least period of the orbit τ [22].

Lemma 3.9. *If ρ is in a projective Anosov representation then the entropy of the representation is equal to the topological entropy of the semi-flow σ^f.*

Proof. We observe that there is a one-one correspondence between the periodic orbits for the semi-flow σ^r and for the shift σ. By Lemma 3.8 and Definition 2.13, we see that $h(\rho) = h(\sigma^r)$. □

4. Extensions of Zeta Functions

We now turn to the results on the meromorphic extensions of zeta functions. The proof of Theorem 2.14 is based on the approach of Ruelle in [26]. We then use Therem 2.14 to prove the corresponding result for $Z(s, \rho)$ in Theorem 2.15.

4.1. **Partitions.** By the construction in the previous section we have associated to the representation ρ a Hölder continuous map $\kappa : \Sigma \to \mathbb{R}P^{d-1}$ which intertwines the shift map on Σ with the projective action, i.e. $\kappa(\sigma x) = \widehat{\rho}(\omega(x_0))\kappa(x)$. In particular, we use κ to define a particular partition $P_i = \kappa([i])$ of the limit set $\Lambda \subset \mathbb{R}P^{d-1}$ (which we recall is a C^1 curve), where

$$[i] = \{x \in \Sigma : x_0 = i\}, \quad 1 \leq i \leq k.$$

We then have that

(1) $\Lambda = \bigcup_{i=1}^k P_i$, and
(2) $P_i \cap P_j$ consists of at most one point, for distinct $i \neq j$.

The analytic extension of the zeta function $\zeta(s, \rho)$ is based on the study of a transfer operator defined on analytic functions. In particular, we can realise the symbolic maps $\Sigma \supset [j] \ni x \mapsto ix \in [i] \subset \Sigma$ (where $A(i, j) = 1$) as local real analytic contractions

$$\psi_i : \coprod_{A(i,j)=1} P_j \to P_i$$

on the disjoint union of the sets.

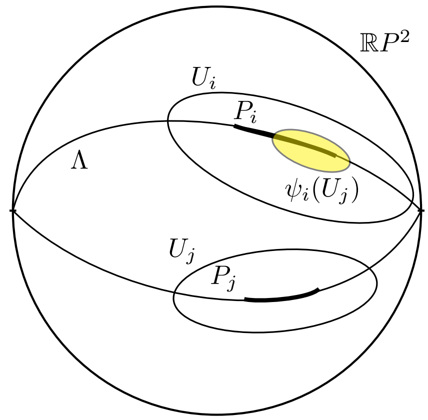

FIGURE 2. The contractions $\psi_i : P_j \to P_i$ satisfy $\psi_i(U_j) \subset U_i$ for the neighbourhoods of the complexifications (suggested in the figure).

4.2. **Transfer operators.** We can consider a Banach space of analytic functions on a neighbourhood of the disjoint union of the partition $\coprod_{i=1}^k P_i$. In particular, let us fix neighbourhoods $U_i \supset P_i$ in the complexification of $\mathbb{R}P^{d-1}$, chosen suitably small as will be indicated below.

This allows us to define a convenient Banach space of analytic functions.

Definition 4.1. We define B_0 to be the space of bounded analytic functions $f : \coprod_{i=1}^k U_i \to \mathbb{C}$ with the supremum norm

$$\|f\| = \max_i \sup_{z \in U_i} |f(z)|,$$

i.e., $B_0 = \left\{ f : \coprod_{i=1}^k U_i \to \mathbb{C} \text{ analytic} : \|f\| < \infty \right\}$.

We can also define a natural family of bounded linear operators on this space.

Definition 4.2. For $s \in \mathbb{C}$, the *transfer operator* $\mathcal{L}_s : B_0 \to B_0$ is defined by

$$\mathcal{L}_s f(z) = \sum_{A(i,j)=1} \det(D\psi_i(z))^s f(\psi_i z), \text{ for } z \in U_j,$$

where $\det(D\psi_i(\cdot))^s$ is understood as the complexification of the real analytic function.

In order for the operator to be well defined we need to show that whenever $A(i,j) = 1$ $(1 \leq i, j \leq k)$ then $\overline{\psi_i(U_j)} \subset U_i$. Since by construction $\psi(I_j) \subset I_i$ it suffices to show that for any $\eta \in P_i$ we have $\|D_\eta \psi\| < 1$. However, associating to η those geodesics (γ) arising from the coding (in Lemma 3.4) we can apply Lemma 2.7. If $ae^{-c} < 1$ then we are done. Otherwise we can choose $n \geq 2$ sufficiently large so that ae^{-cn} and use the standard recoding of Σ by replacing symbols by words of length n.

The advantage of being able to work in the Banach space of bounded analytic functions is that the operators have far better spectral properties than, say, Banach spaces of C^1 functions or Hölder continuous functions. More precisely, the transfer operators are compact operators and, furthermore, satisfy the stronger property of nuclearity. This following follows directly from the work of Ruelle, based on the ideas of Grothendieck.

Lemma 4.3 (Nuclearity: after Grothendieck and Ruelle)**.** *The transfer operators $\mathcal{L}_s : B_0 \to B_0$ are nuclear. More precisely, we can find*

1. $v_m \in B_0$ *with* $\|v_m\|_{B_0} = 1$,
2. $l_m \in B_0^*$ *with* $\|l_m\|_{B_0^*} = 1$, *and*
3. $\lambda_m \in \mathbb{C}$ *with* $|\lambda_m| = O(e^{-cm})$, *for some* $c > 0$,

such that

$$\mathcal{L}_s f = \sum_{m=1}^{\infty} \lambda_m l_m(f) v_m.$$

Proof. This can be deduced from [26]. □

We can also consider the natural generalisation of these operators to the space of analytic j-forms on $\coprod_{i=1}^k U_i$ (for $j = 1, \ldots, d-1$). More precisely, let B_j denote the Banach space of analytic j-forms on $\coprod_{i=1}^k U_i$ whose coefficients are bounded. Then the definition of $\mathcal{L}_s$ naturally extends to define transfer operators $\mathcal{L}_{s,j} : B_j \to B_j$. The analogue of Lemma 4.3 holds for these operators.

4.3. **Traces and determinants.** A particular consequence of Lemma 4.3 is that the operator $\mathcal{L}_s$ (and its powers $\mathcal{L}_s^n$) are trace class, i.e. the eigenvalues are summable: $\sum_{m=1}^{\infty} |\lambda_m| < \infty$.

To relate the transfer operators to the zeta functions we need the following result.

Lemma 4.4. *The trace of $\mathcal{L}_s^n : B_0 \to B_0$ is given by*

$$\mathrm{tr}(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \frac{\det(D\psi_{\underline{i}}(z_{\underline{i}}))^s}{\det(I - D\psi_{\underline{i}}(z_{\underline{i}}))},$$

where:

1. *the summation is over allowed reduced strings* $\underline{i} = (i_1, \cdots, i_n)$ *of length* n *such that* $A(i_1, i_2) = \cdots = A(i_{n-1}, i_n) = A(i_n, i_1) = 1$;
2. $\psi_{\underline{i}}$ *denotes the composition* $\psi_{\underline{i}} = \psi_{i_n} \circ \cdots \circ \psi_{i_1}$; *and*
3. $z_{\underline{i}} \in P_{i_n}$ *is a fixed point for* $\psi_{\underline{i}}$.

Furthermore,

$$\mathrm{tr}(\mathcal{L}_s^n) + \sum_{j=1}^{d} (-1)^j \mathrm{tr}(\mathcal{L}_{s,j}^n) = \sum_{|\underline{i}|=n} \det(D\psi_{\underline{i}}(z_{\underline{i}}))^s. \tag{3.1}$$

Proof. This can be deduced from the general setting in [26]. □

We want to associate to the transfer operators a function of a single complex variable s. We can write

$$\det(I - \mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(\mathcal{L}_s^n)\right).$$

Lemma 4.4 explains the connection with the fixed points of the contractions. Similarly, we can associate to the transfer operators $\mathcal{L}_{s,j}$ the functions

$$\det(I - \mathcal{L}_{s,j}) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}(\mathcal{L}_{s,j}^n)\right)$$

for $1 \leq j \leq d-1$.

The next lemma shows that these functions extend analytically to the entire complex plane. To simplify notation, we shall write $\mathcal{L}_{s,0} = \mathcal{L}_s$.

Lemma 4.5. *The functions $s \mapsto \det(I - \mathcal{L}_{s,j})$ ($0 \leq j \leq d-1$) extend to entire analytic functions of $s \in \mathbb{C}$ of order at most d.*

Proof. This can be (essentially) deduced from [26], although there is a minor correction required, as described in [11]. In particular, one can expand

$$\det(I - \mathcal{L}_{s,j}) = 1 + \sum_{m=1}^{\infty} c_m(s)$$

where

$$c_m(s) = O\left(e^{|s|Dm} e^{-Em^{\left(1+\frac{1}{d-1}\right)}}\right),$$

for some $E > 0$ and $D = \max_j \|D\phi_j\|$. In particular, by considering critical points of the real function $t \mapsto |s|Dt - Et^{\left(1+\frac{1}{d-1}\right)}$ we see that the upper bound on the terms occurs at

$$m = \left[\left(\frac{|s|D}{E(1+\frac{1}{d-1})}\right)^{d-1}\right] \text{ or } \left[\left(\frac{|s|D}{E(1+\frac{1}{d-1})}\right)^{d-1}\right] + 1.$$

giving an upper bound on the terms of $O\left(\exp\left(|s|^d D^d / (E(1+\frac{1}{d-1}))^{d-1}\right)\right)$. Moreover, it is easy to see that this gives an upper bound

$$|\det(I - \mathcal{L}_{s,j})| = O\left(e^{c|s|^d}\right),$$

for some $c > 0$. In particular, we see that the order is given by

$$\limsup_{R \to +\infty} \frac{\log\left(\max_{|s| \leq R} \log|\det(I - \mathcal{L}_{s,j})|\right)}{\log R} \leq d.$$

□

Proof of Theorem 2.14. To complete the proof of Theorem 2.14, we can then use the identity (3.1) to write

$$\zeta(s,\rho) = \frac{\prod_{j \text{ odd}} \det(I - \mathcal{L}_{s,j})}{\prod_{j \text{ even}} \det(I - \mathcal{L}_{s,j})},$$

from which we deduce that $\zeta(s,\rho)$ is a meromophic function which, by Lemma 4.5, may be written as the quotient of two entire functions of order $\leq d$.

Proof of Theorem 2.15. Given $s_0 \in \mathbb{C}$, we can choose $m \in \mathbb{Z}$ with $\mathrm{Re}(s_0) \geq m$. Setting $n = |m| + [h(\rho)] + 1$ we can use (1.4) to write

$$Z(s,\rho) = \frac{Z(s+n,\rho)}{\zeta(s,\rho)\zeta(s+1,\rho)\cdots\zeta(s+n-1,\rho)}. \tag{3.2}$$

For s in a sufficiently small neighbourhood of s_0 we have that $\mathrm{Re}(s+n) > h(\rho)$ and therefore $Z(s+n,\rho)$ converges to an analytic function. Moreover, the denominator is meromorphic by Theorem 2.14 and thus using (3.2) we deduce that $Z(s,\rho)$ is meromorphic.

To obtain the order bound in Theorem 2.15 consider $|s| < m$, say, and set $n = m + [h(\rho)] + 2$. By the proofs of Theorem 2.14 and Lemma 4.5 we can write

$\zeta(s,\rho) = \frac{f(s)}{g(s)}$, where for any $\epsilon > 0$ we can choose $C > 0$ such that $|f(s)|, |g(s)| \leq \exp(C|s|^{d+\epsilon})$ and thus using (3.2) we can write

$$Z(s,\rho) = Z(s+n,\rho)\frac{G(s)}{F(s)} \tag{3.3}$$

where $F(s) = \prod_{k=0}^{n-1} f(s+k)$ and $G(s) = \prod_{k=0}^{n-1} g(s+k)$ are entire functions. In particular, $Z(s+n,\rho)$ is uniformly bounded and since

$$\log|F(s)| \leq C\sum_{k=0}^{m+1} |s+k|^{d+\epsilon} \leq C(m+2)(2m)^{d+\epsilon}$$

the order of $F(s)$ is bounded by

$$\lim_{m\to+\infty} \frac{\log(C(m+2)(2m)^{d+\epsilon})}{\log m} = d+1+\epsilon,$$

and similarly the order for $G(s)$ is bounded by $d+1+\epsilon$. Finally, since $\epsilon > 0$ can be arbitrarily small, we can deduce from (3.3) that $Z(s,\rho)$ is a ratio of two entire functions of order $d+1$.

Remark 4.6 (Equidistribution)**.** For each $t > \delta$ can define a probability measure on Λ by

$$\mu_t = \frac{\sum_{g\in\Gamma\setminus\{1\}} |\mathrm{Jac}D_{\xi_g}\widehat{\rho}(g)|^t \delta_{\xi_g}}{\sum_{g\in\Gamma\setminus\{1\}} |\mathrm{Jac}D_{\xi_g}\widehat{\rho}(g)|^t}$$

and then as t tends to δ this converges to a probability measure μ supported on Λ. This follows easily from the properties of the transfer operator $\mathcal{L}_s$, where μ is related to the maximal eigenprojection of the operator $\mathcal{L}_{h(\rho)}$.

5. Zeros and poles of the zeta functions

Having established that the zeta functions $\zeta(s,\rho)$ and $Z(s,\rho)$ have meromorphic extensions to $\mathbb{C}$ in Theorems 2.14 and 2.15, it is very natural to ask about the location of the zeros and poles of zeta function.

In the case of the classical Selberg zeta function for hyperbolic surfaces the following is well known. Recall from Theorem 2.1 that the Selberg zeta function for a hyperbolic surface is entire with a simple zero at $s = 0$. Furthermore, the following result is well known.

Proposition 5.1 ([15])**.** *There exists $\epsilon > 0$ such that $S(s,\rho)$ has no zeros in $\mathrm{Re}(s) > 1-\epsilon$ other than $s = 1$.*

In this classical case it is even known that the zeros in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$ lie on $[0,1] \cup \left(\frac{1}{2} + i\mathbb{R}\right)$. This is a consequence of the way the zeta function can be extended using the trace formula and the interpretation of the zeros s_n in terms of the eigenvalues $\lambda_n = s_n(1-s_n)$ of the Laplace-Beltrami operator.

Using the relation $R(s,\rho) = S(s+1,\rho)/S(s,\rho)$ we can deduce the corresponding result for $R(s,\rho)$.

Proposition 5.2. *There exists $\epsilon > 0$ such that $R(s,\rho)$ has no zeros or poles in $\mathrm{Re}(s) > 1-\epsilon$ other than a simple pole $s = 1$.*

We can generalize both Proposition 5.1 and Proposition 5.2 to the zeta functions for projective Anosov representations. We will need the following "mixing condition" on the weights $d_\rho(g)$.

Lemma 5.3. *Let ρ be a projective Anosov representation. Then there does not exist $a > 0$ such that $\{d_\rho(g) : g \in \Gamma \setminus \{1\}\} \subset a\mathbb{Z}$.*

Proof. Let $g, h \in \Gamma \setminus \{1\}$ be two distinct elements of the group. For any $N > 0$ we can consider $g^N, h^N \in \Gamma$. The linear maps on $\mathbb{R}^d$ for the associated matrices $\rho(g^N), \rho(h^N) \in \mathrm{SL}(d, \mathbb{R})$ can be written in the form $\lambda(g)^N \pi_g + U_{g^N}$ and $\lambda(h)^N \pi_h + U_{h^N}$, respectively, where $\lambda(g), \lambda(h)$ are the largest simple eigenvalues, $\pi_g, \pi_h : \mathbb{R}^d \to \mathbb{R}^d$ are the eigenprojections onto their one dimensional eigenspaces, $\limsup_{N\to+\infty} \|U_{g^N}\|^{1/N} < \lambda(g)$ and $\limsup_{N\to+\infty} \|U_{h^N}\|^{1/N} < \lambda(h)$.

Let us now consider $g^N h^N \in \Gamma$ and and associated matrix $\rho(g^N h^N)$. The associated linear map will be of the form $\lambda(g^N h^N)\pi_{g^N h^N} + U_{g^N h^N}$ where $\lambda(g^N h^N)$ is the largest simple eigenvalue, $\pi_{g^N h^N} : \mathbb{R}^d \to \mathbb{R}^d$ is the eigenprojection onto their one dimensional eigenspaces, and $\limsup_{N\to+\infty} \|U_{g^N h^N}\|^{1/N} < \lambda(g^N h^N)$. However, since we have the identity $\rho(g^N h^N) = \rho(g^N)\rho(h^N)$ for the matrix representations we can also write the corresponding relationship for the linear maps:

$$\lambda(g^N h^N)\pi_{g^N h^N} + U_{g^N h^N} = \left(\lambda(g^N)\pi_{g^N} + U_{g^N}\right)\left(\lambda(h^N)\pi_{h^N} + U_{h^N}\right). \tag{4.1}$$

In particular, we see that as N becomes larger

$$\lim_{N\to+\infty} \exp\left((d_\rho([g^N h^N]) - d_\rho([g^N]) - d_\rho([h^N])\right) = \lim_{N\to+\infty} \frac{\lambda(g^N h^N)}{\lambda(g^N)\lambda(h^N)} = \langle \pi_h, \pi_g \rangle$$

where $\langle \pi_h, \pi_g \rangle$ is simply the cosine of the angle between the eigenvectors associated to $\lambda(g)$ and $\lambda(h)$, respectively. However, if we assume for a contradiction that the conclusion of the lemma does not hold, then the right hand side of (4.1) must be of the form e^{na}, for some $n \in \mathbb{Z}$. However, the directions for the associated eigenprojections form an infinite set in $\mathbb{R}P^{d-1}$ and have an accumulation point. Thus for suitable choices of g, h we can arrange that $0 < \langle \pi_h, \pi_g \rangle < e^a$, leading to a contradiction. This completes the proof of the lemma. □

We will now obtain our results on the zeta functions. First we will consider the Ruelle-type zeta function $\zeta(s, \rho)$.

Theorem 5.4. *There exists $\epsilon > 0$ such that $\zeta(s, \rho)$ has no zero or poles in $\mathrm{Re}(s) > h(\rho) - \epsilon$ other than $h(\rho)$. Moreover, there exists $\alpha > 0$ such that we can bound $\log|\zeta(s, \rho)| = O(|\mathrm{Im}(s)|^\alpha)$ for s satisfying $h(\rho) - \epsilon < \mathrm{Re}(s) < h(\rho)$ and $|\mathrm{Im}(s)| \geq 1$.*

Proof. The absence of zeros in a half-plane $\mathrm{Re}(s) > h(\rho) - \epsilon$ is a standard result [22]. The pole free strip is a consequence of the method of Dolgopyat [10] and the fact that the limit set $\Lambda \subset \mathbb{R}P^{d-1}$ is a C^1 curve [3]. This method gives bounds on the iterates of the transfer operators $\mathcal{L}_s$ regarded as operators $\mathcal{L}_s : C^1\left(\coprod_{i=1}^k P_i, \mathbb{C}\right) \to C^1\left(\coprod_{i=1}^k P_i, \mathbb{C}\right)$. It is convenient to use the norm

$$\|h\|_{1,t} = \begin{cases} \max\left\{\|h\|_\infty, \frac{\|h'\|_\infty}{|t|}\right\} & \text{if } |t| \geq 1 \\ \max\left\{\|h\|_\infty, \|h'\|_\infty\right\} & \text{if } |t| < 1 \end{cases}$$

on $C^1\left(\coprod_{i=1}^k P_i, \mathbb{C}\right)$. Following Proposition 7.4 of [2], the key ingredients required for the proof are that:

(i) $r : \coprod_{i=1}^k P_i \to \mathbb{R}$ is a C^1 function; and

(ii) $r : \coprod_{i=1}^k P_i \to \mathbb{R}$ is not cohomologous to a constant, i.e. there is no C^1 function $u : \coprod_i P_i \to \mathbb{R}$ and constant $c \in \mathbb{R}$ such that $r = u \circ T - u + c$.

We see that (i) holds by construction and (ii) follows from Lemma 3.8 and Lemma 5.3.

We can apply the results from [10] and [23] to show that there exist constants $\sigma_0 < h$, $C > 0$ and $0 < \beta < 1$ such that whenever $s = \sigma + it$ and $n = p[\log|t|] + l$, where $p \geq 0$ and $0 \leq l \leq [\log|t|] - 1$, then

$$\|\mathcal{L}_{-sr}^n\|_{1,t} \leq C\beta^{p[\log|t|]}e^{lP(-\sigma r)}$$

with respect to the norm $\|h\|_{1,t}$.

By Lemma 2 of [23] (which, apart from the dependence on $|t|$ in the bound, appears in [27]) we have the estimate that for any $x_j \in P_j$, $j = 1, \ldots, k$, and any β_0 satisfying $\max\{\beta, \max_j \|D\psi_j\|_\infty\} < \beta_0 < 1$, there exists $C > 0$ such that

$$\left| \sum_{|\underline{i}|=n} \det(D\psi_{\underline{i}}(z_{\underline{i}}))^s - \sum_{j=1}^k \mathcal{L}_s \chi_{P_j}(x_j) \right| \leq C|t|n\beta_0^n, \quad \forall n \geq 1,$$

where χ_{P_j} is the indicator function for P_j and $\underline{i}$, $\psi_{\underline{i}}$ and $z_{\underline{i}}$ are as in Lemma 4.4. We can use this estimate to get a bound on the logarithm of the absolute value of the zeta function of the form $\log|\zeta(s,\rho)| = O\left(|t|^\alpha\right)$, for some $\alpha > 0$, as in (2.3) of [23]. □

In light of (1.4), and the observation that $Z(s+1,\rho)$ is uniformly bounded on the half-plane $\mathrm{Re}(s) > h(\rho) - \epsilon$, provided $\epsilon < 1$, this gives the following generalization of Proposition 5.1.

Theorem 5.5. *There exists $\epsilon > 0$ such that $Z(s,\rho)$ has no zero or poles in $\mathrm{Re}(s) > h(\rho) - \epsilon$ other than the zero at $s = h(\rho)$. Moreover, there exists $\alpha > 0$ such that we can bound $\log|Z(s,\rho)| = O(|\mathrm{Im}(s)|^\alpha)$ for s satisfying $h(\rho) - \epsilon < \mathrm{Re}(s) < h(\rho)$ and $|\mathrm{Im}(s)| \geq 1$.*

Although Theorem 5.4 is weaker than the full result known for the Selberg zeta function in the case of hyperbolic surfaces it nonetheless has analogous consequences for error terms in counting functions. To conclude the section, we address how the positions of the zeros influence asymptotic formulae.

Definition 5.6. For $T > 0$, we define

$$\pi_\rho(T) = \#\{[g] \in \mathcal{P} : d_\rho([g]) \leq T\},$$

the number of primitive conjugacy classes with weight at most T.

It was shown in [28] (following the approach in [22]) that $\pi_\rho(T)$ satisfies the asymptotic formula

$$\pi_\rho(T) \sim \frac{e^{h(\rho)T}}{h(\rho)T}, \quad \text{as } T \to +\infty.$$

Using Theorem 5.4, we can deduce a stronger result with an error term provided we replace the leading term with a logarithmic integral. As usual, we will write

$$\mathrm{li}(x) := \int_2^x \frac{1}{\log u} du \sim \frac{x}{\log x}, \quad \text{as } x \to +\infty.$$

We then have the following result.

Theorem 5.7. *There exists $\epsilon > 0$ such that*

$$\pi_\rho(T) = \mathrm{li}(e^{h(\rho)T})\left(1 + O(e^{-\epsilon T})\right).$$

Proof. Given Theorem 5.4, we can apply the classical proof from number theory (see [23]). □

6. The L-function in higher rank

We can generalize the definition of the zeta functions to L-functions by incorporating additional information associated to a unitary representation $U : \Gamma \to \mathrm{U}(N)$. More precisely, we make the following definition.

Definition 6.1. We can associate to each projective Anosov representation $\rho : \Gamma \to \mathrm{SL}(d, R)$ and to each unitary representation $R_\chi : \Gamma \to \mathrm{U}(N)$ (with character $\chi = \mathrm{Trace}(R_\chi)$), a Selberg type L-function formally defined by

$$Z(s, \rho, \chi) = \prod_{n=0}^{\infty} \prod_{[g]\in\mathcal{P}} \det\left(1 - e^{-(s+n)d_\rho(g)} R_\chi([g])\right), \quad s \in \mathbb{C},$$

and a Ruelle type L-function formally defined by

$$\zeta(s, \rho, \chi) = \prod_{[g]\in\mathcal{P}} \det\left(1 - e^{-sd_\rho(g)} R_\chi([g])\right)^{-1}, \quad s \in \mathbb{C},$$

where the products converge.

If R_{χ_0} is the trivial representation then we see that these L-functions reduce to the corresponding zeta functions, i.e. $Z(s, \rho, \chi_0) = Z(s, \rho)$ and $\zeta(s, \rho, \chi_0) = \zeta(s, \rho)$. In the particular case of Fuchsian representations then these definitions reduce to the familiar definitions of L-functions for Fuchsian groups [11].

It is easy to see that $Z(s, \rho, \chi)$ and $\zeta(s, \rho, \chi)$ converge for $\mathrm{Re}(s) > h(\rho)$. We can also observe directly from the definitions that

$$\zeta(s, \rho, \chi) = \frac{Z(s+1, \rho, \chi)}{Z(s, \rho, \chi)}. \tag{5.1}$$

The following result generalizes Theorem 2.14 and has a similar proof.

Theorem 6.2. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation and let $R_\chi : \Gamma \to \mathrm{U}(N)$ be a unitary representation. The Ruelle-type L-function $\zeta(s, \rho, \chi)$ has the following properties.*

1. *$\zeta(s, \rho, \chi)$ has a meromorphic extension to the entire complex plane $\mathbb{C}$; and*
2. *$\zeta(s, \rho, \chi)$ has a simple pole at $s = h(\rho)$ if and only if R_χ is trivial.*

The following result generalizes Theorem 2.15 and has a similar proof.

Theorem 6.3. *Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov representation and let $R_\chi : \Gamma \to \mathrm{U}(N)$ be a unitary representation. The generalized L-function $Z(s, \rho, \chi)$ has the following properties.*

1. *$Z(s, \rho, \chi)$ has a meromorphic extension to the entire complex plane $\mathbb{C}$; and*
2. *$Z(s, \rho, \chi)$ has a simple zero at $s = h(\rho)$ if and only if R_χ is trivial.*

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