PAIRS OF PERIODIC ORBITS WITH FIXED HOMOLOGY DIFFERENCE

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Abstract. We obtain an asymptotic formula for the number of pairs of closed orbits of a weak-mixing transitive Anosov flow whose homology classes have a fixed difference.

0. Introduction

Consider M a compact smooth Riemannian manifold and $\phi_t : M \to M$ a transitive Anosov flow on M. Such a manifold has a countable infinity of (prime) periodic orbits γ. We denote the length of such an orbit by $l(\gamma)$. Writing $\pi(T) := \#\{\gamma : l(\gamma) \leq T\}$ the following expansion holds: $\pi(T) \sim e^{hT}/hT$, as $T \to +\infty$, where $h > 0$ is the topological entropy of ϕ [11],[12],[14],[15].

To refine the problem one might try to understand the distribution of periodic orbits with respect to the homology of M. To keep our statements simple, we shall suppose that $H_1(M, \mathbb{Z})$ is infinite and ignore any torsion. Suppose that M has first Betti number $k \geq 1$; then we may fix an identification of $H_1(M, \mathbb{Z})$/torsion with $\mathbb{Z}^k$. For $\alpha \in \mathbb{Z}^k$, write $\pi(T, \alpha) := \#\{\gamma : l(\gamma) \leq T, \ [\gamma] = \alpha\}$, where $[\gamma]$ denotes the homology class of γ (modulo torsion). A variety of behaviours for this counting function are possible. For example, for a geodesic flow (in variable negative curvature) there exists $C > 0$ (independent of α) such that

$$\pi(T, \alpha) \sim C \frac{e^{hT}}{T^{1+k/2}}, \text{ as } T \to +\infty \tag{1}$$

[5],[6],[10],[17],[18] but, for other Anosov flows, $\pi(T, \alpha)$ may grow at a slower rate or even be bounded (or identically zero), depending on the circumstances [7],[20]. Nevertheless, if α is allowed to grow with T at an appropriate linear rate then an asymptotic of the form (1) always holds [2],[9].

In this note, we are going to study the *relative distribution of pairs* of closed orbits in $H_1(M, \mathbb{Z})$. For $\beta \in \mathbb{Z}^k$, define

$$\pi_2^{\beta}(T) := \#\{(\gamma, \gamma') : l(\gamma), l(\gamma') \leq T, \ [\gamma] - [\gamma'] = \beta\}. \tag{2}$$

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Since the asymptotic behaviour of $\pi(T,\alpha)$ is different for different types of weak-mixing transitive Anosov flows one might suspect that $\pi_2^\beta(T)$ has varying asymptotic behaviour as well. We show that this is *not* the case. Surprisingly the asymptotic behaviour is universal. Our main result is the following:

Theorem 1. *Let $\phi_t : M \to M$ be a weak-mixing transitive Anosov flow on a compact smooth Riemannian manifold M with first Betti number $k \geq 1$. Then there exists $\mathcal{C}(\phi) > 0$ such that, for each $\beta \in H_1(M,\mathbb{Z})/\text{torsion} \cong \mathbb{Z}^k$,*

$$\pi_2^\beta(T) \sim \mathcal{C}(\phi)\frac{e^{2hT}}{T^{2+k/2}}, \text{ as } T \to +\infty.$$

Remark. The constant $\mathcal{C}(\phi)$ can be described in terms of the Hessian of an associated entropy function at a special point. To be precise

$$\mathcal{C}(\phi) = \frac{1}{2^k \pi^{k/2} \sigma^k h^2}$$

where σ^{2k} is the determinant of minus the Hessian of this entropy function evaluated at the winding cycle associated to the measure of maximal entropy for ϕ. See section 1 for details.

For a compact hyperbolic surface V of genus $g \geq 2$, the geodesic flow on the sphere bundle SV is a weak-mixing transitive Anosov flow with topological entropy equal to 1. Furthermore, the natural projection $p : SV \to V$ induces an isomorphism between $H_1(SV,\mathbb{Z})/\text{torsion}$ and $H_1(V,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. There is a one-to-one correspondence between periodic orbits for the flow and closed geodesics on the surface, which preserves lengths and respects this isomorphism. Thus $\pi_2^\beta(T)$ counts also the number of pairs of closed geodesics on V, with lengths at most T, and with the two homology classes differing by β. We may recover the following result, previously obtained (using a different method) by the first author in the unpublished preprint [19], which this paper supersedes.

Theorem 2. *Let V be a compact hyperbolic surface of genus g. Then, for each $\beta \in H_1(V,\mathbb{Z}) \cong \mathbb{Z}^{2g}$,*

$$\pi_2^\beta(T) \sim \frac{(g-1)^g}{2^g}\frac{e^{2T}}{T^{2+g}}, \text{ as } T \to +\infty.$$

In the next section, we shall describe the necessary background on Anosov flows, periodic orbits and homology. In section 2, we shall prove Theorems 1 and 2.

1. Anosov Flows and Homology

A C^1 flow $\phi_t : M \to M$ on a smooth compact Riemannian manifold M is called an Anosov flow if the tangent bundle admits a continuous splitting $TM = E^0 \oplus E^s \oplus E^u$, where E^0 is the 1-dimensional bundle tangent to the flow trajectories and where there exist constants $C > 0$ and $\lambda > 0$ such that

(1) $\|D\phi_t(v)\| \leq Ce^{-\lambda t}\|v\|$, for all $v \in E^s$ and $t \geq 0$; and
(2) $\|D\phi_{-t}(v)\| \leq Ce^{-\lambda t}\|v\|$, for all $v \in E^u$ and $t \geq 0$.

We need some background on periodic orbits and homology for Anosov flows. (For more details, see the second author's survey in [12].) Let $\mathcal{M}_\phi$ denote the set of all ϕ_t-invariant probability measures on M and, for $\mu \in \mathcal{M}_\phi$, let $\Phi_\mu \in H_1(M,\mathbb{R})$ denote the associated winding cycle, defined by the duality

$$\langle \Phi_\mu, [\omega] \rangle = \int \omega(\mathcal{X}) d\mu,$$

where $[\omega]$ is the de Rham cohomology class of a closed 1-form ω and $\mathcal{X}$ is the vector field generating ϕ_t. Write $\mathcal{B}_\phi = \{\Phi_\mu : \mu \in \mathcal{M}_\phi\} \subset H_1(M,\mathbb{R})$. (For geodesic flows, $\mathcal{B}_\phi$ is the unit ball for the Gromov-Federer stable norm on homology [13].) The identification $H_1(M,\mathbb{R}) \cong \mathbb{R}^k$ defines a topology on $H_1(M,\mathbb{R})$ by considering the standard topology on $\mathbb{R}^k$ and this induces a topology on $\mathcal{B}_\phi$ also.

Let $\mu_0 \in \mathcal{M}_\phi$ denote the measure of maximal entropy for ϕ_t, i.e., the unique $\mu_0 \in \mathcal{M}_\phi$ for which the measure theoretic entropy $h_\phi(\mu_0)$ is equal to the topological entropy h, and write $\Phi_0 = \Phi_{\mu_0}$; this winding cycle will play a particularly important role.

Let $\mathfrak{p} : H^1(M,\mathbb{R}) \cong \mathbb{R}^k$ be the pressure function defined by the formula $\mathfrak{p}([\omega]) = P(\omega(\mathcal{X})) = \sup\{h_\phi(\mu) + \langle \Phi_\mu, [\omega]\rangle : \mu \in \mathcal{M}_\phi\}$. The interior of $\mathcal{B}_\phi$ may be identified with the set $\{\nabla p(\xi) : \xi \in H^1(M,\mathbb{R})\}$ [2, page 19]. Furthermore, $\nabla p(0) = \int \omega(\mathcal{X}) d\mu_0 = \Phi_0$ [2, page 30], so that Φ_0 lies in the interior of $\mathcal{B}_\phi$.

There is a (real analytic) entropy function $\mathfrak{h} : \mathrm{int}(\mathcal{B}_\phi) \to \mathbb{R}$ defined by

$$\mathfrak{h}(\rho) = \sup\{h_\phi(\mu) : \Phi_\mu = \rho\}.$$

In view of the variational principle $h = \sup\{h_\phi(\mu) : \mu \in \mathcal{M}_\phi\}$, $\mathfrak{h}(\Phi_0) = h$ and if $\rho \neq \Phi_0$ then $\mathfrak{h}(\rho) < h$; in particular, $\nabla \mathfrak{h}(\Phi_0) = 0$. In fact, it is a well-known result that $\mathfrak{h}$ is strictly concave and that $\mathcal{H} = -\nabla^2 \mathfrak{h}(\Phi_0)$ is positive definite. Define a norm $\|\cdot\|$ on $H_1(M,\mathbb{R}) \cong \mathbb{R}^k$ by $\|\rho\|^2 = \langle \rho, \mathcal{H}\rho\rangle$. In particular,

$$\mathfrak{h}(\Phi_0 + \rho) = h - \|\rho\|^2/2 + O(\|\rho\|^3). \tag{3}$$

when $\|\rho\|$ is sufficiently small. Also define $\sigma > 0$ by $\sigma^{-2k} = \det \mathcal{H}$. We note that since $H_1(M,\mathbb{R})$ has finite dimension as a real vector space the norm $\|\cdot\|$ induces the same topology as the one previously considered. The function $\mathfrak{p}$ is Legendre conjugate of the function $-\mathfrak{h}$. In particular, if we set $\xi(\rho) = (\nabla \mathfrak{p})^{-1}(\rho)$, then $\xi(\Phi_0) = 0$.

Remark. The above analysis only applies directly when ϕ is a $C^{1+\epsilon}$ flow, so that the functions $\omega(\mathcal{X})$ are Hölder continuous. For the modifications we required for a flow which is only C^1, see [3].

As in the introduction, for $\alpha \in H_1(M,\mathbb{Z})/\text{torsion}$, we write $\pi(T,\alpha) = \#\{\gamma : l(\gamma) \leq T,\ [\gamma] = \alpha\}$. Now, however, we shall allow α to depend on T (in a linear way). To continue to take values in $H_1(M,\mathbb{Z})$, we shall define an "integer part" on $H_1(M,\mathbb{R})$. Choose a fundamental domain $\mathcal{F}$ for $H_1(M,\mathbb{Z})/\text{torsion}$ as a lattice inside $H_1(M,\mathbb{R})$. Then, for $\rho \in H_1(M,\mathbb{R})$, define $\lfloor \rho \rfloor \in H_1(M,\mathbb{Z})$ by $\rho - \lfloor \rho \rfloor \in \mathcal{F}$.

Proposition 1 [2],[9],[10]. *Let $\phi_t : M \to M$ be a weak-mixing transitive Anosov flow. If $\rho \in \mathrm{int}(\mathcal{B}_\phi)$ and $\alpha_0 \in H_1(M,\mathbb{Z})/\mathrm{torsion}$ then*

$$\pi(T, \alpha_0 + \lfloor \rho T \rfloor) \sim C(\rho) e^{-\langle \xi(\rho), \alpha_0 \rangle} e^{\langle \xi(\rho), T\rho - \lfloor T\rho \rfloor \rangle} \frac{e^{\mathfrak{h}(\rho)T}}{T^{k/2+1}}, \qquad \text{as } T \to +\infty,$$

where $C(\rho) = (\det \nabla^2 \mathfrak{h}(\rho))^{1/2}/((2\pi)^{k/2}\mathfrak{h}(\rho)) > 0$, uniformly for ρ in compact subsets of $\mathrm{int}(\mathcal{B}_\phi)$.

To put this in context, let us consider a fixed homology class α. Suppose first that $0 \in \mathrm{int}(\mathcal{B}_\phi)$; then

$$\pi(T, \alpha) \sim C(0) \frac{e^{\mathfrak{h}(0)T}}{T^{1+k/2}}, \quad \text{as } T \to +\infty$$

[20]. On the other hand, if $0 \notin \mathcal{B}_\phi$ then ϕ_t has a global cross section and there are at most finitely may orbits in each fixed class [4]. If $0 \in \partial \mathcal{B}_\phi$, the situation is not well understood and the growth of $\pi(T,\alpha)$ may be polynomial [1] or exponential. Regardless of these considerations, Proposition 1 gives a universal asymptotic formula for the number of periodic orbits in homology classes which grow like $\Phi_0 T$. To simplify notation, we write

$$\widetilde{\pi}_\alpha(T) = \pi(T, \alpha + \lfloor \Phi_0 T \rfloor).$$

We have the following corollaries of Proposition 1:

Corollary 1. *For $\delta > 0$ sufficiently small,*

$$\lim_{T \to +\infty} \sup_{\|\alpha\| \le \delta T} \left| \frac{T^{k/2+1} \widetilde{\pi}_\alpha(T) e^{\langle \xi(\Phi_0 + \alpha/T), T\Phi_0 - \lfloor T\Phi_0 \rfloor \rangle}}{C(\Phi_0 + \alpha/T) e^{\mathfrak{h}(\Phi_0 + \alpha/T)T}} - 1 \right| = 0.$$

This follows from Proposition 1 by using uniformity when setting $\alpha_0 = 0$ and $\rho = \Phi_0 + \alpha/T$. Since Φ_0 is an interior point of $\mathcal{B}_\phi$ such ρ's are in a compact subset of $\mathcal{B}_\phi$ for δ sufficiently small. The following version of the Central Limit Theorem also holds.

Corollary 2. *For a Jordan set $B \subset \mathbb{R}^k$ whose boundary has zero measure,*

$$\lim_{T \to +\infty} \frac{1}{\pi(T)} \# \left\{ \gamma : l(\gamma) \le T, \ \frac{[\gamma] - \lfloor \Phi_0 T \rfloor}{\sqrt{T}} \in B \right\} = \frac{1}{(2\pi)^{k/2} \sigma^k} \int_B e^{-\|x\|^2/2} dx$$

This is straightforward to derived from Lemma 1 below, which in turn follows from Corollary 1.

2. Proof of Theorems 1 and 2

We now proceed to the proof of Theorem 1. Our argument will be based on the simple yet powerful observation that equation (2) may be replaced by

$$\pi_2^\beta(T) = \sum_{\alpha \in \mathbb{Z}^k} \widetilde{\pi}_\alpha(T) \widetilde{\pi}_{\alpha+\beta}(T) \tag{4}$$

and the properties of $\widetilde{\pi}_\alpha(T)$ contained in Corollaries 1 and 2. In particular, we shall use Corollary 1 to understand $\widetilde{\pi}_\alpha(T)$ for $\|\alpha\| = O(\sqrt{T})$ and Corollary 2 to show that the remaining terms make a negligible contribution.

Our first lemma, shows that in the range $\|\alpha\| = O(\sqrt{T})$, $\widetilde{\pi}_\alpha(T)$ is well approximated by a simpler function than the one given in Corollary 1.

Lemma 1. *For any $\Delta > 0$,*

$$\sup_{\|\alpha\| \le \Delta\sqrt{T}} \left| \frac{hT\widetilde{\pi}_\alpha(T)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{k/2}\sigma^k T^{k/2}} \right| = o\left(\frac{1}{T^{k/2}}\right).$$

Proof. Provided T is sufficiently large, $\Delta\sqrt{T} \le \delta T$, so it follows from Corollary 1 that

$$\sup_{\|\alpha\| \le \Delta\sqrt{T}} \left| \frac{T\widetilde{\pi}_\alpha(T) e^{\langle \xi(\Phi_0+\alpha/T), T\Phi_0 - \lfloor T\Phi_0 \rfloor\rangle}}{C(\Phi_0+\alpha/T)e^{\mathfrak{h}(\Phi_0+\alpha/T)T}} - \frac{1}{T^{k/2}} \right| = o\left(\frac{1}{T^{k/2}}\right).$$

We have $e^{\langle \xi(\Phi_0+\alpha/T), T\Phi_0 - \lfloor T\Phi_0 \rfloor\rangle} = 1 + O(T^{-1/2})$ when $\|\alpha\| \le \Delta\sqrt{T}$ so

$$\sup_{\|\alpha\| \le \Delta\sqrt{T}} \left| \frac{T\widetilde{\pi}_\alpha(T)}{C(\Phi_0+\alpha/T)e^{\mathfrak{h}(\Phi_0+\alpha/T)T}} - \frac{1}{T^{k/2}} \right| = o\left(\frac{1}{T^{k/2}}\right).$$

Note that $C(\Phi_0) = ((2\pi)^{k/2}\sigma^k h)^{-1}$. Since the entropy function $\mathfrak{h}$ is real analytic, we have for $\|\alpha\| \le \Delta\sqrt{T}$,

(i) $|C(\Phi_0+\alpha/T) - C(\Phi_0)| = O(T^{-1/2})$, and, using (3),
(ii) $\mathfrak{h}(\Phi_0+\alpha/T)T = hT - \|\alpha\|^2/2T + O(T^{-1/2})$,

we may replace this by

$$\sup_{\|\alpha\| \le \Delta\sqrt{T}} \left| \frac{hT\widetilde{\pi}_\alpha(T)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}e^{q(\alpha,T)}}{(2\pi)^{k/2}\sigma^k T^{k/2}} \right| = o\left(\frac{1}{T^{k/2}}\right),$$

where $e^{q(\alpha,T)} \in (e^{-cT^{-1/2}}, e^{cT^{-1/2}})$, for some $c > 0$. The result follows by using that $e^{q(\alpha,T)} = 1 + O(T^{-1/2})$. □

We may then use Lemma 1 to find good approximations for $\sum \widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)$ where the sum is over $\|\alpha\| \le \Delta\sqrt{T}$.

Lemma 2. *For any $\Delta > 0$,*

$$\lim_{T\to+\infty} \sum_{\|\alpha\| \le \Delta\sqrt{T}} \left(\frac{(2\pi)^k \sigma^{2k} h^2 T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{e^{2hT}} - \frac{e^{-\|\alpha\|^2/2T}e^{-\|\alpha+\beta\|^2/2T}}{T^{k/2}} \right) = 0.$$

Proof. To shorten some of our formulae, we shall write $e_T(\alpha) = e^{-\|\alpha\|^2/2T}$. We have

$$\begin{aligned} &\left| \frac{T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)^2 e^{2hT}} - \frac{e_T(\alpha)e_T(\alpha+\beta)}{T^{k/2}} \right| \\ \le &\left| \frac{T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)^2 e^{2hT}} - \frac{Te_T(\alpha)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)e^{hT}} \right| + \left| \frac{Te_T(\alpha)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)e^{hT}} - \frac{e_T(\alpha)e_T(\alpha+\beta)}{T^{k/2}} \right|. \end{aligned}$$

Applying Lemma 1, the terms on the Right Hand Side satisfy the estimates

$$o\left(\frac{T\widetilde{\pi}_{\alpha+\beta}(T)}{e^{hT}}\right) = o\left(\frac{1}{T^{k/2}}\right) \quad \text{and} \quad o\left(\frac{e_T(\alpha)}{T^{k/2}}\right) = o\left(\frac{1}{T^{k/2}}\right),$$

respectively, uniformly for $\|\alpha\| \leq \Delta\sqrt{T}$. Summing over $\|\alpha\| \leq \Delta\sqrt{T}$ gives the result. □

Note that, given $\epsilon > 0$, it is possible to choose $\Delta > 0$ sufficiently large that

$$\frac{1}{(2\pi)^{k/2}\sigma^k} \int_{\|x\|>\Delta} e^{-\|x\|^2/2} dx < \epsilon. \tag{5}$$

From Lemma 2 it is clear that we need to understand the asymptotic behaviour of

$$\sum_{\|\alpha\|\leq\Delta\sqrt{T}} e^{-\|\alpha\|^2/2T} e^{-\|\alpha+\beta\|^2/2T}.$$

This behaviour is found in the next lemma:

Lemma 3. *Given $\epsilon > 0$, provided Δ is sufficiently large we have*

$$\pi^{k/2}\sigma^k(1-\epsilon) \leq \lim_{T\to+\infty} \frac{1}{T^{k/2}} \sum_{\|\alpha\|\leq\Delta\sqrt{T}} e^{-\|\alpha\|^2/2T} e^{-\|\alpha+\beta\|^2/2T} \leq \pi^{k/2}\sigma^k(1+\epsilon).$$

Proof. Note that

$$\begin{aligned}\sum_{\|\alpha\|\leq\Delta\sqrt{T}} e^{-\|\alpha\|^2/2T} e^{-\|\alpha+\beta\|^2/2T} &= \sum_{\|\alpha\|\leq\Delta\sqrt{T}} e^{-\|\alpha\|^2/T} e^{-(2\langle\alpha,\mathcal{H}\beta\rangle+\|\beta\|^2)/2T} \\ &= \sum_{\|\alpha\|\leq\Delta\sqrt{T}} e^{-\|\alpha\|^2/T} \left(1 + O\left(\frac{1}{\sqrt{T}}\right)\right).\end{aligned}$$

Since $\int_{\mathbb{R}^k} e^{-\langle x,\mathcal{H}x\rangle} dx = \pi^{k/2}/\sqrt{\det\mathcal{H}}$, applying Lemma 2 of [3] or the proof of Lemma 2.10 in [16] gives

$$\lim_{T\to+\infty} \frac{1}{\pi^{k/2}\sigma^k T^{k/2}} \sum_{\alpha\in\mathbb{Z}^k} e^{-\|\alpha\|^2/T} = 1.$$

Choosing Δ sufficiently large that (5) is satisfied (and since $e^{-\|x\|^2} \leq e^{-\|x\|^2/2}$) we also have

$$\lim_{T\to+\infty} \frac{1}{\pi^{k/2}\sigma^k T^{k/2}} \sum_{\|\alpha\|>\Delta\sqrt{T}} e^{-\|\alpha\|^2/T} = \frac{1}{\pi^{k/2}\sigma^k} \int_{\|x\|>\Delta} e^{-\|x\|^2} dx < \epsilon.$$ □

In order to complete the proof we need a uniform upper bound on $\widetilde{\pi}_\alpha(T)$ in the range where Proposition 1 gives no information. This is provided by the following lemma:

Lemma 4. *There exists $B > 0$ such that*

$$\widetilde{\pi}_\alpha(T) \le B \frac{e^{hT}}{T^{1+k/2}}$$

for all $\alpha \in \mathbb{Z}^k$ and $T > 0$.

Proof. By Corollary 1, if we fix $\delta > 0$ sufficiently small then there exists $T_0 > 0$ such that, for $T \ge T_0$ and $\|\alpha\| \le \delta T$,

$$\widetilde{\pi}_\alpha(T) \le \frac{2C(\Phi_0 + \alpha/T)}{e^{\langle \xi(\Phi_0+\alpha/T), T\Phi_0 - \lfloor T\Phi_0 \rfloor \rangle}} \frac{e^{\mathfrak{h}(\Phi_0+\alpha/T)T}}{T^{1+k/2}} \le B_0 \frac{e^{hT}}{T^{1+k/2}},$$

where

$$B_0 = 2 \sup \left\{ \frac{C(\Phi_0 + \rho)}{e^{\langle \xi(\Phi_0+\rho), \rho' \rangle}} : \|\rho\| \le \delta,\ \rho' \in \mathcal{F} \right\}.$$

To obtain the bound for $\|\alpha\| > \delta T$ we use large deviations theory. For a periodic orbit γ, let μ_γ denote the normalized Lebesgue measure around γ, i.e.,

$$\int f \, d\mu_\gamma = \frac{1}{l(\gamma)} \int_0^{l(\gamma)} f(\phi_t x_\gamma) dt,$$

for any $x_\gamma \in \gamma$. We may choose closed 1-forms $\omega_1, \dots, \omega_k$ such that

$$\frac{[\gamma]}{l(\gamma)} = \left(\int \omega_1(\mathcal{X}) \, d\mu_\gamma, \dots, \int \omega_k(\mathcal{X}) \, d\mu_\gamma \right). \tag{6}$$

Define a set $\mathcal{K} \subset \mathcal{M}_\phi$ by

$$\mathcal{K} = \left\{ \mu \in \mathcal{M}_\phi : \left\| \left(\int \omega_1(\mathcal{X}) \, d\mu_\gamma, \dots, \int \omega_k(\mathcal{X}) \, d\mu_\gamma \right) - \Phi_0 \right\| \ge \frac{\delta}{2} \right\};$$

this is weak* compact. By Theorem 2.1 of [8],

$$\limsup_{T \to +\infty} \frac{1}{T} \log \#\{\gamma : l(\gamma) \le T,\ \mu_\gamma \in \mathcal{K}\} \le h_{\mathcal{K}} := \sup_{\mu \in \mathcal{K}} h_\phi(\mu).$$

Furthermore, since $\mu_0 \notin \mathcal{K}$, $h_{\mathcal{K}} < h$.

Recall that $\mathcal{F}$ is a fundamental domain for $H_1(M, \mathbb{Z})$/torsion in $H_1(M, \mathbb{R})$ and let D denote its diameter with respect to $\| \cdot \|$. Choose $0 < \theta < 1$ and note that

$$\sum_{\|\alpha\| > \delta T} \widetilde{\pi}_\alpha(T) = \#\{\gamma : \theta T < l(\gamma) \le T, \|[\gamma] - \lfloor T\Phi_0 \rfloor\| > \delta T\} + O(e^{\theta h T}), \tag{7}$$

where the implied constant depends only on θ.

Now consider γ with $\theta T < l(\gamma) \le T$. Then $\|[\gamma] - \lfloor T\Phi_0 \rfloor\| > \delta T$ implies that

$$\begin{aligned} \left\| \frac{[\gamma]}{l(\gamma)} - \Phi_0 \right\| &\ge \left\| \frac{[\gamma] - \lfloor T\Phi_0 \rfloor}{l(\gamma)} \right\| - \left\| \frac{\lfloor T\Phi_0 \rfloor}{l(\gamma)} - \frac{T\Phi_0}{l(\gamma)} \right\| - \left\| \frac{T\Phi_0}{l(\gamma)} - \Phi_0 \right\| \\ &> \delta - \frac{D}{\theta T} - (\theta^{-1} - 1)\|\Phi_0\|. \end{aligned} \tag{8}$$

If we choose θ sufficiently close to 1 and $T_1 > 0$ sufficiently large then we may assume that, provided $T \ge T_1$,

$$\delta - \frac{D}{\theta T} - (\theta^{-1} - 1)\|\Phi_0\| \ge \frac{\delta}{2}. \tag{9}$$

Combining (8) and (9), we obtain the estimate

$$\#\{\gamma : \theta T < l(\gamma) \le T, \|[\gamma] - \lfloor T\Phi_0 \rfloor\| > \delta T\} \le \#\{\gamma : l(\gamma) \le T,\ \mu_\gamma \in \mathcal{K}\}.$$

Applying this to (7), there exists T_2 such that, for $T \ge T_2$ and $\|\alpha\| > \delta T$,

$$\widetilde{\pi}_\alpha(T) \le e^{h\kappa + \epsilon}.$$

Increasing T_2 if necessary, we may also suppose that $e^{h\kappa+\epsilon} \le B_0 e^{hT}/T^{1+k/2}$.

Finally, we may choose $B_1 > 0$ so large such that, for $T \le \max\{T_0, T_1, T_2\}$ and any $\alpha \in \mathbb{Z}^k$, $\widetilde{\pi}_\alpha(T) \le B_1 e^{hT}/T^{1+k/2}$. The proposition is thus proved with $B = \max\{B_0, B_1\}$. □

We now combine the preceding lemmas with Corollary 2 to prove Theorem 1.

Proof of Theorem 1. Given $\epsilon > 0$, choose $\Delta > 0$ so that (5) is satisfied. Consider the sum in equation (4). Lemmas 2 and 3 tell us what happens when this sum is restricted to $\|\alpha\| \le \Delta\sqrt{T}$: we need to consider the remaining terms. By Lemma 4,

$$\sum_{\|\alpha\| > \Delta\sqrt{T}} \frac{T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)^2 e^{2hT}} \le \frac{B}{C(\Phi_0)} \sum_{\|\alpha\| > \Delta\sqrt{T}} \frac{T\widetilde{\pi}_\alpha(T)}{C(\Phi_0)e^{hT}}.$$

Thus, by Corollary 2,

$$\limsup_{T\to+\infty} \sum_{\|\alpha\| > \Delta\sqrt{T}} \frac{T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)^2 e^{2hT}} \le \frac{B}{C(\Phi_0)} \left(\int_{\|x\| > \Delta} e^{-\|x\|^2/2} dx \right) < \frac{B}{C(\Phi_0)}\epsilon.$$

By the above estimate and Lemmas 2 and 3,

$$\begin{aligned} \pi^{k/2}\sigma^k(1-\epsilon) &< \liminf_{T\to+\infty} \sum_{\alpha\in\mathbb{Z}^k} \frac{T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)^2 e^{2hT}} \\ &\le \limsup_{T\to+\infty} \sum_{\alpha\in\mathbb{Z}^k} \frac{T^{2+k/2}\widetilde{\pi}_\alpha(T)\widetilde{\pi}_{\alpha+\beta}(T)}{C(\Phi_0)^2 e^{2hT}} < \pi^{k/2}\sigma^k(1+\epsilon) + \frac{B}{C(\Phi_0)}\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof with

$$\mathcal{C}(\phi) = C(\Phi_0)^2 \pi^{k/2} \sigma^k = \frac{1}{2^k \pi^{k/2} \sigma^k h^2}. \qquad \square$$

Remark. It would be interesting to have a version of Theorem 1 where the asymptotic behaviour was uniform in β. A slightly more careful version of our analysis shows that uniformity holds in the range $\|\beta\| = o(\sqrt{T})$ but this is insufficient for most applications. To obtain a stronger result, one would need a deeper analysis of the sum

$$\sum_{\alpha \in \mathbb{Z}^k} e^{-(\|\alpha\|^2 + \|\alpha+\beta\|^2)/2T}.$$

We conclude by proving Theorem 2.

Proof of Theorem 2. All we need to do is to check that the constant $(g-1)^g/2^g$ is correct. For a compact surface of constant curvature -1 and genus g, $h = 1$ and

$$\frac{1}{(2\pi)^g \sigma^{2g}} = C(\Phi_0) = (g-1)^g$$

[17], so that in this case,

$$\mathcal{C}(\phi) = C(\Phi_0)^2 \pi^g \sigma^{2g} = (g-1)^{2g} \pi^g \sigma^{2g} = \frac{(g-1)^g}{2^g},$$

as required. $\square$

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