LARGE DEVIATIONS FOR MAPS WITH INDIFFERENT FIXED POINTS

Mark Pollicott, Richard Sharp and Michiko Yuri

Manchester University and Sapporo University

Abstract. We establish results on the distribution of pre-images with respect to the absolutely continuous invariant measure for certain systems with indifferent periodic points. This proves to be somewhat more delicate than in the uniformly hyperbolic case. We apply these results to the one dimensional Manneville-Pomeau equation and other one dimensional examples. We also consider higher dimensional analogues.

0. Introduction

The statistical properties of uniformly hyperbolic systems have been extensively studied. In the case of endomorphisms there exist results on the distribution of pre-images. By contrast, systems exhibiting intermittent behaviour are not as well understood [1].

In this paper we shall consider the problem of the distribution of pre-images with respect to the natural smooth measure for certain maps with indifferent fixed or periodic points. In particular, we shall prove large deviation results for pre-images weighted by the derivative. More precisely, these results are upper bounds in the level 2 large deviation principle, in the sense of [4]. If we were to consider pre-images without weightings then this would correspond to large deviations from the measure of maximal entropy. Since results for the unweighted pre-images can easily be deduced from the hyperbolic case by conjugacy we shall not consider these here.

In the interests of clarity we shall concentrate on the special case of the piecewise C^1 interval map $T : [0,1] \to [0,1]$ defined by $x \mapsto x+x^{1+s}$ (mod 1), where $0 < s < 1$. This is the important Manneville-Pomeau map which exhibits the essential features of the one dimensional maps that we shall study (cf. [10], [15], [12]). Observe that

(1) $T(0) = 0$ and $|T'(0)| = 1$
(2) $\forall \epsilon > 0 \ \exists \beta > 1 \ \forall x \in (\epsilon, 1]$ we have $|T'(x)| \geq \beta$
(3) $T : [0, a_0] \to [0,1]$ and $T : (a_0, 1] \to (0,1]$ are C^1 diffeomorphisms, where a_0 satisfies $a_0 + a_0^{1+s} = 1$.

It is observed in [5] that if a_n is the smallest solution to $T^n(a_n) = a_0$, then $a_n = O(n^{-1/s})$. In particular, we have that $\sum_{n=0}^{\infty} |a_n| < +\infty$.

For maps satisfying conditions (1)-(3) above and, in addition, properties (A1) and (A2) in Section 1 there is a finite invariant measure μ which is absolutely continuous with respect to Lebesgue measure λ, though the density is unbounded. (This follows from Lemma 2-4 in Section 1 and Propostion 3 in section 5).

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FIGURE 1

Let $\mathcal{M}$ denote the set of all probability measures on $[0,1]$; $\mathcal{M}$ is compact with respect to the weak* topology. Let δ_0 be the Dirac measure supported on the indifferent fixed point 0 then we denote

$$\mathcal{A} = \{\alpha\mu + (1-\alpha)\delta_0 : 0 \le \alpha \le 1\}.$$

Given any point $y \in [0,1]$ and $n \ge 1$ we denote $\delta_{y,n} = \frac{1}{n}(\delta_y + \delta_{Ty} + \ldots + \delta_{T^{n-1}y}) \in \mathcal{M}$. Our main result is the following upper bound for the level 2 large deviation principle.

Theorem 1. *Let $x \in [0,1]$ and let $\mathcal{U} \subset \mathcal{M}$ be a weak* open neighbourhood of the line segment $\mathcal{A}$. Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \left(\frac{\sum_{\substack{T^n y = x \\ \delta_{y,n} \notin \mathcal{U}}} \frac{1}{|(T^n)'(y)|}}{\sum_{T^n y = x} \frac{1}{|(T^n)'(y)|}} \right) < 0.$$

Remark. This result holds not only for the Manneville-Pomeau map, but also for any Markov interval map which is expanding except, at a finite number of indifferent fixed points, and satisfying the technical conditions which are generalizations of (A1) and (A2).

Theorem 1 gives us a quantative estimate on the distribution of pre-images when they are given the natural weighting. An immediate consequence of this result is the corollary below.

Let us write Δ_n for the weighted average of orbital measures

$$\Delta_n := \frac{\sum_{T^n y = x} \frac{1}{|(T^n)'(y)|} \delta_{y,n}}{\sum_{T^n y = x} \frac{1}{|(T^n)'(y)|}}.$$

Theorem 2. *All the weak* limit points of $\{\Delta_n\}_{n=1}^{\infty}$ lie in $\mathcal{A}$.*

The result corresponding to Theorem 1 is well known for hyperbolic systems, where, in fact, one has a stronger result giving large deviations from the smooth measure alone (i.e. one may replace $\mathcal{A}$ by $\{\mu\}$) [7].

1. Measures and Transfer Operators

We need to introduce some results on thermodynamic formalism. In this section we shall obtain the results we need on the pressure function and its relation to the corresponding transfer operator. These results are true under quite general hypotheses. We shall establish the validity of these hypotheses for the Maneville-Pomeau equation in a series of lemmas (Lemmas 2-4).

We let I denote the disjoint union of the two intervals $[0, a_0]$ and $[a_0, 1]$. Given a continuous function $f : I \to \mathbb{R}$ the *pressure* $P(f)$ is defined by

$$P(f) = \sup\{h_m(T) + \int f dm : m \text{ is a } T\text{-invariant probability measure}\}$$

where $h_m(T)$ denotes the entropy of T with respect to m [13], [17].

Definition. Given any $g \in C^0(I)$ we define a *transfer operator* $\mathcal{L}_g : C^0(I, \mathbb{R}) \to C^0(I, \mathbb{R})$ by

$$\mathcal{L}_g h(x) = \sum_{Ty=x} e^{g(y)} h(y).$$

In the special case $g = -\log |T'|$ we shall write $\mathcal{L} = \mathcal{L}_{-\log |T'|}$.

Using the notation $g^n(y) := g(y) + g(Ty) + \ldots + g(T^{n-1}y)$, we have that

$$\mathcal{L}_g^n h(x) = \sum_{T^n y=x} e^{g^n(y)} h(y).$$

In particular, $\mathcal{L}^n h(x) = \sum_{T^n y = x} \frac{h(y)}{|(T^n)'(y)|}$.

The following lemma gives an upper bound for iterates of $\mathcal{L}_g$ in terms of pressure.

Lemma 1. *For any fixed $x \in [0, 1]$ and any $g \in C^0(I)$ we have that*

$$\limsup_{n \to +\infty} \frac{1}{n} \log \mathcal{L}_g^n 1(x) \le P(g).$$

Proof. This is a purely topological statement. It suffices to observe that $T : I \to I$ is topologically conjugate to the (uniformly expanding) doubling map and that the result is well known there [17].

Let $\psi_0 : [0, 1] \to [0, a_0]$ and $\psi_1 : [0, 1] \to [a_0, 1]$ denote the local inverses to T. In particular, $\psi_0^n : [0, 1] \to [0, a_n]$ is the local inverse to T^n which maps onto the interval furthest to the left (containing 0).

We shall show in the subsequent lemmas that the Manneville-Pomeau map satisfies certain technical conditions which in turn lead to the proofs of Proposition 3 in section 5 and Theorem 1.

Lemma 2. *For the Manneville-Pomeau map we have the following property: the function ψ_0''/ψ_0' is monotone decreasing and*

$$\sup_{x \in [0,1]} |\psi_1''(x)/\psi_1'(x)| \le \psi_0''(1)/\psi_0'(1). \qquad (A1)$$

Proof. This is simply a matter of explicit computation, using the formulae $T'(x) = 1 + (1 + s)x^s$, $T''(x) = (1 + s)sx^{s-1}$.

Lemma 3. *For the Manneville-Pomeau map the following property holds: $\exists C > 0$ such that*

$$\frac{|(\psi_0^n)''(x)|}{|(\psi_0^n)'(x)|} \leq C, \forall x \in [1/2, 1], \forall n \geq 1. \tag{A2}$$

Lemma 4. *Let T be the Manneville-Pomeau map. Let $\phi_{n-1} : [0,1] \to I_{n-1}$ be a local inverse to T^{n-1} (onto an interval I_{n-1}). For $x, y \in [0,1]$ we have*

$$\begin{aligned} &|\log|(\phi_{(n-1)} \circ \psi_1)'(x)| - \log|(\phi_{(n-1)} \circ \psi_1)'(x)|| \\ &\leq |\psi_1'(1)/\psi_0'(1)|^N |\frac{(\psi_0^n)''(\psi_1\eta)}{(\psi_0^n)'(\psi_1\eta)}||x-y| \end{aligned}$$

for all $\eta \in (0,1)$, where N satisfies that for all $n \geq N$ we have

$$|\psi_1'|_\infty < \inf_{x \in \psi_0^n \psi_1[0,1]} |\psi_0'(x)|$$

(cf. (WSN-2) in [21]*).*

Proof. This follows by a direct computation (cf. [21] for more details). Note that in [21] the corresponding notation is $I_{a_1} \cap T^{-1} I_{a_2} \cap \ldots \cap T^{-(n-2)} I_{a_{n-1}}$ (instead of I_{n-1}) and $\psi_{a_1 \ldots a_{n-1}}$ (instead of ϕ_{n-1}).

2. The Rohlin Formula

The main result in this section is the following.

Proposition 1 (Rohlin's formula). *Assume that T is the Manneville-Pomeau transformation (or more generally that (1)-(3) and (A1)-(A2) apply) then*

(1) $P(-\log|T'|) = 0$
(2) *If $m \in \mathcal{A}$ then $h_m(T) = \int \log|T'| dm$, in particular*

$$h_\mu(T) = \int \log|T'| d\mu \tag{A3}$$

(3) *If $m \notin \mathcal{A}$ then $h_m(T) < \int \log|T'| dm$.*

Proposition 1 is the crucial step in the proof of Theorem 1. Most of this section is devoted to its proof. From the definition of pressure, it is clear that (2) and (3) imply (1). We shall proceed to prove (2) and (3).

We associate to the transformation $T : [0,1] \to [0,1]$ and the subinterval $[a_0, 1]$ the induced transformation $T_0 : [a_0, 1] \to [a_0, 1]$. For a point $x \in [a_0, 1]$ we shall define the return time $n(x)$ to be the unique positive integer n such that

$$\begin{cases} T^i(x) \notin [0, a_0] \text{ for } i = 1, \ldots, n-1 \\ T^n(x) \in [0, a_0] \end{cases}.$$

We shall then write $T_0(x) = T^{n(x)}(x)$.

The following results relate the invariant measures of T and T_0.

Lemma 5. *Let $E(T)$ denote the set of ergodic T-invariant probability measures on $[0,1]$ and let $E(T_0)$ denote the set of ergodic T_0-invariant probability measures on $[a_0,1]$. The map $E(T)-\{\delta_0\} \to E(T_0) : m \mapsto m|_{[a_0,1]}/m([a_0,1])$ is injective. (Observe that if $m \in E(T)$ and $m \neq \delta_0$ then $m([a_0,1]) > 0$.)*

Proof. This follows from the Rohlin Tower argument. If m is in $E(T)-\{\delta_0\}$ then it is isomorphic to a measure $\hat{m}$ on the Rohlin Tower over $[a_0,1]$ the measure $\bar{m} = \frac{m|_{[a_0,1]}}{m([a_0,1])}$ is the base measure (i.e. $\hat{m} = d\bar{m} \times d\mathbb{N}$). The only additional consideration is the fixed point 0 (which is the only point the Rohlin tower does not project to). This is why the associated Dirac measure δ_0 has to be added in.

Remark. Unfortunately, Lemma 5 does not give a bijection, because of the problem of *infinite* (i.e. non-normalizable) invariant measures on $[0,1]$.

The key idea of the proof of Proposition 1 is to establish the corresponding result for the induced transformation T_0. In fact replacing T by T_0 then merely scales both sides in Proposition 2 by the constant $\int n(x)d\bar{\mu}(x)$. Thus it suffices to show the equality (A3) with T replaced by T_0 and μ replaced by $\hat{\mu}$.

We need to consider the more general situation in which μ is replaced by more general ergodic measures m. Our aim is to show that there is a strict inequality in (A3) when $m \neq \mu, \delta_0$.

We claim that it suffices to consider the corresponding inequality with T_0 replacing T and $\bar{m}$ replacing m. If $\bar{m}$ is the ergodic T_0-invariant probability measure on $[a_0,1]$ corresponding to the ergodic T-invariant probability measure m on $[0,1]$ (using Lemma 5) then by Abramov's theorem[3] we see that $h(\bar{m}) = \left(\int n(x)d\bar{m}(x)\right)h(m)$, and by Kac's theorem [3] we see that $\int n(x)d\bar{m}(x) = 1$. Moreover, we see that

$$m[a_0,1] \times \left(\int \log|\hat{T}_0'(x)|d\bar{m}(x)\right) = \int \log|T'(x)|dm(x)$$

since we can use the Birkhoff ergodic theorem and the observation that

$$\begin{aligned}\int \log|T'(x)|dm(x) &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N-1} \log|T'| \circ T^i \\ &= \lim_{M\to\infty} \left(\frac{\sum_{i=1}^{M-1} \log|T_0'| \circ T_0^i}{\sum_{i=1}^{M-1} n(T_0^i x)}\right) \\ &= \frac{\int \log|T_0'(x)|d\bar{m}(x)}{\int n(x)d\bar{m}(x)}\end{aligned}$$

for $\bar{m}$-a.e. $x \in [a_0,1]$.

Thus, to show that $h(m) < \int \log|T'(x)|dm$ is equivalent to showing that $h(\bar{m}) < \int \log|T_0'(x)|d\bar{m}$.

We shall also require the following result.

Lemma 6. *The map $T_0 : [a_0,1] \to [a_0,1]$ has an abolutely continuous invariant measure $\bar{\nu}$ such that the Radon-Nikodym derivative is continuous and $0 < C_1 \leq \rho(x) = \frac{d\bar{\nu}(x)}{dx} \leq C_2$.*

Proof. The corresponding result for the so-called jump transformation appears in [5], [15]. The proof for T_0 is similar.

All that remains is to show that the analogous statement to Proposition 1 holds when we replace T by T_0 (and m with $\bar{m}$). We shall follow the approach described in [17]. We can define a Perron-Frobenius operator $\mathcal{M} : C^0([a_0,1]) \to C^0([a_0,1])$ by $\mathcal{M}h(x) = \sum_{T_0 y = x} \frac{h(y)}{|T_0'(y)|}$. It is well known that ρ is an eigenvector for $\mathcal{M}$, corresponding to the eigenvalue 1. Define

$$\log g(x) = -\log |T_0'(x)| - \log \rho(T_0 x) + \log \rho(x)$$

and observe that by the change of variables formula we have that $\sum_{T_0 y = x} g(y) = 1$.

We can write $[a_0, 1] = \cup_{n=1}^{\infty} C_n$ as a union of intervals $C_n = \{x \in [a_0,1] : n(x) = n\}$. These intervals have disjoint interiors and the transformation $T_0 : C_n \to [a_0, 1]$ is a local homeomorphism.

If $\bar{m}$ is any T_0-invariant measure then by [11, Lemma 3.3] we have for a.e. x:

$$\begin{aligned} &\sum_{n=0}^{\infty} \bar{m}(C_n | T_0^{-1}\mathcal{B})(x) \log \bar{m}(C_n | T_0^{-1}\mathcal{B})(x) \\ &\geq \sum_{n=0}^{\infty} \bar{m}(C_n | T_0^{-1}\mathcal{B})(x) \log g(y_n), \end{aligned} \tag{1}$$

where $\{y_n\} = C_n \cap T_0^{-1}(x)$ and $\mathcal{B}$ denotes the Borel σ-algebra. Moreover, by convexity we have equality if and only if $\log g(y_n) = \log \bar{m}(C_n | T_0^{-1}\mathcal{B})(x)$, for $n \geq 0$ [11, Lemma 3.3].

Since

$$h_{\bar{m}}(T_0) = \int \left(\sum_{n=0}^{\infty} \bar{m}(C_n | T_0^{-1}\mathcal{B})(x) \log \bar{m}(C_n | T_0^{-1}\mathcal{B})(x) \right) d\bar{m}(x)$$

and

$$\int \log g(x) d\bar{m}(x) = \int \left(\sum_{n=0}^{\infty} \bar{m}(C_n | T_0^{-1}\mathcal{B})(x) \log g(y_n) \right) d\bar{m}(x)$$

we see that $h_{\bar{m}}(T_0) = \int \log g d\bar{m}$ implies an equality in (1) almost everywhere. In particular, we conclude that $\bar{m}(C_n | T_0^{-1}\mathcal{B})(x) = \log g(y_n)$ a.e. However, since this uniquely determines the measure $\bar{m}$, we conclude that $\bar{m} = \bar{\mu}$ [16, p.133]. This completes the proof of Proposition 1.

For the proof of Theorem 1, we need an estimate on iterates of the operator $\mathcal{L}$. This is given by the next result.

Proposition 2. *If* $0 \leq x \leq 1$ *then* $\lim_{n \to +\infty} \frac{1}{n} \log \mathcal{L}^n 1(x) = 0$.

Proof. The upper bound $\limsup_{n \to +\infty} \frac{1}{n} \log \mathcal{L}^n 1(x) \leq 0$ follows from Lemma 1 and part (1) of Proposition 1.

Consider first the case $x = 0$. Observing that $\mathcal{L}^n 1(0) \geq 1/|(T^n)'(0)| = 1$, we also have that $\liminf_{n \to +\infty} \frac{1}{n} \log \mathcal{L}^n 1(0) \geq 0$.

On the other hand if $0 < x \leq 1$ then $\mathcal{L}^n 1(x) \geq 1/|(T^n)'(\psi_0^n x)|$ and so by the chain rule

$$\frac{1}{n} \log \mathcal{L}^n 1(x) \geq -\frac{1}{n} \sum_{i=0}^{n-1} \log |(T)'(\psi_0^i x)| \to 0,$$

as $n \to +\infty$ (since $|T'(\psi_0^n x)| \to 1$, as $n \to +\infty$).

Remark. In fact, for Proposition 1 (3) it suffices to know that the only ergodic measure m with $h_m(T) > 0$, satisfying $h_m(T) = \int \log |T'(x)| dm(x)$ is the measure μ. This was proved by Ledrappier [9]. His result is proved in far greater generality, but does not easily lend itself to extension to higher dimensional systems.

A second example. Another map for which the hypotheses have been checked is due to the third author [21] (cf. also [14])

$$T_\beta(x) = \begin{cases} \frac{x}{(1-x^\beta)^{1/\beta}} & 0 \le x \le 2^{-1/\beta} \\ \frac{x}{(\frac{1}{2})^{\frac{1}{\beta}}-1} - \frac{1}{1-(\frac{1}{2})^{\frac{1}{\beta}}} & 2^{-1/\beta} \le x \le 1. \end{cases}$$

3. Proof of Theorem 1

We can define a map $Q : C^0(I) \to \mathbb{R}$ by $Q(g) = P(-\log |T'| + g)$. For $\nu \in \mathcal{M}$, we then denote the *Legendre transform* of $Q(g)$ by

$$I(\nu) = \sup_{g \in C^0(I)} (\int g d\nu - Q(g)).$$

Our proof will be based upon the following estimate.

Lemma 7. *Let $\mathcal{K} \subset \mathcal{M}$ be a weak* closed (and hence compact) subset. Then*

$$\limsup_{n \to +\infty} \frac{1}{n} \log \left(\frac{\sum_{\substack{T^n y = x \\ \delta_{y,n} \in \mathcal{K}}} \frac{1}{|T^n(y)|}}{\sum_{T^n y = x} \frac{1}{|T^n(y)|}} \right) \le - \inf_{\nu \in \mathcal{K}} I(\nu).$$

Proof. Write $\rho = \inf_{\nu \in \mathcal{K}} I(\nu)$ and fix a choice of $\epsilon > 0$. From the definition of ρ, for every $\nu \in \mathcal{K}$, there exists $g \in C^0(I)$ such that

$$\int g d\nu - Q(g) > \rho - \epsilon.$$

Thus we have that

$$\mathcal{K} \subset \bigcup_{g \in C^0([0,1])} \left\{ \nu \in \mathcal{M} : \int g d\nu - Q(g) > \rho - \epsilon \right\}$$

and by weak* compactness we can choose a finite subcover

$$\mathcal{K} \subset \bigcup_{i=1}^{k} \left\{ \nu \in \mathcal{M} : \int g_i d\nu - Q(g_i) > \rho - \epsilon \right\}.$$

Therefore we have the inequality

$$\sum_{\substack{T^n y = x \\ \delta_{y,n} \in \mathcal{K}}} \frac{1}{|(T^n)'(y)|} \le \sum_{i=1}^{k} \left(\sum_{\substack{T^n y = x \\ \frac{1}{n} g_i^n(y) - Q(g_i) > \rho - \epsilon}} \frac{1}{|(T^n)'(y)|} \right)$$
$$\le \sum_{i=1}^{k} e^{-n(Q(g_i) + (\rho - \epsilon))} \left(\sum_{T^n y = x} e^{-(\log|T'|)^n(y) + g_i^n(y)} \right).$$

Taking limits we get that

$$\limsup_{n \to +\infty} \frac{1}{n} \log \left(\frac{\sum_{\substack{T^n y = x \\ \delta_{y,n} \in \mathcal{K}}} \frac{1}{|(T^n)'(y)|}}{\sum_{T^n y = x} \frac{1}{|(T^n)'(y)|}} \right)$$
$$\le \sup_{1 \le i \le k} \left\{ -Q(g_i) - \rho + \epsilon + \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sum_{T^n y = x} e^{-(\log|T'|)^n(y) + g_i^n(y)} \right) \right.$$
$$\left. - \liminf_{n \to +\infty} \frac{1}{n} \log \left(\sum_{T^n y = x} e^{-(\log|T'|)^n(y)} \right) \right\}$$
$$\le \sup_{1 \le i \le k} \{ -Q(g_i) - \rho + \epsilon + P(-\log|T'| + g_i) \}$$
$$= -\rho + \epsilon,$$

by Lemma 1 and Proposition 2. Since $\epsilon > 0$ can be chosen arbitrarily small this completes the proof of the lemma.

We next want to show that if $\mathcal{K}$ does not intersect $\mathcal{A}$ then $\inf_{\nu \in \mathcal{K}} I(\nu) > 0$. This will follow from the next lemma.

We shall write $\mathcal{M}_T$ for the set of T-invariant probability measures in $\mathcal{M}$.

Lemma 8.

(i) *If $\nu \notin \mathcal{A}$ then $I(\nu) > 0$.*

(ii) *The map $\nu \to I(\nu)$ is lower semi-continuous on $\mathcal{M}_T$. Futhermore, on $\mathcal{M} - \mathcal{M}_T$, $I(\nu)$ is bounded below by the continuous function $\int \log|T'| d\nu$.*

Proof. For part (i) we have that

$$I(\nu) = \sup_{g \in C^0(I)} \left(\int g d\nu - P(-\log|T'| + g) \right)$$
$$= \sup_{g \in C^0(I)} \left(\int (g + \log|T'|) d\nu - P(g) \right)$$
$$= \sup_{g \in C^0(I)} \left(\int g d\nu - P(g) \right) + \int \log|T'| d\nu$$
$$= - \inf_{g \in C^0(I)} \left(P(g) - \int g d\nu \right) + \int \log|T'| d\nu.$$

If $\nu \in \mathcal{M}_T$ then, by the variational principle, this is equal to $-h_\nu(T) + \int \log |T'| d\nu$ [14, pp.221-222]. If $\nu \notin \mathcal{A}$ then, by Proposition 1 we see that $-h_\nu(T) + \int \log |T'| d\nu > 0$. On the other hand, if $\nu \in \mathcal{M} - \mathcal{M}_T$, then

$$\inf_{g \in C^0(I)} (P(g) - \int g d\nu) < 0$$

[14, pp.221-222] therefore $I(\nu) \geq \int \log |T'| d\nu > 0$.

For the proof of (ii) we first notice that, since entropy is upper semi-continuous, $I(\nu) = -h_\nu(T) + \int \log |T'| d\nu$ for $\nu \in \mathcal{M}_T$, which is lower semi-continuous. We then complete the proof with the the lower bound in the proof of (i) above. This completes the proof of the lemma.

Since $\mathcal{K}$ is compact, we can conclude that if $\mathcal{A} \cap \mathcal{K} = \emptyset$ then $\rho = \inf_{\nu \in \mathcal{K}} \{I(\nu)\} > 0$. Theorem 1 now follows by setting $\mathcal{K} = \mathcal{M} - \mathcal{U}$.

4. Proof of Theorem 2

To prove Theorem 2 we shall use the following estimates.

Given $m \notin \mathcal{A}$ there exists an open neighbourhood $\mathcal{U}$ of $\mathcal{A}$ such that $m \notin \mathcal{U}$; $g \in C^0(I)$ and $\epsilon > 0$, such that

$$\int g d\nu - \int g dm \geq \epsilon, \quad \forall \nu \in \mathcal{U}.$$

There exists $C > 0$ and $0 < \eta < 1$ such that

$$\frac{\sum_{\substack{T^n y = x \\ \delta_{y,n} \notin \mathcal{U}}} \frac{1}{|T'(y)|}}{\sum_{T^n y = x} \frac{1}{|T'(y)|}} \leq C \eta^n,$$

or equivalently,

$$\frac{\sum_{\substack{T^n y = x \\ \delta_{y,n} \in \mathcal{U}}} \frac{1}{|T'(y)|}}{\sum_{T^n y = x} \frac{1}{|T'(y)|}} \geq 1 - C \eta^n.$$

Assume for a contradiction there exists a subsequence n_i such that $\Delta_{n_i} \to m \notin \mathcal{A}$. This means that for every $g \in C^0(I)$

$$\frac{\sum_{T^n y = x} \frac{1}{|T'(y)|} \frac{g^n(y)}{n}}{\sum_{T^n y = x} \frac{1}{|T'(y)|}} \to \int g dm, \qquad \text{as } n_i \to +\infty.$$

However, we have the estimate

$$
\begin{aligned}
&\left|\frac{\sum_{T^n y=x} \frac{1}{|T'(y)|}\frac{g^n(y)}{n}}{\sum_{T^n y=x} \frac{1}{|T'(y)|}} - \int gdm\right| \\
&= \left|\frac{\sum_{T^n y=x} \frac{1}{|T'(y)|}\left(\frac{g^n(y)}{n} - \int gdm\right)}{\sum_{T^n y=x} \frac{1}{|T'(y)|}}\right| \\
&\geq \left|\frac{\sum_{\substack{T^n y=x \\ \delta_{y,n}\in\mathcal{U}}} \frac{1}{|T'(y)|}\left(\frac{g^n(y)}{n} - \int gdm\right)}{\sum_{T^n y=x} \frac{1}{|T'(y)|}}\right| - \left|\frac{\sum_{\substack{T^n y=x \\ \delta_{y,n}\notin\mathcal{U}}} \frac{1}{|T'(y)|}\left(\frac{g^n(y)}{n} - \int gdm\right)}{\sum_{T^n y=x} \frac{1}{|T'(y)|}}\right| \\
&\geq \left(\frac{\sum_{\substack{T^n y=x \\ \delta_{y,n}\in\mathcal{U}}} \frac{1}{|T'(y)|}}{\sum_{T^n y=x} \frac{1}{|T'(y)|}}\right)\epsilon - 2C||g||_\infty \eta^n \\
&\geq (1 - C\eta^n)\epsilon - 2C||g||_\infty\eta^n \\
&\geq \frac{\epsilon}{2} \text{ for all sufficiently large } n.
\end{aligned}
$$

This is a contradiction.

5. Higher dimensional maps

In this final section we shall show that the analogue of Theorem 1 holds in higher dimensional settings. Consider $T : X \to X$ where X is compact subset of $\mathbb{R}^k$. Assume that X can be written as a countable union of connected sets $\{X_a\}$ in $\mathbb{R}^k$ such that

(1) each X_a has a piecewise smooth boundary;
(2) for each X_a, $T : X_a \to X$ is a C^1 diffeomorphism; and
(3) $T^n\left(\mathrm{int}(X_{a_1}) \cap \mathrm{int}(T^{-1}X_{a_2}) \cap \ldots \cap \mathrm{int}(T^{-(n-1)}X_{a_n})\right)$ is one of a finite number of given open sets in $\mathcal{U} = \{U_1, \ldots, U_N\}$. (This condition is called the *Finite Range Condition*).

Definition. The open sets $X_{a_1\ldots a_n} = \mathrm{int}(X_{a_1})\cap\mathrm{int}(T^{-1}X_{a_2})\cap\ldots\cap\mathrm{int}(T^{-(n-1)}X_{a_n})$ are called *cylinders of length* n.

Given $C > 1$ we let $D_{n,C} \subset X$ be the union of those cylinders $X_{a_1\ldots a_n}$ for which the following condition fails for $i = 1, \ldots, n$:

$$\sup_{x,y\in X_{a_1\ldots a_i}} \frac{|\det DT^i(x)|}{|\det DT^i(y)|} < C.$$

$D_{n,C}$ contains cylinders of length n touching indifferent periodic points (see [19, Proposition 3.1]).

Definition. For $C \geq 1$ we let $R(C,T)$ denote the set of those cylinders $X_{a_1\ldots a_n}$ for which the following conditions is satisfied:

$$\sup_{x,y\in X_{a_1\ldots a_n}} \frac{|\det DT^n(x)|}{|\det DT^n(y)|} < C.$$

The following conditions are sufficient for the existence of an ergodic (in fact exact) T-invariant probability measure μ equivalent to Lebesgue measure:

(i) $Q = \{X_a\}$ is a generating partition (i.e., for almost all points $x \neq y$ we can find $n \geq 0$ such that $T^n x$ and $T^n y$ are in different elements of $\{X_a\}$);

(ii) there exists $C \geq 1$ such that $R(C,T) \neq \emptyset$ and $X_{b_1 \ldots b_l a_1 \ldots a_n} \in R(C,T)$ whenever $X_{a_1 \ldots a_n} \in R(C,T)$; and

(iii) for each $U_i \in \mathcal{U}$ there exists $X_{a_1 \ldots a_s} \subset U_i$ such that $X_{a_s} \in R(C,T)$ and $T^s X_{a_1 \ldots a_s} = X$.

(iv) $\sum_{n=1}^{\infty} \lambda(D_{n,C}) < +\infty$ where λ is Lebesgue measure on $\mathbb{R}^k$.

Proposition 3. *Assume that (1)-(3) and (i)-(iv) all hold. There exists an ergodic absolutely continuous T-invariant measure μ.*

Example. We now consider a simple example for which conditions (1)-(3) and (i)-(iv) are valid. Let $X = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1, -y \leq x < -y + 1\}$ and define $T : X \to X$ by

$$T(x,y) = \left(\frac{1}{x} - \left[\frac{1-y}{x} \right] + \left[-\frac{y}{x} \right], - \left[-\frac{y}{x} \right] - \frac{y}{x} \right).$$

It was shown in [18] that there exists an ergodic absolutely continuous invariant measure for this example. (cf. [8] for other examples.)

Proof of Proposition 3. The proofs are outlined in [19, p.1095]. (This follows the same lines as the detailed proofs for specific examples in [6].) The key to the proof of the existence of μ is to construct a jump transformation

$$T_R : \cup_{X_{b_1 \ldots b_j} \in R(C,T)} X_{b_1 \ldots b_j} \to X$$

defined by

$$T_R x = T^j x \text{ for } x \in B_j = \cup \{X_{b_1 \ldots b_j} : X_{b_1 \ldots b_j} \subset D_{j-1,C}, X_{b_1 \ldots b_j} \in R(C,T)\}$$

Under the conditions (i), (ii), (iii), and (iv) the transformations T_R on $X_R = X - \left(\cup_{m=0}^{\infty} T_R^{-m} \cap_{n \geq 0} D_{n,C} \right)$ admits an ergodic invariant measure $\nu \sim \lambda$ for which the density is bounded away from zero and infinity. We can define the T-invariant measure μ by $\mu(E) = \sum_{n=0}^{\infty} \nu(D_{n,C} \cap T^{-n} E)$, where $D_{0,C} = X$, for all measurable sets E.

In order to obtain a higher dimensional version of Rohlin's formula we need the following additional condition:

(v) Q is finite and T is continuous.

To simplify our problem, we shall assume that $\cap_{n=0}^{\infty} D_{n,C}$ consists only of one indifferent periodic orbit.

The following Rohlin type result was established in [19] (cf. [20]).

Let $\mathcal{M}(X,T)$ denote the set of T-invariant probability measure on the σ-algebra of the Borel subsets of X (and $E(X,T)$ denote the ergodic measures). Let $\mathcal{A}$ denote the convex hull of μ and the set of measures supported on $\cap_{n=0}^{\infty} D_{n,C}$.

Proposition 4, [19, Theorem 5.1]**.** *Assume that conditions (1)-(3) and (i)-(v) hold. Assume further that* $\inf_{x\in X}|\det DT(x)|>0$*. Then*

1. $P(-\log|\det DT|)=0$,
2. *If* $m\in\mathcal{A}$ *then*
$$h_m(T)=\int\log|\det DT(x)|dm(x),$$
3. *If* $m\notin\mathcal{A}$ *then*
$$h_m(T)<\int\log|\det DT(x)|dm(x).$$

Proof. Part (1) follows from parts (2) and (3). Part (2) follows from Lemmas 5.1 and 5.2 in [19].

Finally, we prove part (3). From Theorem 8.1 in [20], we already know that

$$\begin{aligned}0&=\mu\left(I_\mu(\mathcal{B}|T^{-1}\mathcal{B})+\log\left(\frac{h}{h\circ T}\right)-\log|\det DT|\right)\\&\geq m\left(I_m(\mathcal{B}|T^{-1}\mathcal{B})+\log\left(\frac{h}{h\circ T}\right)-\log|\det DT|\right).\end{aligned}$$

Furthermore, for $m\in E(X,T)$ with $m\notin\mathcal{A}$ we have, by the Rohlin tower argument in the proof of Proposition 1,

$$\begin{aligned}&m\left(I_m(\mathcal{B}|T^{-1}\mathcal{B})+\log\left(\frac{h}{h\circ T}\right)-\log|\det DT|\right)\\&=h_m(T)-\int\log|\det DT(x)|dm(x)<0.\end{aligned}$$

Proposition 5. *Assume that conditions (1)-(3) and (i)-(v) hold. For* $f\in C(X)$ *we have that*

$$\lim_{n\to+\infty}\frac{1}{n}\log\mathcal{L}^n1(x)=0$$

for all $x\in X$*.*

Proof. The upper bound $\limsup_{n\to+\infty}\frac{1}{n}\log\mathcal{L}^n1(x)\leq 0$ follows from the higher dimensional analogue of Lemma 1 and part (1) of Proposition 4.

We now turn to the lower bound. Consider first the case x is an indifferent periodic point. Observing that $\mathcal{L}^n1(x)\geq 1/|\det D(T^n)(x)|=1$, we also have that $\liminf_{n\to+\infty}\frac{1}{n}\log\mathcal{L}^n1(x)\geq 0$.

More generally, for any $x\in X$ we observe that there exist local inverses ψ_n to T^n, $n\geq 1$, such that all the accumulation points of $\psi_n(x)$ lie in $\cap_{n=1}^\infty D_{C,n}$. Thus, for any $x\in X$ we see that $\mathcal{L}^n1(x)\geq 1/|\det D(T^n)(\psi_n x)|$ and so by the chain rule

$$\frac{1}{n}\log\mathcal{L}^n1(x)\geq-\frac{1}{n}\sum_{i=0}^{n-1}\log|\det(DT)(T^i\psi_n x)|\to 0,$$

as $n\to+\infty$ (since $|\det(DT)(\psi_n x)|\to 1$, as $n\to+\infty$).

Thus we have the following result.

Theorem 3. *Assume that* $T : X \to X$ *satisfies (1)-(3), (i) - (v), and* $\inf_{x\in X} |\det DT(x)| > 0$. *Let* $x \in X$ *and let* $\mathcal{U} \subset \mathcal{M}$ *be a weak* open neighbourhood of the line segment* $\mathcal{A}$. *Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \left(\frac{\sum_{\substack{T^n y = x \\ \delta_{y,n} \notin \mathcal{U}}} \frac{1}{|\det DT^n(y)|}}{\sum_{T^n y = x} \frac{1}{|\det DT^n(y)|}} \right) < 0.$$

Proof. The proof is completely analogous to that in section 3.

Example (Brun's map, [19]*).* Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_2 \le x_1 \le 1\}$ and for $i = 0, 1, 2$, $X_i = \{(x_1, x_2) \in X : x_i + x_1 \ge 1 \ge x_{i+1} + x_1\}$, where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$T(x_1, x_2) = \begin{cases} \left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_1}\right) & \text{on } X_0 \\ \left(\frac{1}{x_1} - 1, \frac{x_2}{x_1}\right) & \text{on } X_1 \\ \left(\frac{x_2}{x_1}, \frac{1}{x_1} - 1\right) & \text{on } X_2. \end{cases}$$

Thus $TX_i = X$ and $Q = \{X_0, X_1, X_2\}$. The point $(0, 0)$ is an indifferent fixed point and T is a piecewise C^2 map. The invariant density for T takes the form

$$h(x_1, x_2) = \frac{1}{2x_1(1 + x_2)}$$

[14]. The map T satisifies (1) - (3) and (i) - (v) (see [19]).

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Mark Pollicott, Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK.

Richard Sharp, Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK.

Michiko Yuri, Department of Business Administration, Sapporo University, Nishioka, Toyohira-ku, Sapporo 062, JAPAN.