ANGULAR SELF-INTERSECTIONS FOR CLOSED GEODESICS ON SURFACES

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Abstract. In this note we consider asymptotic results for self-interections of closed geodesics on surfaces for which the angle of the intersection occurs in a given arc. We do this by extending Bonahon's definition of interesction forms for surfaces.

0. Introduction

In recent years, many authors have considered the self-intersection of geodesics [BS], [Bo1], [Po] [La]. In this note we consider the problem of estimating the number of self-intersections for which the angle of intersection lies in a given arc. Let V be a compact surface with negative curvature. Let γ denote a closed geodesic on V of length $l(\gamma)$. It is a classical result of Huber [Hu1] and Margulis [Ma] that the number $N(T)$ of closed geodesics of length at most T satisfies the asymptotic formula $N(T) \sim e^{hT}/hT$, as $T \to +\infty$, i.e., the ratio of the two sides converges to one. (Here $h > 0$ denotes the topological entropy of the geodesic flow over V.) In the case where V has constant curvature, Huber [Hu2] obtained the stronger estimate $N(T) = \mathrm{li}(e^{hT})(1 + O(e^{-\delta T}))$, for some $\delta > 0$, while for variable curvature this was established only in recent years [PS], [Do].

For $0 \leq \theta_1 < \theta_2 \leq \pi$ we let $i_{\theta_1,\theta_2}(\gamma)$ denote the number of self-intersections of the closed geodesic γ such that the absolute value of the angle of intersection lies in the interval $[\theta_1, \theta_2]$.

Theorem 1. *Given $0 \leq \theta_1 < \theta_2 \leq \pi$, there exists $I = I(\theta_1, \theta_2)$ and $\delta > 0$ such that, for any $\epsilon > 0$,*

$$\#\left\{\gamma : l(\gamma) \leq T, \frac{i_{\theta_1\theta_2}(\gamma)}{l(\gamma)^2} \in (I - \epsilon, I + \epsilon)\right\} = \mathrm{li}\left(e^{hT}\right)\left(1 + O(e^{-\delta T})\right).$$

If $\theta_1 = 0$ and $\theta_2 = \pi$ then $i_{0,\pi}(\gamma)$ is simply the total number of self-intersections of γ. In this particular case, Anantharaman [An] has observed this result with $I(0, \pi) = i(\tilde{m}, \tilde{m})/2$, where $\tilde{m}$ is the transverse measure associated to the measure of maximal entropy for the geodesic flow over V and $i(\cdot, \cdot)$ is the intersection form introduced by Bonahon [Bo1] (see also [Ot]). Our proof follows the same basic approach as that of [La] and [An], in that we apply a standard large deviation result for closed orbits. The distinction is that we need to introduce two new ingredients:

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the angular intersection bundle and the associated angular intersection form. This allows us to restrict to counting the self-intersections of geodesics to those whose angle of intersection lies in a given interval.

We can interpret the constant $I(\theta_1, \theta_2)$ in terms of an *angular intersection form*, generalising $i(\cdot,\cdot)$. For a surface of constant negative curvature, it can be easily described explicitly and we have the formula

$$I(\theta_1, \theta_2) = \frac{1}{2\pi \mathrm{Area}(V)} \int_{\theta_1}^{\theta_2} \sin\theta d\theta.$$

In particular, using $\mathrm{Area}(V) = 4\pi(g-1)$, where $g \geq 2$ is the genus of V, we can verify a heuristic estimate of Sieber and Richter [SR] that

$$\sum_{l(\gamma)\leq T} i_{\theta_1,\theta_2}(\gamma) \sim \left(\int_{\theta_1}^{\theta_2} \sin\theta d\theta\right) \frac{T^2}{8\pi^2(g-1)} N(T). \tag{0.1}$$

Quantum Chaos provides an interesting motivation for this result associated to the study of pair correlations of energy levels [Ke]. The associated form factor, or Fourier transform, leads to correlations of closed geodesics and (0.1) is relevant to the leading term of the off-diagonal contribution.

We can also consider the distribution of the self-intersection points of the geodesic γ on the surface V. We let $\mathcal{S}(\gamma;\theta_1,\theta_2) \subset V$ denote the set of self-intersection points of γ such that the absolute value of the angle of intersection lies in the interval $[\theta_1, \theta_2]$. In particular, $i_{\theta_1,\theta_2}(\gamma) = \#\mathcal{S}(\gamma;\theta_1,\theta_2)$. Given a continuous function $f : V \to \mathbb{R}$, we let $A(f,\gamma;\theta_1,\theta_2)$ denote the sum of f over the points of self-intersection of γ lying in $\mathcal{S}(\gamma;\theta_1,\theta_2)$, i.e.,

$$A(f,\gamma;\theta_1,\theta_2) = \sum_{x\in\mathcal{S}(\gamma;\theta_1,\theta_2)} f(x).$$

Theorem 2. *Given $0 \leq \theta_1 < \theta_2 \leq \pi$, there exists $I_f = I(f;\theta_1,\theta_2)$ and $\delta > 0$ such that, for any $\epsilon > 0$,*

$$\#\left\{\gamma : l(\gamma) \leq T, \frac{A(f,\gamma;\theta_1,\theta_2)}{l(\gamma)^2} \in (I_f - \epsilon, I_f + \epsilon)\right\} = \mathrm{li}(e^{hT})\left(1 + O\left(e^{-\delta T}\right)\right).$$

In particular, when $f = 1$ Theorem 2 reduces to Theorem 1.

Sections 1 and 2 contain preliminary material. Section 3 contains the proofs of Theorem 1 and Theorem 2.

1. Intersection Bundles and Forms

1.1 The geodesic flow. Let SV be the unit tangent bundle of V and let $\pi : SV \to V$ denote the canonical projection. For $x \in V$, we write $S_xV = \pi^{-1}(x)$. Consider the geodesic flow $\phi_t : SV \to SV$ on the unit tangent bundle of V. This is an example of a weak-mixing hyperbolic flow [As]. There is a one-to-one correspondence between oriented closed geodesics on V and periodic orbits for ϕ. We shall write $h > 0$ for the topological entropy of ϕ and, given a ϕ-invariant *probability* measure μ, we shall write $h(\mu)$ for the measure theoretic entropy of ϕ with respect to μ.

There is a unique invariant probability measure m, called the measure of maximal entropy, for which $h(m) = h$.

We let $\mathcal{F}$ denote the foliation of SV by ϕ-orbits. Given any ϕ-invariant finite measure μ (not necessarily a probability measure) we can consider the associated transverse measure $\tilde{\mu}$ for $\mathcal{F}$. The set of these transverse measures $\mathcal{C} = \{\tilde{\mu} : \mu \text{ is a } \phi\text{-invariant measure}\}$ is called the *space of currents*. Each $\tilde{\mu} \in \mathcal{C}$ is normalized by the requirement that (locally) $\mu = \tilde{\mu} \times dt$, where dt is one-dimensional Lebesgue measure along orbits in $\mathcal{F}$.

1.2 Bundles. Let $E = SV \oplus SV - \Delta$ be the Whitney sum of the bundle SV with itself minus the diagonal $\Delta = \{(x, v, v) : x \in V, v \in S_xV\}$ and let $p : E \to V$ denote the canonical projection. In particular, points of the four dimensional vector bundle E (with two dimensional fibres) consist of triples (x, v, w), where $x \in V$ and $v, w \in S_xV$. Let $p_1 : E \to SV$ be defined by $p_1(x, v, w) = (x, v)$ and $p_2 : E \to SV$ be defined by $p_2(x, v, w) = (x, w)$. Following Bonahon [Bo1], we consider the two transverse foliations (with two dimensional leaves) of E given by $\mathcal{F}_1 = p_1^{-1}(\mathcal{F}) = \mathcal{F}_2 = p_2^{-1}(\mathcal{F})$.

Definition. Given $0 \le \theta_1 < \theta_2 \le \pi$, we define the *angular intersection bundle* E_{θ_1,θ_2} by $E_{\theta_1,\theta_2} = \{(x, v, w) \in E : \angle v, w \in [\theta_1, \theta_2]\}$, where $0 \le \angle v, w \le \pi$ denotes the angle between the two vectors. This is a closed subbundle of E.

It is sometimes convenient to consider an equivalent geometric definition of the bundle E_{θ_1,θ_2} is in terms of pairs of geodesics on the universal cover $\widetilde{V}$ of V. In particular, consider the space $\mathcal{G}_{\theta_1,\theta_2}$ of pairs of oriented geodesics $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ on $\widetilde{V}$ which intersect, at some point $\widetilde{x} \in \widetilde{V}$, say, and with an angle of intersection in the interval $[\theta_1, \theta_2]$. There is a natural identification E_{θ_1,θ_2} with $\mathcal{G}_{\theta_1,\theta_2}/\pi_1(V)$, where the quotient is with respect to the diagonal action of the fundamental group (cf. [Bo2]).

1.3 Intersection forms. Given currents $\tilde{\mu}, \tilde{\mu}' \in \mathcal{C}$, we can take the lifts $\widehat{\mu}_1 := p_1^{-1}\tilde{\mu}$ and $\widehat{\mu}_2' := p_2^{-1}\tilde{\mu}'$, which are transverse measures to the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ for E, respectively. Bonahon defined the *intersection form* $i : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the total mass of the E with respect to the product measure $\widehat{\mu}_1 \times \widehat{\mu}_2'$, i.e., $i(\tilde{\mu}, \tilde{\mu}') = (\widehat{\mu}_1 \times \widehat{\mu}_2')(E)$ [Bo1]. By analogy, we can define the following.

Definition. We define an *angular intersection form* $i_{\theta_1,\theta_2} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the total mass of the E_{θ_1,θ_2} with respect to the product measure $\widehat{\mu}_1 \times \widehat{\mu}_2'$, i.e., $i_{\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu}') = (\widehat{\mu}_1 \times \widehat{\mu}_2')(E_{\theta_1,\theta_2})$.

In the geometric picture, the space of currents corresponds to measures on $\mathcal{G}_{\theta_1,\theta_2}/\pi_1(V)$. For example, given a geodesic γ on V the associated measure is simply a finite sum of Dirac measures supported on the quotient of pairs of closed geodesics on $\widetilde{V}$ consisting of lifts of γ and its $\pi_1(V)$ images (cf. [Bo2]).

More generally, in order to study the spatial distribution of the intersection points, we can introduce a weighted form. Let $f : V \to \mathbb{R}$ be a continuous function.

Definition. We define the *weighted angular intersection form* $i_{f;\theta_1,\theta_2} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the integral of $f \circ p$ restricted to E_{θ_1,θ_2} with respect to the product measure $\widehat{\mu}_1 \times \widehat{\mu}_2'$, i.e., $i_{f;\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu}') = \int f \circ p \; d(\widehat{\mu}_1 \times \widehat{\mu}_2')$.

2. Intersection forms and closed geodesics

Given a closed geodesic γ, let μ_γ denote the unique invariant measure of total mass $l(\gamma)$ supported on the corresponding periodic orbit and let $\tilde{\mu}_\gamma$ be the corresponding transverse measures for the orbit foliation $\mathcal{F}$, which are normalized to be a finite sum of Dirac measures on transverse sections. We shall write $\widehat{\mu}_{\gamma,1} = p_1^{-1}\tilde{\mu}_\gamma$ and $\widehat{\mu}_{\gamma,2} = p_2^{-1}\tilde{\mu}_\gamma$.

Lemma 1. *Let γ and γ' be a pair of closed geodesics. Then*

(i) $(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2})$ *is equal to the number of intersections between γ and γ' with angle between θ_1 and θ_2, and*

(ii) *given a continuous function $f : V \to \mathbb{R}$, $\int_{E_{\theta_1,\theta_2}} f \circ p \, d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma',2})$ is equal to the summation of f over the points of intersection of γ and γ' with angle between θ_1 and θ_2.*

Proof. The proof is modelled on [Bo1, p.111]. We can choose flow boxes $B_1, B_2 \subset SV$ for ϕ such that, provided they are sufficiently small, for $v_1 \in B_1$ and $v_2 \in B_2$ the associated geodesic arcs intersect transversally at at most one point. We denote $B_1 \oplus B_2 := p_1^{-1}(B_1) \cap p_2^{-1}(B_2) \subset E$. As in [Bo1], one can see that $(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma',2})(B_1 \oplus B_2)$ is precisely the number of times γ crosses γ' with tangent vectors in B_1 and B_2, respectively. Covering SV by a partition consisting of such flow boxes, and summing over those $B_1 \oplus B_2$ which approximate E_{θ_1,θ_2} gives an estimate to the number of times the geodesics intersect with angle in $[\theta_1, \theta_2]$, i.e., if $d(\cdot, \cdot)$ denotes the Hausdorff distance between closed subsets of E,

$$\sum_{d(E_{\theta_1,\theta_2}, B_1 \oplus B_2) < \epsilon} (\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma',2})(B_1 \oplus B_2)$$

can be made arbitrarily close to $i_{\theta_1,\theta_1}(\tilde{\mu}_\gamma, \tilde{\mu}_{\gamma'})$ by choosing $\epsilon > 0$ sufficiently small. By taking the size of the flow boxes arbitrarily small, the result follows.

For part (ii), if suffices to show that, for any small disk $D \subset V$, we have that $(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma',2})(E_{\theta_1,\theta_2} \cap p^{-1}(D))$ is equal to the number of intersections between γ and γ' lying inside D and with angle between θ_1 and θ_2. However, this follows from the preceeding argument if we further restrict the summation to those $B_1 \oplus B_2$ whose projections $\pi(B_1)$ and $\pi(B_2)$ approximate D. □

As immediate consequences of the above lemma we have that the number of self-intersections of γ with angle lying in the interval $[\theta_1, \theta_2]$ is given by $i_{\theta_1,\theta_2}(\gamma) = (\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})(E_{\theta_1,\theta_2})/2$ and that $A(f, \gamma; \theta_1, \theta_2) = \int_{E_{\theta_1,\theta_2}} f \circ p \, d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})/2$.

The next result will show that $\tilde{\mu} \mapsto i_{\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu})$ is continuous in a neighbourhood of the transverse measure $\tilde{m}$ associated to the measure of maximal entropy m. Let $\widehat{m}_1 = p_1^{-1}\tilde{m}$ and $\widehat{m}_2 = p_2^{-1}\tilde{m}$.

Lemma 2. *Let γ_n be a sequence of closed geodesics and let $\mu_{\gamma_n}/l(\gamma_n)$ be the (normalised) probability measure on SV. Assume that $\mu_{\gamma_n}/l(\gamma_n)$ converges in the weak* topology to the measure of maximal entropy m. Then*

(1) $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2}) \to (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2})$, *as $n \to +\infty$; and*

(2) *given a continuous function $f : V \to \mathbb{R}$, $\int_{E_{\theta_1,\theta_2}} (f \circ p) \, d(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2}) \to \int_{E_{\theta_1,\theta_2}} (f \circ p) \, d(\widehat{m}_1 \times \widehat{m}_2)$, as $n \to +\infty$.*

Proof. We recall Bonahon's proof that $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(E) \to (\widehat{m}_1 \times \widehat{m}_2)(E)$, as $n \to +\infty$, [Bo1, pp. 112-114]. In his proof, he approximated $E - U$, where U is a neighbourhood of the diagonal Δ, by unions of sets $B_1 \oplus B_2$, where B_1 and B_2 are sufficiently small flow boxes. He then showed that $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(B_1 \oplus B_2) \to (\widehat{m}_1 \times \widehat{m}_2)(B_1 \oplus B_2)$, as $n \to +\infty$. By similarly approximating E_{θ_1,θ_2} by unions $B_1 \oplus B_2$ we see that part (1) holds.

For part (2), if suffices to show that, for any small disk $D \subset V$, we have that $(\widehat{\mu}_{\gamma_n,1} \times \widehat{\mu}_{\gamma_n,2})(E_{\theta_1,\theta_2} \cap p^{-1}(D)) \to (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2} \cap p^{-1}(D))$, as $n \to +\infty$. In the preceeding argument we may further restrict the union to those $B_1 \oplus B_2$ whose projections $\pi(B_1)$ and $\pi(B_2)$ approximate D. This completes the proof. □

3. Proof of Theorems

Theorems 1 and 2 will follow from a large deviation result for periodic orbit measures established by Kifer [Ki]. Kifer's result is valid for any hyperbolic flow and so, in particular, for the geodesic flow $\phi_t : SV \to SV$.

Lemma 3 [Ki]**.** *Let $\mathcal{U}$ be an open neighbourhood of the measure of maximal entropy m in the set of ϕ-invariant probability measures on SV. Then*

$$\frac{1}{N(T)}\#\{\gamma : l(\gamma) \le T \text{ and } \mu_\gamma / l(\gamma) \notin \mathcal{U}\} = O(e^{-\delta T}),$$

as $T \to +\infty$, where $\delta = \inf_{\nu\in\mathcal{U}^c}\{h - h(\nu)\}$.

In particular, if $m \in \mathcal{U}$ then $\delta > 0$. In our context, Lemma 3 gives the following estimates.

Lemma 4.
(1) *Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{1}{N(T)}\#\{\gamma : l(\gamma) \le T \text{ and } |l(\gamma)^{-2}(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})(E_{\theta_1,\theta_2}) - (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2})| \ge \epsilon\}$$
$$= O(e^{-\delta T}), \text{ as } T \to +\infty; \text{ and}$$

(2) *Let $f : V \to \mathbb{R}$ be a continuous function. Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{1}{N(T)}\#\left\{\gamma : l(\gamma) \le T \text{ and } \left|\frac{1}{l(\gamma)^2}\int_{E_{\theta_1,\theta_2}} f \circ p \; d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2}) - \int_{E_{\theta_1,\theta_2}} f \circ p \; d(\widehat{m}_1 \times \widehat{m}_2)\right| \ge \epsilon\right\} = O(e^{-\delta T}),$$

as $T \to +\infty$.

Proof. To prove part (1), we cannot directly apply Lemma 3 with

$$\mathcal{U} = \{\nu : |(\widehat{\nu}_1 \times \widehat{\nu}_2)(E_{\theta_1,\theta_2}) - (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2})| < \epsilon\},$$

since, as the indicator function for E_{θ_1,θ_2} is not continuous, this set is not open. The measure m is known to be non-atomic and so we can deduce that $\widehat{m}_1 \times \widehat{m}_2$ is also non-atomic. In particular, we can choose continuous functions $\psi_1 \le \chi_{E_{\theta_1,\theta_2}} \le \psi_2$ such that

$$\int \psi_2 \, d(\widehat{m}_1 \times \widehat{m}_2) - \tau \le (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2}) \le \int \psi_1 \, d(\widehat{m}_1 \times \widehat{m}_2) + \tau,$$

for some $0 < \tau < \epsilon$, and we can then deduce the required bound by considering

$$\mathcal{U}' = \left\{\nu : \int \psi_2 \, d(\widehat{\nu}_1 \times \widehat{\nu}_2) - \int \psi_2 \, d(\widehat{\nu}_1 \times \widehat{\nu}_2) < \epsilon - \tau\right\}$$
$$\bigcap \left\{\nu : \int \psi_1 \, d(\widehat{\nu}_1 \times \widehat{\nu}_2) - \int \psi_1 \, d(\widehat{\nu}_1 \times \widehat{\nu}_2) > -\epsilon + \tau\right\}$$

and noting that

$$\#\{\gamma : l(\gamma) \le T \text{ and } \mu_\gamma / l(\gamma) \notin \mathcal{U}\} \le \#\{\gamma : l(\gamma) \le T \text{ and } \mu_\gamma / l(\gamma) \notin \mathcal{U}'\}.$$

Clearly, $m \in \mathcal{U}'$, so $\delta > 0$, as required.

If f is non-negative then the proof of part (2) is similar. By choosing continuous functions $\psi_3(x,v) \le (f \circ p)(x) \chi_{E_{\theta_1,\theta_2}}(v) \le \psi_4(x,v)$ such that

$$\int \psi_4 \, d(\widehat{m}_1 \times \widehat{m}_2) - \tau \le \int_{E_{\theta_1,\theta_2}} f \circ p d(\widehat{m}_1 \times \widehat{m}_2) \le \int \psi_3 \, d(\widehat{m}_1 \times \widehat{m}_2) + \tau,$$

for some $0 < \tau < \epsilon$, we can deduce the required bound by considering

$$\mathcal{U}'' = \left\{\gamma : \frac{1}{l(\gamma)^2} \int \psi_4 \, d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2}) - \int \psi_4 \, d(\widehat{m}_1 \times \widehat{m}_2) < \epsilon - \tau\right\}$$
$$\bigcap \left\{\gamma : \frac{1}{l(\gamma)^2} \int \psi_3 \, d(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2}) - \int \psi_3 \, d(\widehat{m}_1 \times \widehat{m}_2) > -\epsilon + \tau\right\}$$

and noting that

$$\#\{\gamma : l(\gamma) \le T \text{ and } \mu_\gamma / l(\gamma) \notin \mathcal{U}\} \le \#\{\gamma : l(\gamma) \le T \text{ and } \mu_\gamma / l(\gamma) \notin \mathcal{U}''\}.$$

Clearly, $m \in \mathcal{U}''$, so $\delta > 0$, as required. To obtain the general case, consider the positive and negative parts of f separately. □

Proof of Theorem 1. Write $I(\theta_1,\theta_2) = i_{\theta_1,\theta_2}(\tilde{m},\tilde{m})/2$ and recall that $i_{\theta_1,\theta_2}(\gamma) = (\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})(E_{\theta_1,\theta_2})/2$. We can apply part (1) of Lemma 4 to deduce that, except for an exceptional set with cardinality of order $O(e^{(h-\delta)T})$, the set of closed geodesics of length at most T satisfy $|l(\gamma)^{-2} i_{\theta_1,\theta_2}(\gamma) - I(\theta_1,\theta_2)| < \epsilon$. Theorem 1 then follows by applying the asymptotic counting results described in the introduction. □

Proof of Theorem 2. This is similar to the proof of Theorem 1. We can write $I(\theta_1,\theta_2,f) = i_{\theta_1,\theta_2,f}(\tilde{m},\tilde{m})/2$ and denote $i_{\theta_1,\theta_2,f}(\gamma) = i_{\theta_1,\theta_2}(\gamma) A(f,\gamma;\theta_1\theta_2)$. We can apply part (1) of Lemma 4 to deduce that, except for an exceptional set with cardinality of order $O(e^{(h-\delta)T})$, the set of closed geodesics of length at most T satisfy $|l(\gamma)^{-2} i_{\theta_1,\theta_2,f}(\gamma) - I(\theta_1,\theta_2,f)| < \epsilon$. Theorem 2 then follows by applying the asymptotic counting results described in the introduction. □

An expression for $I(\theta_1,\theta_2) = (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2})/2$ can be given fairly explicity. Let m_V be the projection onto the surface of m. For each $x \in V$, let m_x be the induced measure on the fibre S_xV. Note that m_x cannot have atoms (for m_V-a.e. x) since otherwise invariance of the measure would imply the existence of closed orbits with non-zero m measure. However, since the measure of maximal entropy m is fully supported such orbits would immediately contradict the ergodicity of m.

Lemma 5. *We can write*

$$I(\theta_1,\theta_2) = \int_V \int \left(\int_{\substack{\{(v_1,v_2)\in S_xV\times S_xV \ : \\ \angle v_1,v_2\in[\theta_1,\theta_2]\}}} \sin(\angle v_1, v_2)\ d(m_x \times m_x)(v_1,v_2) \right) dm_V(x),$$

where $\angle v_1, v_2$ is the angle between the vectors $v_1, v_2 \in S_xV$ are in the same fibre.

Proof. The intersection form is defined in terms of the total mass of the angular intersection bundle with respect to a product measure. One can approximate this value by approximating the bundle E_{θ_1,θ_2} by the unions of small sets of the form $B_1 \oplus B_2 = p_1^{-1}(B_1) \cap p_2^{-1}(B_2)$ for pairs of flow boxes B_1, B_2 in SV whose projections onto V intersect transversely at a single point. By choosing the pairs of flow boxes B_1, B_2 sufficiently small we can choose the angle of intersection of any geodesics arcs associated to B_1 and B_2, respectively, to be arbitrarily close to a constant θ, say. The mass with respect to the product measure of the small set $B_1 \oplus B_2$ is now particularly easy to estimate. It is given by the usual formula for the area of a parallelogram (height $\times$ base $\times \sin\theta$). Finally, we can take the union over all such sets $B_1 \oplus B_2$ in the approximation to E_{θ_1,θ_2}. The estimate then follows by approximation. □

In the particular case of constant curvature, one easily sees that the expression in Lemma 5 reduces to (0.1).

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