PERIODIC ORBITS AND HOLONOMY FOR HYPERBOLIC FLOWS

Mark Pollicott and Richard Sharp

University of Warwick and University of Manchester

0. Introduction

One of the most natural problems in ergodic theory to understand the asymptotic behaviour of various orbital averages. Such an approach is particularly fruitful in the case of hyperbolic systems.

Consider the case of (weak mixing) hyperbolic flows $\phi_t : \Lambda \to \Lambda$. The number of (prime) periodic orbits τ with least period $l(\tau) \leq T$ satisfies the remarkably simple asymptotic formula

$$\pi(T) := \sum_{l(\tau) \leq T} 1 \sim \frac{e^{hT}}{hT}, \text{ as } T \to +\infty$$

where $h > 0$ denotes the topological entropy of the flow and the symbol $\sim$ denotes that the ratio of the two sides converges to unity [12]. Moreover, there is a simple hypothesis on lengths of closed orbits which ensures that there is a polynomial error term, i.e., there exists $\delta > 0$ such that

$$\pi(T) = \frac{e^{hT}}{hT} + O\left(\frac{e^{hT}}{T^{1+\delta}}\right), \text{ as } T \to +\infty$$

[14]. More precisely, we have this error term if we can choose three closed orbits of least periods l_1, l_2, l_3 such that $\theta = (l_1 - l_2)/(l_2 - l_3)$ is diophantine, i.e., there exists $C > 0$ and $\beta > 0$ such that $|q\theta - p| \geq Cq^{-(1+\beta)}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

In this paper we shall consider compact groups extensions of hyperbolic flows. Let G be a compact Lie group. Let $\widehat{\phi}_t : \widehat{\Lambda} \to \widehat{\Lambda}$ be a topologically weak mixing G-extension of a hyperbolic flow $\phi_t : \Lambda \to \Lambda$ with projection $\pi : \widehat{\Lambda} \to \Lambda$. Given a closed ϕ-orbit and $x \in \tau$ there exists a unique element $g \in G$ such that, for $\widehat{x} \in \pi^{-1}(x)$, $\phi_{l(\tau)}\widehat{x} = g\tilde{x}$. If we choose another point $x' \in \tau$ then the corresponding group element is conjugate to g. We call the conjugacy class $[g]$ in G the holonomy class of τ, which we denote by $[\tau]$. In [11], the following general equidistribution result was established: if $\chi \in \widehat{G}$ is a non-trivial character then

$$\lim_{T \to +\infty} \frac{1}{\pi(T)} \sum_{l(\tau) \leq T} \chi([\tau]) = 0.$$

We shall concentrate on the particularly simple case $G = SO(2)$. (In particular, $[\tau]$ is a single element of $SO(2)$, called the holonomy element.) In order to state our result we need to formulate an inhomogeneous diophantine condition. Assume that the three closed orbits above have associated holonomy elements $e^{2\pi i\Theta_1}, e^{2\pi i\Theta_2}, e^{2\pi i\Theta_3}$. Let $\Theta = (\Theta_2 - \Theta_3) + \theta(\Theta_1 - \Theta_2)$, then we require that there exists $C > 0$ and $\beta > 0$ such that $|q\theta - p + \Theta| \geq Cq^{-(1+\beta)}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. For any irrational number θ the set of real numbers Θ satisfying this identity has full Hausdorff dimension.

Theorem 1. *Let $\widehat{\phi}_t : \widehat{\Lambda} \to \widehat{\Lambda}$ be a weak mixing $SO(2)$-extension of a hyperbolic flow. Assume that we can choose three closed orbits satisfying the inhomogeneous diophantine condition. Let χ be a non-trivial character. Then there exists $\delta > 0$ such that*

$$\frac{1}{\pi(T)} \sum_{l(\tau) \leq T} \chi([\tau]) = O\left(T^{-\delta}\right), \text{ as } T \to +\infty. \tag{0.1}$$

The notation $f(T) = O(g(T))$ means that the ratio $|f(T)|/g(T)$ is bounded above. Unfortunately, we are not able to describe the dependence on χ of the implied constant in (0.1). Such information would be required to replace characters by arbitrary smooth functions.

Theorem 1 has natural generalizations to other compact groups G. However, there are a number of additional technical details (not least the formulation of the hypotheses) which we do not discuss here.

Example 1. Perhaps the best know example of a hyperbolic flow is a geodesic flow $\phi_t : M \to M$, where $M = SV$ is the unit-tangent bundle on a compact negatively curved manifold V. Provided that the base manifold V is 1/4-pinched, i.e., that all the curvatures lie in an interval $[-\kappa, -\kappa/4]$, for some $\kappa > 0$, then there exists $\epsilon > 0$ such that

$$\pi(T) = \mathrm{li}(e^{hT}) + O(e^{(h-\epsilon)T}), \text{ as } T \to +\infty, \tag{0.2}$$

where $\mathrm{li}(x) = \int_2^x (\log u)^{-1} du$. For compact manifolds with constant negative sectional curvature this is a classical result of Huber [8], whose proof made use of the Selberg trace formula. For the case of variable curvature, the leading asymptotic for $\pi(T)$ is due to Margulis [10], while the present authors obtained (0.2) in [13].

When V has dimension $d \geq 3$ we can consider the frame flow $\widehat{\phi}_t : \widehat{M} \to \widehat{M}$, where $\widehat{M}$ is the bundle of (positively oriented) orthonormal d-frames over V and $\widehat{\phi}$ is defined by parallel transporting frames [3], [4]. Sarnak and Wakayama [16] consider the particular case of locally symmetric manifolds with negative sectional curvature, where they proved in that case that the error term was exponential. We know of no results on error terms in the general case of manifolds of variable curvature.

It is simple to choose hyperbolic matrices $A, B, C \in SL(2, \mathbb{C})$ which generate a Schottky group Γ acting on the three dimensional Poincaré upper half disk and for which the corresponding closed geodesics satisfy the inhomogeneous diophantine condition. (In particular, the length and the holonomy can be read off from the action of the matrix at a fixed point of the matrix.) The geodesic flow restricted to the the non-wandering set is a weak mixing hyperbolic flow, and Theorem 1 applies.

Example 2. Consider the usual shift map $\sigma : X \to X$ on the space $X = \prod_{n=-\infty}^{\infty}\{0,1\}$ and a locally constant function $r : X \to \mathbb{R}$ defined by

$$r(x) = \begin{cases} \omega & \text{if } x_0 = 0 \\ 1 & \text{if } x_0 = 1 \end{cases}$$

where $\omega > 0$. We denote $X^r = \{(x,t) : 0 \leq t \leq r(x)\}$ with the identification $(x, r(x)) \sim (\sigma x, 0)$. We define a suspension flow $\sigma_t^r : X^r \to X^r$ locally by $\sigma_t^r(x,u) = (x, u+t)$, for $t \in \mathbb{R}$.

There are hyperbolic flows $\phi_t : \Lambda \to \Lambda$ which are conjugate to $\sigma_t^r : \Sigma^r \to \Sigma^r$ [1]. If ω is irrational then the flow is weak mixing. Given a Hölder continuous function, $\Theta : X \to G = SO(2)$ we can associate a skew product $\widehat{\sigma}_t^r : X^r \times G \to X^r \times G$ defined by $\widehat{\sigma}_t^r(x,u,g) = (\sigma_t^r(x,u), \Theta(x)g)$. This is topologically weak mixing if there are only trivial solutions to $F(\sigma x) = e^{iar(x)}F(x)$, with $a > 0$ and $F \in C(X^r)$.

If ω is diophantine then Theorem 1 applies. On the other hand, if ω is very well approximable then no such error term will hold for either the hyperbolic flow or the G-extension [12].

1. Hyperbolic flows

We begin by recalling the definition of a hyperbolic flow $\phi_t : \Lambda \to \Lambda$. Let $\phi_t : M \to M$ be a C^1 flow on on a C^∞ manifold M and let $\Lambda \subset M$ be a ϕ-invariant compact set. We say that $\phi_t : \Lambda \to \Lambda$ is a hyperbolic flow if

(i) there is a splitting $T_\Lambda M = E^0 \oplus E^s \oplus E^u$ such that:
  (a) there exist constants $C, \lambda > 0$ with $||D\phi_t|E^s||, ||D\phi_{-t}|E^u|| \leq Ce^{-\lambda t}$, for all $t \geq 0$;
  (b) E^0 is one dimensional and tangent to the flow;

(ii) Λ contains a dense orbit;

(iii) the periodic orbit in Λ are dense (and Λ consists of more than a single closed orbit);

(iv) there exists an open set $U \supset \Lambda$ such that $\Lambda = \cap_{t=-\infty}^{\infty}\phi_t U$.

Suppose that G is a compact Lie group and that $\pi : \widehat{M} \to M$ is a principal G-bundle over M. If $\widehat{\Lambda} = \pi^{-1}(\Lambda)$ and $\widehat{\phi}_t : \widehat{\Lambda} \to \widehat{\Lambda}$ is a flow satisfying $\phi_t \circ \pi = \pi \circ \widehat{\phi}_t$ then we say that $\widehat{\phi}_t$ is a G-extension of ϕ_t.

To prove Theorem 1 we need to analyse hyperbolic flows via a symbolic model (of the same kind as that in Example 2 above). Given a $k \times k$ zero-one aperiodic matrix A we can define a shift space

$$X_A = \left\{x = (x_n) \in \{1, \ldots, k\}^{\mathbb{Z}} : A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}\right\}$$

with a metric $d(x,y) = \sum_{n=-\infty}^{\infty}(1 - \delta(x_n, y_n))/2^{|n|}$, where $\delta(i,j)$ is the Kronecker symbol. The shift map $\sigma : X_A \to X_A$ given by $(\sigma x)_n = x_{n+1}$ is a homeomorphism.

Assume that $r : X_A \to \mathbb{R}$ a strictly positive Hölder continuous function. Then we define the r-suspension space

$$X_A^r = \{(x,t) : 0 \leq t \leq r(x)\},$$

subject to the identification $(x, r(x)) \sim (\sigma x, 0)$. We also define the associated suspended flow $\sigma_t^r : X_A^r \to X_A^r$ defined locally by $\sigma_t^r(x,u) = (x, u+t)$, and respecting the above identifications.

It follows by results of Ratner and Bowen that any hyperbolic flow can be modelled by such a symbolic flow. More precisely, we have the following result.

Proposition 1.1. [2],[15]. *Given a hyperbolic flow $\phi_t : \Lambda \to \Lambda$ there exists an aperiodic zero-one matrix A, a positive Hölder continuous function $r : X_A \to \mathbb{R}$ and a Hölder continuous map $p : X_A^r \to \Lambda$ such that*

(1) *p is a semi-conjugacy, i.e., $p \circ \sigma_t^r = \phi_t \circ p$,*
(2) *p is bounded-to-one, and*
(3) *p one-to-one on a residual set.*

Unfortunately, there is not a one-to-one correspondence between periodic orbits for σ_t^r and ϕ_t. However, the following result is sufficient for our purposes [2].

Lemma 1.2. *Let $\upsilon(T)$ denote the number of σ_t^r-periodic orbits of least period at most T. Then $\pi(T) = \upsilon(T) + O(e^{(h-\epsilon)T})$, for some $\epsilon > 0$.*

The symbolic model can be understood in the following way. The image $p(X_A \times \{0\})$ is a disjoint union $\coprod_{i=1}^k T_i$ of local cross sections for the flow. The return time between $p(x,0)$ and $p(\sigma x, 0)$ is equal to $r(x)$. Above each section, we can trivialize the bundle $\widehat{M}$ and write it as $T_i \times G$. Using this trivialization, we can define a Hölder continuous function $\Theta : X_A \to G$ by the formula

$$\widehat{\phi}_{r(x)}(p(x,0), e) = (p(\sigma x, 0), \Theta(x)),$$

where $e \in G$ is the identity element. (In other words, $\Theta(x)$ the skewing function associated to the hyperbolic flow as the point $p(x,0)$ flows to $p(\sigma x, 0)$.)

We can define a corresponding one-sided shift space

$$X_A^+ = \{x \in \{1, \ldots, k\}^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}^+\}$$

with a metric $d(x,y) = \sum_{n=0}^{\infty} \frac{1-\delta(x_n,y_n)}{2^n}$. The shift map $\sigma : X_A^+ \to X_A^+$ given by $(\sigma x)_n = x_{n+1}$ is a local homeomorphism. There is an obvious one-to-one correspondence between periodic orbits for the one-sided and two-sided shift.

We shall need to relate functions on X_A to functions on X_A^+. To do this need the following definition. Let Y be either $\mathbb{R}$ or a compact Lie group. Two functions $f, f' : X_A \to Y$ are said to be cohomologous if there exists a continuous function $u : X_A \to Y$ such that $f' = (u \circ \sigma)^{-1} \cdot f \cdot u$ (where $\cdot$ denotes the group operation in Y). A function is called a coboundary if it is cohomologous to the identity. Two functions are cohomologous if and only if their values around periodic orbits agree. The following useful lemma is standard.

Lemma 1.3 [12]**.** *Given a Hölder continuous function $f : X_A \to Y$, where Y is either $\mathbb{R}$ or a compact Lie group, there exists a cohomologous Hölder continuous function $f' : X_A \to Y$ which depends only on future co-ordinates (i.e. $f'(x) = f'(y)$, if $x_n = y_n$ for $n \geq 0$) and so may be identified with a function defined on X_A^+.*

As a result of this lemma, we can replace r and Θ by Holder continuous functions $r : X_A^+ \to \mathbb{R}$ and $\Theta : X_A^+ \to G$ without affecting the values around periodic orbits. Choose an exponent $\alpha > 0$ so that $r \in C^\alpha(X_A^+, \mathbb{R})$ and $\Theta \in C^\alpha(X_A^+, G)$.

For each $\xi \in \mathbb{R}$, we can define a bounded linear operator $\mathcal{L}_\xi : C^\alpha(X_A^+, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C})$, called the *transfer operator*, by

$$\mathcal{L}_\xi w(x) = \sum_{\sigma y = x} e^{-\xi r(y)} w(y).$$

This has an isolated simple eigenvalue equal to its spectral radius, written $e^{P(-\xi r)}$ (where $P(-\xi r)$ is the topological pressure of the function $-\xi r$). If $h > 0$ is the topological entropy of ϕ_t (necessarily equal to the topological entropy of the suspended flow) then $e^{P(-hr)} = 1$, so $\mathcal{L}_h$ has spectral radius 1. The following lemma is a consequence of the Ruelle operator theorem [12].

Lemma 1.4. *By adding a coboundary to r, if necessary, we may assume that $\mathcal{L}_h$ fixes the constant function* 1.

We can define a transfer operator for complex parameters and representations of general compact Lie groups. Let $R_\chi : G \to U(d)$ be a unitary representation. For each $s \in \mathbb{C}$, we can define a new transfer operator $\mathcal{L}_{s,\chi} : C^\alpha(X_A^+, \mathbb{C}^d) \to C^\alpha(X_A^+, \mathbb{C}^d)$ by

$$\mathcal{L}_{s,\chi} w(x) = \sum_{\sigma y = x} e^{-sr(y)} R_\chi(\Theta(y)) w(y).$$

In the particular case that $G = SO(2)$ then $\chi \in \widehat{SO(2)}$ is of the form $\chi(\theta) = \chi_m(\theta) := e^{2\pi i m\theta}$, for some $m \in \mathbb{Z}$. Then, given $s \in \mathbb{C}$, we can define a weighted transfer operator $\mathcal{L}_{s,\chi} : C^\alpha(X_A^+, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C})$ by

$$\mathcal{L}_{s,\chi} w(x) = \sum_{\sigma y = x} e^{-sr(y)} e^{2\pi i m\Theta(y)} w(y).$$

Subject to the inhomogeneous diophantine condition discussed in the introduction, the estimate we have on the transfer operator is the following.

Lemma 1.5. *There exist constants $\gamma > 0$, $t_0 > 0$, $C_1 > 0$ and $N > 0$, such that whenever $|t| \geq t_0$ we have that, for all $n \geq 1$,*

$$\|\mathcal{L}_{s,\chi}^{2nN}\|_\alpha \leq C_1 e^{2nNP(-Re(s)r)} |Im(s)| \left(1 - \frac{1}{|Im(s)|^\gamma}\right)^{n-1}.$$

Lemma 1.5 is proved in section 3 by a modification of the argument in [6].

Information about periodic orbits for ϕ_t and their holonomy classes may be encoded in a family of functions of a complex variable called L-functions; the properties of these will be key to our subsequent analysis. Given a G-extension $\widehat{\phi_t}$ of a hyperbolic flow ϕ_t and unitary representation $R_\chi : G \to U(n)$ we can associate an L-function:

$$L(s,\chi) = \prod_\tau \det\left(1 - R_\chi([\tau]) e^{-shl(\tau)}\right)^{-1} \quad \text{for } s \in \mathbb{C},$$

where the product is taken over all prime periodic orbits for ϕ_t. This converges provided $Re(s) > 1$. In the particular case that $G = SO(2)$ this takes the simpler form

$$L(s,\chi) = \prod_\tau \det\left(1 - \chi([\tau]) e^{-shl(\tau)}\right)^{-1} \quad \text{for } s \in \mathbb{C}.$$

The proof of Theorem 1 depends on the following results on the analytic domain of the L-functions.

Lemma 1.6.

(1) *There exists $\epsilon > 0$ such that $L(s,\chi)$ has a non-zero meromorphic extension to a half-plane $Re(s) > 1 - \epsilon$.*

(2) *There exist constants $c, t_0, \gamma, \alpha > 0$ such that $L(s,\chi)$ is analytic in a region $Re(s) > 1 - c|Im(s)|^{-\gamma}$, $|Im(s)| \geq t_0$ with a bound*

$$\left|\frac{L'(s,\chi)}{L(s,\chi)}\right| = O(|Im(s)|^{\alpha})$$

Proof. The proof of part (1) is easily established using standard techniques. In particular, we need only obvious modifications to the proof of Lemma 3 in [11]. The bound in part (2) comes from studying the symbolic version of the L-function defined by

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Z_n(s,\chi)\right)$$

where

$$Z_n(s,\chi) := \sum_{\sigma^n x = x} \chi(\Theta^n(x)) e^{-shr^n(x)}.$$

Using the bounds on the transfer operator in Proposition 1.5 we can bound

$$Z_n(s,\chi) = O\left(e^{nP(-Re(s)hr)}|Im(s)|\left(1 - \frac{1}{|Im(s)|^{\gamma}}\right)^{[n/N]}\right)$$

for $n \geq 1$. The argument is very similar to that for zeta functions, cf. the proof of Proposition 3 in [14]. We can then derive the bound on the logarithmic derivative of the L-function using an analogous method to that used for zeta functions, cf. Proposition 4 in [14] □

2. Proof of Theorem 1

For ease of notation, it will be convenient to consider non-prime periodic orbits $\tau' = \tau^n$, where τ is prime, and let $l(\tau') = nl(\tau)$, $[\tau'] = [\tau]^n$ and $\Lambda(\tau') = l(\tau)$. With this notation the logarithmic derivative $L'(s,\chi)/L(s,\chi)$ may be written

$$\frac{L'(s,\chi)}{L(s,\chi)} = -h \sum_{\tau'} \Lambda(\tau')\chi([\tau'])e^{-shl(\tau')},$$

for $Re(s) > 1$.

We shall apply the following identity [9, p.31]:

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds = \begin{cases} 0 & \text{for } x \leq 1 \\ \frac{1}{k!}(1-1/x)^k & \text{for } x \geq 1, \end{cases}$$

valid for any $d > 0$. We shall choose $k > \alpha$ (where α is the exponent appearing in Lemma 1.6) and $d > 1$. Then we can integrate $L'(s,\chi)/L(s,\chi)$ term by term to get

$$\sum_{e^{hl(\tau')} \leq x} \chi([\tau'])\Lambda(\tau')(x - e^{l(\tau')})^k = \frac{-k!}{2\pi i h}\int_{d-i\infty}^{d+i\infty} \frac{L'(s,\chi)}{L(s,\chi)} \frac{x^{s+k}}{s(s+1)\cdots(s+k)} ds. \tag{2.1}$$

We can use this to deduce the following.

Lemma 2.1. *There exists $\beta > 0$ such that*

$$\sum_{e^{hl(\tau')}\le x} \chi([\tau'])\Lambda(\tau')(x-e^{hl(\tau')})^k = O(x^{k+1}/(\log x)^\beta)$$

Proof. Using the identity (2.1), we can move the line of integration to a piecewise linear curve connecting $d+i\infty$, $d+iR$, $c+iR$, $c-iR$, $d-iR$, $d-i\infty$, where $c = h - R^{-\gamma}$. We can bound the integrals over the piecewise linear pieces by:

$$\begin{aligned}\left|\frac{k!}{2\pi ih}\int_{d\pm iR}^{d\pm i\infty}\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^{s+k}}{s(s+1)\cdots(s+k)}ds\right| &= O\left(x^{d+k}\int_R^\infty \frac{1}{t^{k-\alpha+1}}dt\right)\\ &= O\left(x^{d+k}\frac{1}{R^{k-\alpha+1}}\right),\end{aligned}$$

$$\left|\frac{k!}{2\pi ih}\int_{c\pm iR}^{d\pm iR}\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^{s+k}}{s(s+1)\cdots(s+k)}ds\right| = O\left(x^{c+k}\frac{1}{R^{k-\alpha+1}}dt\right),$$

$$\begin{aligned}\left|\frac{k!}{2\pi ih}\int_{c-iR}^{c+iiR}\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^{s+k}}{s(s+1)\cdots(s+k)}ds\right| &= O\left(x^{c+k}\int_1^R \frac{1}{t^{k-\alpha+1}}dt\right)\\ &= O\left(x^{c+k}\right).\end{aligned}$$

If we let $d = 1 + (\log x)^{-1}$ and $R = (\log x)^K$ for any $0 < K < 1/\gamma$ then we see that the bound in the statement holds for any $\beta < (k-\alpha+1)/\gamma$. $\square$

We would like to derive an asymptotic expression for the summatory function

$$\sum_{e^{hl(\tau)}\le x}\chi([\tau])l(\tau),$$

where the sum is now taken over prime orbits. First, we remove the multiple orbits from the estimate in Lemma 2.1. We have

$$\begin{aligned}\sum_{e^{hl(\tau')}\le x}\chi([\tau'])\Lambda(\tau')(x-e^{hl(\tau')})^k = &\sum_{e^{hl(\tau)}\le x}\chi([\tau])l(\tau)(x-e^{hl(\tau)})^k\\ &+\sum_{n\ge 2}\sum_{e^{hnl(\tau)}\le x}\chi([\tau^n])l(\tau)(x-e^{hnl(\tau)})^k.\end{aligned}\tag{2.2}$$

Let N_0 be a lower bound for the numbers $e^{hl(\tau)}$; then $e^{hnl(\tau)} \le x$ implies that $n \le (\log x)/N_0$. Thus there are only $O(\log x)$ terms in the second expression on the Right Hand Side of (2.2) and we have the estimate

$$\sum_{n\ge 2}\sum_{e^{hnl(\tau)}\le x}\chi([\tau^n])l(\tau)(x-e^{hnl(\tau)})^k = O(x^{k+1/2}(\log x)^2).$$

Thus we only need to consider prime periodic orbits τ.

Assume first that $k \geq 2$. For any $\delta > 0$ we can write

$$\sum_{e^{hl(\tau)} \leq x(1+\delta)} \chi([\tau]) l(\tau) (x(1+\delta) - e^{hl(\tau)})^k - \sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})^k$$
$$= \sum_{x \leq e^{hl(\tau)} \leq x(1+\delta)} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})^k + kx\delta \sum_{e^{l(\tau)} \leq x} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})^{k-1}$$
$$+ \sum_{l=2}^{k} (x\delta)^l \frac{k!}{l!(k-l)} \sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})^{k-l}$$

By Lemma 2.1 the Left Hand Side of this expression is $O(x^{k+1}/(\log x)^\beta)$. Since we have the basic asymptotic

$$\sum_{x \leq e^{hl(\tau)} \leq x(1+\delta)} l(\tau) \sim \delta x$$

the first term on the Right Hand Side is $O\left((x\delta)^{k+1}\right)$, and the lth term in the final summation is $O(x^{k+1}\delta^l)$. Dividing through by $kx\delta$ gives

$$\sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})^{k-1} = O\left(\frac{x^k}{\delta(\log x)^\beta}, \delta x^k\right).$$

Thus if we choose $\delta = 1/(\log x)^{\beta/2}$, say, then we have that

$$\sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})^{k-1} = O(x^k/(\log x)^{\beta/2}).$$

Proceeding inductively, we can can assume the asymptotic in the Lemma 2.1 in the case $k = 1$. Then for any $\delta > 0$ let us write

$$\sum_{e^{hl(\tau)} \leq x(1+\delta)} \chi([\tau]) l(\tau) (x(1+\delta) - e^{hl(\tau)}) - \sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) (x - e^{hl(\tau)})$$
$$= \delta x \sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) - \sum_{x \leq e^{hl(\tau)} \leq x(1+\delta)} \chi([\tau]) l(\tau) (x(1+\delta) - e^{hl(\tau)})$$

We can use the Lemma to bound the Left Hand Side by $O(x^2/(\log x)^\beta)$. The last term on the Right Hand Side is bounded by

$$\sum_{x \leq e^{hl(\tau)} \leq x(1+\delta)} \chi([\tau]) l(\tau) (x(1+\delta) - e^{hl(\tau)}) = O\left(\delta^2 x^2\right)$$

using that $\sum_{x \leq e^{hl(\tau)} \leq x(1+\delta)} l(\tau) = O(\delta x)$. In particular, rearranging the terms and dividing by δx we see that

$$\sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) = O\left(\frac{1}{\delta} \frac{x}{(\log x)^\beta}, \delta x\right)$$

If we choose $\delta = 1/(\log x)^{\beta/2}$ then this becomes

$$\sum_{e^{hl(\tau)} \leq x} \chi([\tau]) l(\tau) = O\left(\frac{x}{(\log x)^{\beta/2}}\right)$$

or, equivalently,

$$\psi_\chi(T) := \sum_{l(\tau) \leq T} \chi([\tau]) l(\tau) = O\left(\frac{e^{hT}}{T^{\beta/2}}\right).$$

Now write $\pi_\chi(T) = \sum_{l(\tau) \leq T} \chi([\tau])$ and observe that

$$\begin{aligned} \pi_\chi(T) &= \int_1^T \frac{1}{u}\, d\psi_\chi(u) + O(1) \\ &= \left[\frac{\psi_\chi(u)}{u}\right]_1^T + \int_1^T \psi_\chi(u) \frac{d}{du}\left(-\frac{1}{u}\right) du + O(1). \end{aligned}$$

We deduce from this that

$$\pi_\chi(T) = O\left(\frac{e^{hT}}{T^{1+\delta}}\right),$$

for some $\delta > 0$. Finally, we note that

$$\frac{1}{\pi(T)} \sum_{l(\tau) \leq T} \chi([\tau]) = \frac{\pi_\chi(T)}{\pi(T)} = O\left(\frac{1}{T^{1+\delta}}\right).$$

This completes the proof of Theorem 1.

3. Proof of Lemma 1.5

Let $\|\cdot\|$ be the strong norm on $C^\alpha(X_A^+, \mathbb{C})$ given by $\|\cdot\| = |\cdot|_\alpha + \|\cdot\|_\infty$, where

$$|w|_\alpha = \sup\left\{\frac{|w(x) - w(y)|}{d(x,y)^\alpha} \,:\, x \neq y, \text{ for } x, y \in X_A^+\right\}$$

and $\|w\|_\infty = \sup\left\{|w(x)| \,:\, x \in X_A^+\right\}$.

We shall write $s \in \mathbb{C}$ in terms of its real and imaginary parts as $s = \sigma + it$. We can assume, by adding a coboundary to $-\sigma r$, if necessary, that $\mathcal{L}_\sigma 1 = e^{P(-\sigma r)} 1$, and by replacing $-\sigma r$ by $-\sigma r - P(-\sigma r)$ we can further assume $\mathcal{L}_\sigma 1 = 1$.

With this simplification we now need to prove the following result.

Lemma 3.1. *There exist $\gamma > 0$, $t_0 > 0$, $C_1 > 0$, $N > 0$, $m_0 \geq 1$ such that for all $m \geq m_0$, $|t| > t_0$ we can bound*

$$\|\mathcal{L}_{\sigma+it,\chi}^{2Nm}\| \leq C_1 |t| \left(1 - \frac{1}{|t|^\gamma}\right)^{m-1}. \tag{3.1}$$

For later convenience let $N = N(t) := [B \log t]$, where B will be chosen later. The next lemma follows from the spectral gap for $\mathcal{L}_\sigma$.

Lemma 3.2. *Let μ be an equilibrium measure for $-\sigma r$. There exists $0 < \delta < 1$ such that, for all $0 \leq m \leq 2N$,*

$$\|\mathcal{L}^{2N}_{\sigma+it,\chi} w\|_\infty \leq \int |\mathcal{L}^m_{\sigma+it,\chi} w| d\mu + O(\|\mathcal{L}^m_{\sigma+it,\chi} w\| \delta^{2N-m}).$$

Proof. In fact, δ has only to be chosen larger than the radius of the rest of the spectrum once the maximal eigenvalue has been removed. Given $x \in X_A^+$ we can write that, for $0 \leq n \leq N$ we have

$$\begin{aligned}|\mathcal{L}^{2N}_{\sigma+it,\chi} w(x)| &\leq \mathcal{L}^{2N-m}_\sigma(|\mathcal{L}^n_{\sigma+it,\chi} w|)(x) \\ &= \int |\mathcal{L}^{2N}_{\sigma+it,\chi} w| d\mu + O(\|\mathcal{L}^m_{\sigma+it,\chi} w\| \delta^{2N-m}). \quad \square\end{aligned}$$

The next lemma is a version of the standard Basic Inequality (cf. [12]).

Lemma 3.3. *There exists $C_2 > 0$ and $0 < \theta < 1$ such that*

$$\|\mathcal{L}^n_{\sigma+it,\chi} w\| \leq C_2 |t| \|w\|_\infty + \theta^n |w|_\alpha.$$

The final ingredient is the following.

Lemma 3.4. *Assume that*

(1) $\|w\|_\infty = 1$ *and* $\|w\|_\alpha \leq |t|$; *and*
(2) *there exists* $\gamma_1 > 0$ *and* x *such that* $|\mathcal{L}^{2N}_{\sigma+it,\chi} w(x)| \leq 1 - |t|^{-\gamma_1}$

then there exists $\gamma > 0$ such that

$$\|\mathcal{L}^{2N}_{\sigma+it,\chi} w\|_\infty \leq 1 - \frac{1}{|t|^\gamma}$$

for $|t|$ sufficiently large.

Proof. By Lemma 3.3 and assumption (1), we know that

$$\|\mathcal{L}^n_{\sigma+it,\chi} w\| \leq (C_2 + 1)|t|,$$

for all $n \geq 0$. If $y \in B_n(x) := \{y : y_i = x_i, 0 \leq i \leq n-1\}$, where $n = 1 + \left[\log(2(C_2+1)|t|^{\gamma_1+1}) / \log\theta\right]$ is chosen so that

$$\theta^{n+1} \leq \frac{1}{2(C_2+1)|t|^{\gamma_1+1}} < \theta^n$$

(where $0 < \theta < 1$ is as in Lemma 3.3). Moreover, if $B > 0$ is sufficiently large we can assume $n \leq N$. Then we have that

$$\begin{aligned}&|\mathcal{L}^{2N}_{\sigma+it,\chi} w(y)| \\ &\leq |\mathcal{L}^{2N}_{\sigma+it,\chi} w(x)| + (C_2+1)|t|\theta^n \\ &\leq \left(1 - |t|^{-\gamma_1}\right) + (C_2+1)|t|\theta^n \\ &\leq \left(1 - \frac{1}{2|t|^{\gamma_1}}\right)\end{aligned}$$

by hypothesis (1). Since μ is a Gibbs measures there exists $c > 0$ such that $D = (\|r\|_\infty + c)/|\log\theta|$ satisfies

$$\mu(B_n(x)) \geq \theta^{nD} \geq \left(\frac{1}{2(C_2+1)|t|^{\gamma_1+1}}\right)^D$$

[12]. We then have that

$$\begin{aligned}
&\int |\mathcal{L}^{2N}_{\sigma+it,\chi} w| d\mu \\
&\leq \int_{B_n(x)^c} |\mathcal{L}^{2N}_{\sigma+it,\chi} w| d\mu + \int_{B_n(x)} |\mathcal{L}^{2N}_{\sigma+it,\chi} w| d\mu \\
&\leq 1 - \mu(B_n(x)) \left(1 - \frac{1}{2|t|^{\gamma_1}}\right) \\
&\leq 1 - \left(\frac{1}{2(C_2+1)|t|^{\gamma_1+1}}\right)^D \left(1 - \frac{1}{2|t|^{\gamma_1}}\right) \\
&\leq 1 - \frac{1}{|t|^{\gamma_2}}
\end{aligned}$$

provided $\gamma_2 > D(\gamma_1+1)+\gamma_1$, and $|t|$ is sufficiently large. Thus we see from Lemma 3.3 with $m = N$ that

$$\begin{aligned}
\|\mathcal{L}^{2N}_{\sigma+it,\chi} w\|_\infty &\leq \left(1 - |t|^{-\gamma_2}\right) + O(|t|\delta^N) \\
&\leq 1 - |t|^{-\gamma}
\end{aligned}$$

provided $\gamma > \gamma_2$ and $B > 0$ is chosen sufficiently large (i.e., $B > (1+\gamma)/|\log\delta|$). □

We will prove Lemma 3.1 by contradiction. Therefore, we assume for a contradiction for all $\gamma > 0$ and $C_1 > 0$, there exists $m_k \to +\infty$ and t_k with $|t_k| \to +\infty$ such that

$$\|\mathcal{L}^{m_k}_{\sigma+it_k,\chi}\| > C_1|t_k| \left(1 - \frac{1}{|t_k|^\gamma}\right)^{m_k-1}. \tag{3.2}$$

Assuming that (3.2) holds, we first observe the following first step in the proof of the Lemma 3.1.

Lemma 3.5. *Condition (3.2) implies that there exist $\gamma_3 > 0$, $C_3 > 0$, t_k with $|t_k| \to +\infty$, $w_k \in C^\alpha(X_A^+, \mathbb{C})$ with $\|w_k\|_\infty = 1$, $|w_k|_\alpha \leq |t_k|$ such that*

$$\inf_{x\in X_A^+} |\mathcal{L}^n_{\sigma+it_k,\chi} w_k(x)| \geq C_3 \left(1 - \frac{1}{|t_k|^{\gamma_3}}\right) \tag{3.3}$$

for $0 \leq n \leq N$.

Proof of Lemma 3.5. The proof of this lemma is also by contradiction. Assume for contradiction that (3.3) fails. The contrapositive statement is that for all $\gamma > 0$ and $C > 0$ for all sufficiently large $|t|$ and $w \in C^\alpha(X_A^+, \mathbb{C})$ with $\|w\|_\infty = 1$, $|w|_\alpha \leq t$, there exists $0 \leq n \leq N$ such that

$$\inf_{x\in X_A^+} |\mathcal{L}^n_{\sigma+it,\chi} w_k(x)| \leq C \left(1 - \frac{1}{|t|^\gamma}\right), \tag{3.4}$$

Using Lemma 3.4 we can deduce that there exists $C > 0$ such that

$$\|\mathcal{L}^{2Nm}_{\sigma+it,\chi} w\|_\infty \le C|t| \left(1 - \frac{1}{|t|^\gamma}\right)^m,$$

for all $m \ge 1$. Next we can deduce norm convergence (using Lemma 3.3):

$$\begin{aligned}
&\|\mathcal{L}^{2Nm}_{\sigma+it,\chi} w\| \\
&\le C_2|t|||\mathcal{L}_{\sigma+it,\chi} w||_\infty + \theta^{(m-1)}||\mathcal{L}_{\sigma+it,\chi} w||_\infty \\
&\le C_2|t| \left(\left(1 - \frac{1}{|t|^\gamma}\right)^{m-1} + \theta^{(m-1)}\right) \\
&\le C_2|t| \left(1 - \frac{1}{|t|^{\gamma_3}}\right)^{m-1},
\end{aligned}$$

since $||w|| \ge ||w||_\infty = 1$, for $\gamma_3 > \gamma$ and m sufficiently large. In particular, this contradicts (3.2), and completes the proof. $\square$

With the proof of Lemma 3.5 complete, we can proceed with the proof of Lemma 3.1. We can now assume that (3.3) holds. Fix $0 < \gamma' < \gamma$ and let $A < (\gamma - \gamma')/(\sigma||r||_\infty) < B/2$. Let $n_k = [A \log |t_k|]$ and then $2n_k \le N = N(t_k)$. For the functions w_k, we can write

$$\begin{aligned}
w_k(x) &= R_{k,0}(x) e^{i\theta_0(x)} \\
\mathcal{L}^{n_k}_{\sigma+it_k,\chi} w_k(x) &= R_{k,1}(x) e^{i\theta_1(x)} \\
\mathcal{L}^{2n_k}_{\sigma+it,\chi} w_k(x) &= R_{k,2}(x) e^{i\theta_2(x)},
\end{aligned}$$

where $R_i(\cdot)$, $\theta_i(\cdot)$ $(i = 1, 2, 3)$ are real valued functions corresponding to the moduli and the arguments. In particular, by (3.3) (and since $\mathcal{L}_\sigma 1 = 1$) we have that

$$\begin{aligned}
1 - R_{k,1}(x) &= \sum_{\sigma^{n_k} y = x} e^{-\sigma r^{n_k}(y)} \left(1 - R_{k,0}(y) e^{i\theta_0(x)} e^{i(t_k r^{n_k}(y) + 2\pi\Theta^{n_k}(y))} e^{-i\theta_1(y)}\right) \\
&= O(|t_k|^{-\gamma}).
\end{aligned} \tag{3.5}$$

Lemma 3.6. *Whenever $y \in X_A^+$ satisfies $\sigma^{n_k} y = x$ then*

$$|1 - e^{i\theta_1(y)} e^{i(t_k r^{n_k}(y) + 2\pi\Theta^{n_k}(y))} e^{-i\theta_2(y)}| = O(|t_k|^{-\gamma'})$$

$$|1 - e^{i\theta_0(y)} e^{i(t_k r^{n_k}(y) + 2\pi\Theta^{n_k}(y))} e^{-i\theta_1(y)}| = O(|t_k|^{-\gamma'})$$

Proof. The estimate (3.5) implies that, for each y such that $\sigma^{n_k} y = x$, we have that

$$e^{-\sigma r^{n_k}(y)} |1 - R_{k,0}(y) e^{i\theta_0(x)} e^{i(t_k r^{n_k}(y) + 2\pi\Theta^{n_k}(y))} e^{-i\theta_1(y)}| = O(|t_k|^{-\gamma}).$$

Thus

$$\begin{aligned}
|1 - e^{i\theta_0(x)} e^{i(t_k r^{n_k}(y) + 2\pi\Theta^{n_k}(y))} e^{-i\theta_1(y)}| &= O(e^{n_k \sigma ||r||_\infty} |t_k|^{-\gamma}) \\
&= O(|t_k|^{-\gamma'}).
\end{aligned}$$

Similarly, replacing θ_1 by θ_2, and θ_0 by θ_1 gives the second bound. $\square$

We can now finish the proof of Lemma 3.1. Fix a point z and choose y sufficiently close to z that $|\theta_0(z) - \theta_0(y)| = O(|t_k|^{-\gamma'})$ and $|\theta_1(z) - \theta_1(y)| = O(|t_k|^{-\gamma'})$. Thus

$$\theta_1(y) - \theta_0(y) = Y + O(|t_k|^{-\gamma'}),$$

where $Y = \theta_1(z) - \theta_0(z)$. For n_k large we can choose $y \in \sigma^{-n_k}x$. Thus, by Lemma 3.6,

$$\begin{aligned} &e^{i(t_k r^{n_k}(x) + 2\pi\Theta^{n_k}(x) - \theta_0(x) + \theta_0(\sigma^{n_k}x))} \\ &= e^{i(t_k r^{n_k}(x) + 2\pi\Theta^{n_k}(x) + \theta_1(x) - \theta_1(\sigma^{n_k}x))} \\ &= e^{i(Y + t_k r^{n_k}(y) + 2\pi\Theta^{n_k}(y) + \theta_2(y) - \theta_2(\sigma^{n_k}x) + \theta_1(x))} + O(|t_k|^{-\gamma'}) \\ &= e^{iY} + O(|t_k|^{-\gamma'}). \end{aligned}$$

Let τ_1, τ_2, τ_3 be the three periodic orbits mentioned in the introduction, with least periods l_1, l_2, l_3 and holonomies $e^{2\pi i\Theta_1}, e^{2\pi i\Theta_2}, e^{2\pi i\Theta_3}$. For simplicity, suppose that these correspond to fixed points $\sigma x_i = x_i$, $i = 1, 2, 3$, of the shift map. Evaluating the above expression on these periodic points, we deduce that $t_k n_k l_i + 2\pi n_k \Theta_i - 2\pi m_i + Y + O(|t_k|^{-\gamma'})$, where $m_i \in \mathbb{Z}$. Let $\alpha = (l_2 - l_3)(l_1 - l_2)$ and $\Theta = (\Theta_2 - \Theta_3) + \alpha(\Theta_1 - \Theta_2)$. Let $p = m_2 - m_3$ and $q = m_1 - m_2$. Then $|q\alpha - p + \Theta| = O(|t_k|^{-\gamma'}) = O(q^{-\gamma'})$. This contradicts the inhomogeneous diophantine condition.

Remark. To generalize this approach to more general compact Lie groups G, we need to study the spectra of transfer operators $\mathcal{L}_{s,\chi} : C^\alpha(X_A^+, \mathbb{C}^d) \to C^\alpha(X_A^+, \mathbb{C}^d)$, where $R_\chi : G \to U(d)$ is a unitary representation. The inhomogeneous diophantine condition would then correspond to the existence of periodic orbits $\sigma^{n_i}x_i = x_i$ for which there are no vectors $w_i \in \mathbb{C}^d$ such that $\langle w_i, e^{itr^{n_i}(x_i)} R_\chi(\Theta^{n_i}(x_i)) w_i \rangle = ||w_i|| + O(|t|^{-\gamma})$.

4. Examples with Exponential Bounds

To complete this note we will briefly describe a mechanism that gives faster exponential mixing. We begin with a simple class of examples.

Examples. Let $T : K \to K$ be a C^1expanding map of the circle of degree $d \geq 2$. Assume that $r : K \to \mathbb{R}^+$ is a C^1 function such that for each $x \in K$ we can (locally) choose $y_1 = y_1(x), y_2 = y_2(x)$ such that $T(y_1) = x = T(y_2)$ and such that $R(x) = r(y_1(x)) - r(y_2(x))$ has derivative bounded away from zero. Let $\Theta : K \to SO(2)$ be any C^1 function for which the associated skew product is mixing.

To establish exponential error terms we would need the following stronger version of Lemma 1.6. For the above examples, this may be proved in a similar way to corresponding results for zeta functions associated to negatively curved surfaces [13] (using ideas originating in [5], cf. also [7]).

Lemma 4.1. *Assume there exists $\sigma_1 < h$ such that $L(\sigma + it, \chi)$ is non-zero and analytic in the half-plane $\sigma > \sigma_1$. Furthermore, assume there exists $t_0 > 0$ and $0 < \gamma < 1$ such that, in the region $\sigma > \sigma_1$, $|t| > t_0$,*

$$\left| \frac{L'(\sigma + it, \chi)}{L(\sigma + it, \chi)} \right| = O(|t|^\gamma), \quad |t| \to +\infty.$$

Then there exists $\epsilon > 0$ such that $\pi_\chi(T) = O(e^{-\epsilon T})$ as $T \to +\infty$.

The proof follows from standard arguments in complex analysis. In particular, an effective version of the Perron Formula [17, p.132] may be used to relate $L'(s,\chi)/L(s,\chi)$ to the function $\psi_\chi(T)$ (defined in section 2) and show that it satisfies the estimate $\psi_\chi(T) = O(e^{(h-\epsilon)T})$, for some $\epsilon > 0$. The result then follows easily (with a smaller value of ϵ).

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Mark Pollicott, Department of Mathematics, University of Warwick, Coventry CV4 7AL, UK

E-mail address: `mpollic@maths.warwick.ac.uk`

Richard Sharp, School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK

E-mail address: `sharp@maths.man.ac.uk`