DISTRIBUTION OF PERIODIC ORBITS IN THE HOMOLOGY GROUP OF A KNOT COMPLEMENT

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Abstract. Consider a transitive Anosov flow on a closed 3-manifold. After removing a finite set of null-homologous periodic orbits, we study the distribution of the remaining periodic orbits in the homology of the knot complement.

1. Introduction

In a recent paper [19], McMullen discussed the way in which periodic orbits $\mathscr{P}$ of the geodesic flow on the unit-tangent bundle of a compact negatively curved surface (and more general Anosov flows on 3-manifolds) behave like prime numbers, in the sense that they obey a *Chebotarev law*. More precisely, if M_N is the knot complement obtained by removing N periodic orbits $\{K_1, \ldots, K_N\}$ and G is a finite quotient of $\pi_1(M_N)$ then each of the remaining periodic orbits $\gamma \in \mathscr{P}_N := \mathscr{P} \setminus \{K_1, \ldots, K_N\}$ induces a well-defined conjugacy class $\langle \gamma \rangle_G$ in G and these conjugacy classes are equidistributed in the sense that, writing $\ell(\gamma)$ for the least period of γ, for every conjugacy class $C \subset G$ we have

$$\lim_{T \to \infty} \frac{|\{\gamma \in \mathscr{P}_N : \ell(\gamma) \leq T \, \langle \gamma \rangle_G = C\}|}{|\{\gamma \in \mathscr{P}_N : \ell(\gamma) \leq T\}|} = \frac{|C|}{|G|},$$

where $|A|$ denotes the cardinality of A. (See also Ueki [29] for results on modular knots.) The aim of this paper is to provide precise asymptotic information about how the periodic orbits in $\mathscr{P}_N$ link with the removed orbits $\{K_1, \ldots, K_N\}$. This may be viewed as a question about how the orbits are distributed in the homology of the knot complement $H_1(M_N, \mathbb{Z})$. The distribution of periodic orbits in the first homology group of M is well understood ([1], [3], [12], [13], [15], [27]) and the above problem may be viewed as a generalisation of this theory. Many of the ideas introduced by McMullen will be invaluable to our analysis.

We shall begin by recalling some classical distribution results for periodic orbits. Let M be a smooth closed (compact and without boundary) oriented Riemannian 3-manifold and let $X^t : M \to M$ be a weak-mixing transitive Anosov flow. This flow has a countably infinite set $\mathscr{P}$ of prime periodic orbits: we denote a typical such orbits by γ and its least period by $\ell(\gamma)$.

The following asymptotic formula is due to Margulis [17], [18]:

$$|\{\gamma \in \mathscr{P} : \ell(\gamma) \le T\}| \sim \frac{e^{h(X)T}}{h(X)T}, \quad \text{as } T \to \infty,$$

where $h(X)$ is the topological entropy of X.

We may also consider how the periodic orbits are distributed with respect to the homology of M. Let $b = b_1(M)$ denote the first Betti number of M, which we shall assume to be at least 1. For $\gamma \in \mathcal{P}$, we shall write $[\gamma]$ for the torsion-free part of the homology class of γ in $H_1(M, \mathbb{Z})$; we may identify this with an element of $\mathbb{Z}^b$. Write $\mathscr{C}$ for the closure of the set of normalised homology classes,

$$\mathscr{C} = \overline{\left\{ \frac{[\gamma]}{\ell(\gamma)} : \gamma \in \mathscr{P} \right\}} \subset \mathbb{R}^b. \tag{1.1}$$

This is a compact convex set with non-empty interior. We say that X^t is *homologically full* if the interior of $\mathscr{C}$ contains zero. Assuming for the moment that X is homologically full, write $\mathscr{P}_0 = \{\gamma \in \mathscr{P} : [\gamma] = 0\}$. Then there exists $0 < h^* \le h(X)$ and $C > 0$ such that

$$|\{\gamma \in \mathscr{P}_0 : \ell(\gamma) \le T\}| \sim C\frac{e^{h^*T}}{T^{1+b/2}}, \quad \text{as } T \to \infty,$$

and a similar formula for $[\gamma] = \alpha$ for any fixed $\alpha \in H_1(M, \mathbb{Z})$ (though the constant C may change) [27].

Dropping the assumption that X is homologically full, more general results hold where, instead of restricting γ to be null-homologous (or to lie in a fixed homology class), we consider γ such that $[\gamma]$ is approximately $T\rho$, where $\rho \in \mathrm{int}(\mathscr{C})$ (Lalley [15] for geodesic flows over surfaces, Babillot and Ledrappier [3] for the general case). The right hand side then takes the form $C_\rho(T)e^{H(\rho)T}T^{-(1+b/2)}$, where $H : \mathrm{int}(\mathscr{C}) \to \mathbb{R}^+$ is a natural entropy function and $C_\rho(T)$ an explicit continuous function which is bounded above and bounded below away from zero.

The object of this paper will be to consider analogous problems for the knot complements introduced above, inspired by the approach of McMullen [19]. Let $K_1, \ldots, K_n \in \mathscr{P}_0$ be a finite set of (integral) null-homologous periodic orbits, which we think of as knots in M, and consider the knot complement $M_N := M \setminus \{K_1, \ldots, K_N\}$. Then, for any $\gamma \in \mathscr{P}_N := \mathscr{P} \setminus \{K_1, \ldots, K_N\}$, we can consider the torsion-free part of its homology class $[\gamma]_N \in H_1(M_N, \mathbb{Z})/\text{torsion}$. Let

$$\mathscr{C}_N = \overline{\left\{ \frac{[\gamma]_N}{\ell(\gamma)} : \gamma \in \mathscr{P}_N \right\}} \subset H_1(M_N, \mathbb{R}) \cong \mathbb{R}^{b+N}.$$

Write

$$\pi_N(T, \lfloor \rho T \rfloor) := |\{\gamma \in \mathscr{P}_N : \ell(\gamma) \le T,\ [\gamma]_N = \lfloor \rho T \rfloor\}|,$$

where $\lfloor \cdot \rfloor$ is defined by choosing a fundamental domain $\mathfrak{F}_N$ for $H_1(M_N, \mathbb{Z})$ as a lattice in $H_1(M_N, \mathbb{R})$ and setting $\lfloor \rho \rfloor$ to be the unique element of $H_1(M_N, \mathbb{Z})/\text{torsion}$ satisfying $\rho - \lfloor \rho \rfloor \in \mathfrak{F}_N$. Here is a sample theorem.

Theorem 1.1. *Let M be a smooth closed oriented Riemannian 3-manifold and let $X^t : M \to M$ be a transitive Anosov flow which mixes exponentially fast. There is a strictly positive real analytic function $\mathfrak{h}_N : \text{int}(\mathscr{C}_N) \to \mathbb{R}^+$ such that, for each $\rho \in \text{int}(\mathscr{C}_N)$, we have*

$$\pi_N(T, \lfloor \rho T \rfloor) \sim c_\rho(T) \frac{e^{\mathfrak{h}_N(\rho)T}}{T^{1+(b+N)/2}},$$

as $T \to \infty$, where $c_\rho(T)$ is bounded above and bounded below away from zero.

Remark 1.2. In fact, the result holds under a weaker hypothesis than exponential mixing. See section 2 below.

We will now outline the content of the rest of the paper. In section 2, we define Anosov flows and introduce the assumptions for our main result. In section 3, we discuss the homology of a link complement, when the link is given by finitely many null-homologous periodic orbits in M. In particular we show that the first homology group is generated by the homology classes of the remaining periodic orbits. In section 4, we see how the homology of orbits is encoded by the modelling of our Anosov flow as a symbolic suspension flow. In section 5, we describe some relevant tools from the thermodynamic formalism for suspension flows. In section 6 we state our main results and show how they follow from the main result of [3].

2. Anosov flows

Let M be a smooth, connected, oriented, closed (compact and without boundary) Riemannian 3-manifold and let $X^t : M \to M$ be a C^1 flow generated by the vector field X. We will assume X^t is Anosov, meaning that the tangent bundle has a continuous DX^t-invariant splitting $TM = E^0 \oplus E^s \oplus E^u$, where E^0 is the one-dimensional bundle spanned by X and where there exist constants $C, \lambda > 0$ such that

(i) $\|DX^t v\| \leq Ce^{-\lambda t}\|v\|$, for all $v \in E^s$ and $t > 0$;
(ii) $\|DX^{-t} v\| \leq Ce^{-\lambda t}\|v\|$, for all $v \in E^u$ and $t > 0$.

This class of flows was introduced by Anosov [2]; for a good modern reference see [10]. In addition, we assume that $X^t : M \to M$ is topologically transitive, i.e. that there is a dense orbit.

We say that the flow is *topologically weak-mixing* if the equation $\psi \circ X^t = e^{iat}\psi$, for $\psi \in C(M, \mathbb{C})$ and $a \in \mathbb{R}$, has only the trivial solution where ψ is constant and $a = 0$. We say it is *topologically mixing* if for all non-empty open sets $U, V \subset M$, there exists t_0 such that $X^t(U) \cap V \neq \varnothing$ for all $t \geq t_0$. For Anosov flows (and more general hyperbolic flows) these conditions are equivalent and are also equivalent to the flow being mixing with respect to the equilibrium state μ of any Hölder continuous function, i.e. that

$$\rho_{F,G}(t) := \int F \circ X^t \, G \, d\mu - \int F \, d\mu \int G \, d\mu \to 0,$$

as $t \to \infty$, for all $F, G \in L^2(\mu)$. A classical result of Plante [23] is that a transitive Anosov flow fails to be topologically weak-mixing if and only if it is the constant time suspension of an Anosov diffeomorphism.

An Anosov flow has a countably infinite set $\mathscr{P}$ of prime periodic orbits. For $\gamma \in \mathscr{P}$, $\ell(\gamma)$ denotes the least period of γ. We have that the flow fails to be mixing if and only if $\{\ell(\gamma) : \gamma \in \mathcal{P}\}$ is contained in a discrete subgroup of $\mathbb{R}$. If $X^t : M \to M$ is mixing then $|\{\gamma \in \mathscr{P} : \ell(\gamma) \leq T\}| \sim e^{hT}/hT$, where h is the topological entropy [17],[18],[20].

A transitive Anosov flow is (bounded-to-one) semiconjugate to a suspension flow over a subshift of finite type. (This will be discussed in greater detail in section 4.) We shall consider flows such that the roof function cannot be chosen to be locally constant; we call such flows *not locally constant*. This is an open dense condition which is implied by the flow having *good asymptotics* in the sense defined in [9]. (We omit the rather complicated definition.) In particular, this holds if the flow is exponentially mixing, i.e. that $\rho_{F,G}(t)$ converges to zero exponentially fast for the equilibrium state of any Hölder continuous function and all sufficiently regular F and G. The class of exponentially mixing transitive Anosov flows includes geodesic flows over compact surfaces of (not necessarily constant) negative curvature [7]. In fact, Tsujii and Zhang have shown that any *smooth* mixing Anosov flow on a closed 3-manifold is exponentially mixing [28].

3. Knot complements

We now wish to consider M with a finite number of null-homologous periodic orbits removed. (For convenience, we suppose that they are integrally null-homologous, so we end up with integral linking numbers.) Let $K_1, \ldots, K_N$ be null-homologous knots in M (for the moment, they do not need to be periodic orbits). We need to understand the homology of the complement $M_N = M \setminus \bigcup_{i=1}^N K_i$. We will denote the homology class of γ in $H_1(M_N, \mathbb{R})$ by $[\gamma]_N$.

For each $i = 1, \ldots, N$, replace K_i with a tubular neighbourhood $\mathscr{N}(K_i)$ (an embedded solid torus), and (slightly abusing notation) let

$$M_N := M \setminus \bigcup_{i=1}^N \mathrm{int}(\mathscr{N}(K_i)).$$

For each $i = 1, \ldots, N$, let $\mathfrak{m}_i$ be a meridian in $\partial \mathscr{N}(K_i)$.

Lemma 3.1. *Let $K_1, \ldots, K_N$ be null-homologous knots in M. Then*

$$H_1(M_N, \mathbb{R}) \cong H_1(M, \mathbb{R}) \oplus \mathbb{R}^N.$$

In particular, the first Betti number of M_N *is* $b+N$*. Furthermore, the classes* $[\mathfrak{m}_1]_N, \ldots, [\mathfrak{m}_N]_N$ *are linearly independent in* $H_1(M_N, \mathbb{R})$*.*

Proof. The first statement follows from the homology long exact sequence

$$\longrightarrow H_2(M_N, \partial M_N) \longrightarrow H_1(\partial M_N) \longrightarrow H_1(M_N) \longrightarrow H_1(M_N, \partial M_N) \longrightarrow$$

and the 'half lives, half dies' principle (Lemma 3.5 of [11]), which tells us that the dimension of the image of the boundary homomorphism $\partial : H_2(M_N, \partial M_N) \to H_1(\partial M_N)$ is equal to N.

For the second statement, we are indebted to a Mathematics Stack Exchange post by Kyle Miller. By construction, the longitude $\mathfrak{l}_i$ is null-homologous in $M \setminus \mathrm{int}(\mathscr{N}(K_i))$ and so we can choose an embedded surface $S_i \subset M \setminus \mathrm{int}(\mathscr{N}(K_i))$ with boundary $\mathfrak{l}_i$. We may assume that S_i intersects the other knots transversally and so we obtain an embedded surface $S_i' = S_i \cap M_N$ that represents $[\mathfrak{l}_i]$ as a linear combination of the homology classes of the meridians $[\mathfrak{m}_1]_N, \ldots, [\mathfrak{m}_N]_N$. Since $[\mathfrak{m}_1]_N, \ldots, [\mathfrak{m}_N]_N, [\mathfrak{l}_1]_N, \ldots, [\mathfrak{l}_N]_N$ is a basis for $H_1(\partial M_N, \mathbb{R})$, $[\mathfrak{m}_1]_N, \ldots, [\mathfrak{m}_N]_N$ span the image of the inclusion homomorphism $i_* : H_1(\partial M_N, \mathbb{R}) \to H_1(M_N, \mathbb{R})$, and $\dim \ker i_* = N$, it follows that $[\mathfrak{m}_1]_N, \ldots, [\mathfrak{m}_N]_N$ are linearly independent in $H_1(M_N, \mathbb{R})$. □

We will need the following result. (The analogous statement for $H_1(N, \mathbb{Z})$ may be found in [21].)

Proposition 3.2. *The torsion-free part of the group* $H_1(M_N, \mathbb{Z})$ *is generated by* $\{[\gamma]_N : \gamma \in \mathscr{P}_N\}$*.*

Proof. This is a corollary of the main result in [19], where an analogue of the Chebotarev density theorem is proved for weak-mixing Anosov flows. More precisely, for a finite group G, and a surjective homomorphism $\phi : \pi_1(M_N) \to G$, given any conjugacy class C in G we have

$$\lim_{T \to \infty} \frac{|\{\gamma \in \mathscr{P}_N : \ell(\gamma) \le T,\, \phi(\gamma) \in C\}|}{|\{\gamma \in \mathscr{P}_N : \ell(\gamma) \le T\}|} = \frac{|C|}{|G|}. \tag{3.1}$$

If $H_1(M_N, \mathbb{Z})$/torsion is not generated by the orbits in $\mathscr{P}_N$ then there is some proper cofinite subgroup $H \le H_1(M_N, \mathbb{Z})$, such that

$$\{[\gamma]_N : \gamma \in \mathscr{P}_N\} \subset H.$$

If we set $G = H_1(M_N, \mathbb{Z})/H$ then we have a quotient homomorphism $\phi : \pi_1(M_N) \to G$ and the formula (3.1) is contradicted, since all orbits in $\mathscr{P}_N$ satisfy $\phi(\gamma) = 0$. □

4. Symbolic dynamics

In this section, we discuss how to model our flows by symbolic systems, which (as we will see in the next section) will also keep track of the homology and linking numbers of periodic orbits. This is classical material, due independently to Bowen and Ratner. We begin by defining subshifts of finite type.

4.1. **Subshifts of finite type and coding.** Let $k \geq 2$, and let Γ be a directed graph on k vertices $V(\Gamma)$, labelled $1, \ldots, k$, and with directed edges $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. (In particular, for each ordered pair of vertices (i,j) there is at most one directed edge from i to j.) Define

$$\Sigma(\Gamma) := \{(x_n)_{n\in\mathbb{Z}} \in V(\Gamma)^{\mathbb{Z}} : (x_n, x_{n+1}) \in E(\Gamma) \text{ for all } n\},$$

i.e. the set of bi-infinite walks on Γ. The shift map $\sigma : \Sigma(\Gamma) \to \Sigma(\Gamma)$ is defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$.

We give $V(\Gamma)$ the discrete topology, $V(\Gamma)^{\mathbb{Z}}$ the product topology, and $\Sigma(\Gamma)$ the subspace topology. The following metrics are compatible with this topology. Fix $0 < \theta < 1$. For $x, y \in X$ set $d_\theta(x,y) = \theta^{n(x,y)}$, where

$$n(x,y) = \sup\{n \in \mathbb{Z}^+ : x_i = y_i \ \forall |i| < n\}.$$

With this metric in place, let F_θ denote the set of real-valued d_θ-Lipschitz functions on $\Sigma(\Gamma)$.

We will say that Γ is *aperiodic* if there exists $N \geq 1$ such that, for each ordered pair of vertices (i,j), there is a path of length N from i to j. This condition is equivalent to $\sigma : \Sigma(\Gamma) \to \Sigma(\Gamma)$ being topologically mixing (i.e. that there exists $N \geq 1$ such that for all non-empty open sets $U, V \subset \Sigma(\Gamma)$, we have $\sigma^n(U) \cap V \neq \varnothing$ for all $n \geq N$).

Given a strictly positive function $r \in F_\theta$, we can suspend $\Sigma(\Gamma)$ with r as roof function, to obtain the space

$$\Sigma(\Gamma, r) = \{(x,t) \in \Sigma(\Gamma) \times \mathbb{R} : 0 \leq t \leq r(x)\}/(x, r(x)) \sim (\sigma(x), 0).$$

The suspension flow $\sigma_t^r : \Sigma(\Gamma, r) \to \Sigma(\Gamma, r)$ is defined by $\sigma_t^r[x,s] = [x, s+t]$, where $[x,s]$ is the equivalence class of (x,s) in $\Sigma(\Gamma, r)$. According to the classic results of Bowen and Ratner from the 1970s, Anosovs flows (and more general hyperbolic flows) may be modelled by suspension flows of this type.

Theorem 4.1 (Bowen [4], Ratner [24])**.** *Let $X^t : M \to M$ be a topologically weak-mixing transitive Anosov flow. Then there exists a suspension flow $(\sigma^r, \Sigma(\Gamma, r))$, over a topologically mixing $(\sigma, \Sigma(\Gamma))$, and a continuous surjection $\pi : \Sigma(\Gamma, r) \to M$ such that*

- (i) *For all $t \in \mathbb{R}$, $\pi \circ \sigma_t^r = X^t \circ \pi$;*
- (ii) *There exists $N \in \mathbb{N}$ such that $\#\pi^{-1}(y) \leq N$ for all $y \in M$;*
- (iii) *σ^r is topologically weak-mixing and transitive.*

The map π is constructed as follows. Using the structure of the stable and unstable manifolds of X, one can construct a Markov section. In particular, this gives a finite set $\{R_1, \ldots R_k\}$ of codimension one cross-sections to the flow (or rectangles), such that every orbit of X^t intersects $R = \bigcup_{i=1}^k R_i$ infinitely often in the past and future, and these intersections are transversal. Letting $\tau : R \to R$ be the first return map to R (or Poincaré map), define a directed graph Γ on k vertices $\{1, 2, \ldots, k\}$ as follows. First, denote by ij the directed edge from vertex i to vertex j. Include ij in the directed edge set

$E(\Gamma)$ whenever $R_i \cap \tau^{-1}(R_j) \neq \varnothing$. This yields a connected directed graph, with at most two edges (of opposite direction) between two vertices. The rectangles are chosen with the properties that there exists $T > 0$ such that $M \subset X^{[0,T]}R$, and that , for each $x \in \Sigma(\Gamma)$, $\bigcap_{n\in\mathbb{Z}} \tau^n(R_{x_n})$ contains exactly one point. With this, we can define $\pi' : \Sigma(\Gamma) \to R$ by $\pi'(x) \in \bigcap_{n\in\mathbb{Z}} \tau^n(R_{x_n})$. Setting $r : R \to \mathbb{R}^+$ to be the first return time to R, we can extend π' to a map $\pi : \Sigma(\Gamma, r) \to M$, by $\pi[x, s] = X^s(\pi'(x))$.

4.2. **Homology of periodic orbits via symbolic dynamics.** We need to understand how periodic orbits for X^t are encoded by π and how homology and linking may be captured by the symbolic model.

Given $1 \leq i, j \leq k$, let E_{ij} be the set of points on flow lines going from $\mathrm{int}(R_i)$ to $\mathrm{int}(R_j)$, and let

$$U = \left(\bigcup_{i=1}^{k} \mathrm{int}(R_i)\right) \cup \left(\bigcup_{1\leq i,j\leq k} E_{ij}\right).$$

Let $\mathscr{P}_{\sigma^r}$ denote the set of prime periodic orbits for σ^r. Given a periodic orbit $\gamma \in \mathscr{P}$, there is a periodic orbit $\eta \in \mathscr{P}_{\sigma^r}$, with $\ell(\gamma) = \ell(\eta)$, such that $\gamma = \pi(\eta)$. Furthermore, η is unique as long as $\gamma \subset U$, i.e. γ does not pass through the edge of some rectangle. In the particular case of a 3-manifold, uniqueness holds except for a finite number of orbits. (In higher dimensions, there many be infinitely many non-uniquely coded orbits but they exhibit a slower rate of exponential growth.) Thus π induces a one-to-one correspondence for periodic orbits in $\mathscr{P}$ and $\mathscr{P}_{\sigma^r}$, up to removing finitely many.

An additional feature of the symbolic dynamics is that, as argued in [19], we can arrange for the removed periodic orbits $K_1, \ldots, K_N$ to sit in the boundary of the sections. Indeed, suppose that, for some $1 \leq n \leq N$, $x \in K_n \cap \mathrm{int}(R_i)$ (by transversality, there can only be finitely many such x). Then we may split R_i into two pieces $R_{i,1}$ and $R_{i,2}$ at the point x, so that

$$\{R_1, \ldots, R_{i-1}, R_{i,1}, R_{i,2}, R_{i+1}, \ldots, R_k\}$$

still forms a Markov section with the same properties as above. With this in mind, we will henceforth assume that $K_1, \ldots, K_N$ are in the boundary of R.

Let $\bar{\Gamma}$ be the graph obtained by removing the direction from edges of Γ, but retaining any multiple edges between vertices. (So if ij and ji are both directed edges then $\bar{\Gamma}$ has two undirected edges between i and j.) One can then choose a natural embedding $\iota : \bar{\Gamma} \hookrightarrow U \subset M_N$. The following result appears as Lemma 5.1 of [19].

Proposition 4.2. *The embedding $\iota : \bar{\Gamma} \to U$ induces a surjective homomorphism $\iota_* : \pi_1(\bar{\Gamma}) \to \pi_1(M_N)$.*

We now describe how the homology of periodic orbits can be encoded using the symbolic dynamics. For each $i \in \{1, \ldots, k\}$, fix a path $p(1, i)$ from

1 to i in $\bar{\Gamma}$, noting that paths in $\bar{\Gamma}$ need not follow the directions of edges of Γ. That such paths exist for all i is a consequence of transitivity of σ. Let $p(i,1)$ be the path from i to 1 obtained by following $p(1,i)$ backwards. For any vertex j which satisfies that $ij \in E(\Gamma)$, let $e(i,j)$ be the corresponding edge between i,j in $\bar{\Gamma}$. Then we can form a loop $K(i,j)$ in $\bar{\Gamma}$ given by

$$K(i,j) = 1 \xrightarrow{p(1,i)} i \xrightarrow{e(i,j)} j \xrightarrow{p(j,1)} 1,$$

which has a homotopy class $\langle K(i,j)\rangle \in \pi_1(\bar{\Gamma})$. From Proposition 4.2, we have a surjective homomorphism $\iota_* : \pi_1(\bar{\Gamma}) \to \pi_1(M_N)$. We also have the Hurewicz homomorphism

$$q : \pi_1(M_N) \to H_1(M_N, \mathbb{Z})$$

and the projection homomorphism

$$\mathrm{proj} : H_1(M_N, \mathbb{Z}) \to H_1(M_N, \mathbb{Z})/\mathrm{torsion} \cong \mathbb{Z}^{b+N}.$$

Let $\rho = \mathrm{proj} \circ q \circ \iota_* : \pi_1(\bar{\Gamma}) \to \mathbb{Z}^{b+N}$, and define $f : \Sigma(\Gamma) \to \mathbb{Z}^{b+N}$ by

$$f(x) = \rho([K(x_0, x_1)]). \tag{4.1}$$

Clearly, f is locally constant. Further, as ρ is a homomorphism, we have that for a periodic point $x = \overline{x_0x_1\ldots x_{n-1}}$, the Birkhoff sum

$$\begin{aligned} f^n(x) :&= f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x) \\ &= \rho([K(x_0,x_1)]_N[K(x_1,x_2)]_N \cdots [K(x_{n-2},x_{n-1})]_N[K(x_{n-1},x_0)]_N) \\ &= \rho([c(x_0,x_1,\ldots,x_{n-1},x_0)]_N), \end{aligned}$$

where $c(x_0,x_1,\ldots,x_{n-1},x_0)$ is the cycle

$$x_0 \xrightarrow{e(x_0,x_1)} x_1 \xrightarrow{e(x_1,x_2)} x_2 \to \ldots \to x_{n-2} \xrightarrow{e(x_{n-2},x_{n-1})} x_{n-1} \xrightarrow{e(x_{n-1},x_0)} x_0.$$

This construction leads to the following.

Lemma 4.3. *Let $\eta \in \mathscr{P}_{\sigma^r}$, and $x \in \Sigma(\Gamma)$ the corresponding periodic point for σ, with period $n \in \mathbb{N}$. Then $\ell(\pi(\eta)) = \ell(\eta) = r^n(x)$, and $f^n(x) = [\pi(\eta)]_N$.*

5. Thermodynamic formalism for suspension flows

To formulate and prove our results, we will need to use the machinery of thermodynamic formalism. Since we need to deal with linking numbers and homology of the knot complement M_N, we will not be able to use thermodynamic functions defined with respect to the Anosov flow $X^t : M \to M$. Rather, we will work with the shift $\sigma : \Sigma(\Gamma) \to \Sigma(\Gamma)$ and suspension flow $\sigma_t^r : \Sigma(\Gamma, r) \to \Sigma(\Gamma, r)$.

Let $\mathscr{M}_\sigma$ denote the space of σ-invariant Borel probability measures on $\Sigma(\Gamma)$. For a continuous function $\varphi : \Sigma(\Gamma) \to \mathbb{R}$, we define its *pressure* (with respect to σ) $P_\sigma(\varphi)$ by

$$P_\sigma(\varphi) = \sup_{m \in \mathscr{M}_\sigma} \left\{ h_m(\sigma) + \int \varphi \, dm \right\},$$

where $h_m(\sigma)$ is the measure theoretic entropy. If φ is Hölder continuous then the supremum is attained at a unique measure m_φ, which is called the equilibrium state for φ; this measure is ergodic and fully supported.

Now let $\mathscr{M}_{\sigma^r}$ denote the space of σ^r-invariant Borel probability measures on $\Sigma(\Gamma, r)$. There is a natural bijection between $\mathscr{M}_\sigma$ and $\mathscr{M}_{\sigma^r}$ given by $m \mapsto \widetilde{m}$, where $\widetilde{m}$ is given locally by

$$\frac{m \times \mathrm{Leb}}{\int r\, dm}.$$

For a continuous function $\Phi : \Sigma(\Gamma, r) \to \mathbb{R}$, we define its pressure (with respect to σ^r) $P_{\sigma^r}(\Phi)$ by

$$P_{\sigma^r}(\Phi) = \sup_{\mu \in \mathscr{M}_{\sigma^r}} \left\{ h_\mu(\sigma^r) + \int \Phi \, d\mu \right\}. \tag{5.1}$$

One can put a metric on $\Sigma(\Gamma, r)$ [5] and consider Hölder continuous functions with respect to this metric, but for our purposes it is sufficient to consider functions satisfying the following (strictly weaker) condition. We say that Φ is *fibre-Hölder* if $\varphi : \Sigma(\Gamma) \to \mathbb{R}$, defined by

$$\varphi(x) = \int_0^{r(x)} \Phi(x, t)\, dt, \tag{5.2}$$

is Hölder continuous. If Φ is fibre-Hölder then the supremum in (5.1) is attained at a unique measure μ_Φ, which is called the equilibrium state for Φ; this measure is ergodic and fully supported. Furthermore, μ_Φ is given locally by the product

$$\frac{m_{-P(\Phi)r+\varphi} \times \mathrm{Leb}}{\int r\, dm_{-P(\Phi)r+\varphi}},$$

where φ is the function in (5.2).

It will also be convenient to pass from functions on $\Sigma(\Gamma)$ to functions on $\Sigma(\Gamma, r)$, so we 'lift' functions in the following way. Choose a smooth function $\kappa : [0,1] \to \mathbb{R}^+$ satisfying $\kappa(0) = \kappa(1) = 0$ and $\int_0^1 \kappa(t)\, dt = 1$. Let $\varphi : \Sigma(\Gamma) \to \mathbb{R}$ be Hölder continuous and define $\widetilde{\varphi} : \Sigma(\Gamma, r) \to \mathbb{R}$ by

$$\widetilde{\varphi}(x, t) = \frac{\varphi(x)}{r(x)} \kappa\left(\frac{t}{r(x)}\right).$$

Then

$$\varphi(x) = \int_0^{r(x)} \widetilde{\varphi}(x, t)\, dt$$

and $\widetilde{\varphi}$ is automatically fibre-Hölder. Furthermore, for a periodic orbit η of σ^r, corresponding to a point $x \in \Sigma(\Gamma)$ of least period n,

$$\frac{\varphi^n(x)}{r^n(x)} = \frac{1}{\ell(\eta)} \int_\eta \widetilde{\varphi},$$

where $\int_\eta \tilde{\varphi} = \int_0^{\ell(\eta)} \tilde{\varphi}(\sigma_t^r x_\eta)\, dt$, for $x_\eta \in \eta$. This integral defines a probability measure, μ_η, by

$$\int \Phi \, d\mu_\eta = \frac{1}{\ell(\eta)} \int_\eta \Phi.$$

We now wish to consider pressure functions of a particular form. Fix a fibre-Hölder function $F : \Sigma(\Gamma, r) \to \mathbb{R}^d$ (for some $d \geq 1$) and let $f : \Sigma(\Gamma) \to \mathbb{R}^d$ be defined by the higher dimensional analogue of (5.2), i.e.

$$f(x) = \int_0^{r(x)} F(x,t)\, dt.$$

We then have a set $\mathscr{C}_F \subset \mathbb{R}^d$ defined in the following equivalent ways:

$$\mathscr{C}_F = \overline{\left\{ \frac{1}{\ell(\eta)} \int_\eta F \, : \, \eta \in \mathscr{P}_{\sigma^r} \right\}} = \left\{ \int F d\mu \, : \, \mu \in \mathscr{M}_{\sigma^r} \right\}$$
$$= \left\{ \frac{\int f \, dm}{\int r \, dm} \, : \, m \in \mathscr{M}_\sigma \right\}.$$

We write $F = (F_1, \ldots, F_d)$ and, for $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, write $\langle u, F \rangle = u_1 F_1 + \cdots + u_d F_d$. We then define the pressure function $\mathfrak{p}_F : \mathbb{R}^d \to \mathbb{R}$ by

$$\mathfrak{p}_F(u) = P_{\sigma^r}(\langle u, F \rangle).$$

This is real analytic and convex and satisfies

$$\nabla \mathfrak{p}_F(u) = \int F \, d\mu_{\langle u, F \rangle}.$$

We have the following standard result (see [14], for example).

Lemma 5.1. *The following are equivalent.*

- *(i) For all $u \in \mathbb{R}^d \setminus \{0\}$, the function $\langle u, F \rangle : \Sigma(\Gamma, r) \to \mathbb{R}$ is not cohomologous to a constant.*
- *(ii) The pressure function $\mathfrak{p}_F$ is strictly convex.*
- *(iii)* $\mathrm{int}(\mathscr{C}_F) \neq \varnothing$.

Assuming that any (and hence all) of the statements in Lemma 5.1 hold, we define an entropy function $\mathfrak{h}_F : \mathrm{int}(\mathscr{C}_F) \to \mathbb{R}$ by

$$\mathfrak{h}_F(\rho) = \sup \left\{ h_\mu(\sigma) \, : \, \mu \in \mathscr{M}_{\sigma^r}, \, \int F \, d\mu = \rho \right\}.$$

This leads to the following lemma.

Lemma 5.2. *The map $u \mapsto \nabla \mathfrak{p}_F(u)$ is a smooth diffeomorphism between $\mathbb{R}^d$ and $\mathrm{int}(\mathscr{C}_F)$. Furthermore, $\mathfrak{h}_F$ is a smooth function on $\mathrm{int}(\mathscr{C}_F)$ and $(\nabla \mathfrak{p}_F)^{-1} = -\nabla \mathfrak{h}_F$.*

Proof. This essentially follows from Theorem 26.5 of [25]. Since

$$\mathfrak{p}_F(u) = h_{\mu_{\langle u, F \rangle}}(\sigma^r) + \left\langle u, \int F \, d\mu_{\langle u, F \rangle} \right\rangle$$

for a unique measure $\mu_{\langle u,F\rangle} \in \mathscr{M}_{\sigma^r}$, we have

$$\mathfrak{p}_F(u) = \sup_{z \in \mathscr{C}_F} \Big(\mathfrak{h}_F(z) + \langle u, z\rangle\Big),$$

with the supremum attained uniquely at $z = \int F \, d\mu_{\langle u,F\rangle}$. This shows that the functions $\mathfrak{p}_F$ and $-\mathfrak{h}_F$ are Legendre duals. Theorem 26.5 of [25] then gives that $\nabla \mathfrak{p}_F$ is a diffeomorphism onto its image with inverse $-\nabla \mathfrak{h}_F$. To complete the proof, it only remains to show that $\nabla \mathfrak{p}_F(\mathbb{R}^d) = \mathrm{int}(\mathscr{C}_F)$.

We now use an argument adapted from the proof of Lemma 7 in [16]. Let $z \in \mathrm{int}(\mathscr{C}_F)$ and choose $\epsilon > 0$ such that $B(z, 2\epsilon) \subset \mathscr{C}_F$. Then, for $u \neq 0$, $z + (u/\|u\|)\epsilon \in \mathscr{C}_F$. Thus

$$\langle u, z\rangle + \|u\|\epsilon \leq \sup_{w \in \mathscr{C}_F} \langle u, w\rangle = \sup\left\{\left\langle u, \int F \, d\mu \right\rangle \,:\, \mu \in \mathscr{M}(\sigma^r)\right\} \leq \mathfrak{p}_F(u).$$

Let $e_z : \mathbb{R}^d \to \mathbb{R}$ be defined by $e_z(u) = \mathfrak{p}_F(u) - \langle u, z\rangle$. Then, by the above inequality, $e_z(u) \geq 0$ for all $u \in \mathbb{R}^d$. Thus e_z has a finite minimum attained at a unique $u(z) \in \mathbb{R}^d$. Hence $\nabla e_z(u(z)) = 0$, which is exactly $\nabla \mathfrak{p}_F(u(z)) = z$. □

We will continue to use the notation of this proof: given $\rho \in \mathrm{int}(\mathscr{C}_F)$, let $u(\rho) \in \mathbb{R}^d$ be defined by $\nabla \mathfrak{p}_F(u(\rho)) = \rho$. Furthermore, we will write $\mu_\rho \in \mathscr{M}_{\sigma^r}$ for the equilibrium state for $\langle u(\rho), F\rangle$.

6. Counting results

The main results of the paper will follow from counting results for periodic orbits subject to constraints initiated by Lalley [14] (see also Sharp [26]) and given in streamlined form by Babillot and Ledrappier [3]. The results in [3] are stated for hyperbolic flows but the machinery all works for suspension flows.

We make the following definitions. Let $\mathscr{A}_F$ be the closed subgroup of $\mathbb{R}^d$ generated by

$$\left\{\int_\eta F \,:\, \eta \in \mathscr{P}_{\sigma^r}\right\}$$

and let $\widetilde{\mathscr{A}_F}$ be the closed subgroup of $\mathbb{R}^{d+1}$ generated by

$$\left\{\left(\ell(\eta), \int_\eta F\right) \,:\, \eta \in \mathscr{P}_{\sigma^r}\right\}.$$

We choose a fundamental domain $\mathfrak{F}$ for $\mathscr{A}_F$ and, for $\rho \in \mathbb{R}^d$, we define $\lfloor \rho \rfloor \in \mathscr{A}_F$ by $\rho - \lfloor \rho \rfloor \in \mathfrak{F}$.

Let $g_0 : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support and let $g : \mathscr{A}_F \to \mathbb{R}$ be finitely supported. Following [3], we will study the functional N_T^ρ defined by

$$N_T^\rho(g_0 \otimes g) = \sum_{\eta \in \mathscr{P}_{\sigma^r}} g_0(\ell(\eta) - T) g\left(\int_\eta F - \lfloor T\rho \rfloor\right).$$

The following is a special case of Theorem 1.2 in [3].

Proposition 6.1. *Suppose that $\widetilde{\mathscr{A}_F} = \mathbb{R} \times \mathscr{A}_F$. Then, for all $\rho \in \mathrm{int}(\mathscr{C}_F)$, we have*

$$N_T^\rho(g_0 \otimes g) \sim \frac{\sqrt{|\det \nabla^2 \mathfrak{h}_F(\rho)|}}{(2\pi)^{d/2}} a_\rho(g_0, g) \frac{e^{\mathfrak{h}_F(\rho)T + \langle u(\rho), T\rho - \lfloor T\rho \rfloor \rangle}}{T^{1+d/2}},$$

where

$$a_\rho(g_0, g) = \int_{\mathbb{R}} e^{\mathfrak{p}_F(u(\rho))x} g_0(x)\, dx \sum_{y \in \mathscr{A}_F} e^{-\langle u(\rho), y\rangle} g(y),$$

as $T \to \infty$.

We shall apply this result when $d = b + N$ and $F = \widetilde{f}$, where $f : \Sigma(\Gamma) \to \mathbb{Z}^{b+N}$ is the locally constant function defined in (4.1). Hence, by Proposition 3.2, $\mathscr{A}_F = \mathbb{Z}^d$. However, there are some additional complications that we need to account for. First, we note that

$$\sum_{\gamma \in \mathscr{P}_N} g_0(\ell(\gamma) - T) g\left([\gamma]_N - \lfloor T\rho \rfloor\right) = N_T^\rho(g_0 \otimes g) + O(1),$$

so the removed orbits do not cause a problem. Second, we need to show that $\widetilde{\mathscr{A}_F} = \mathbb{R} \times \mathscr{A}_F$. We do this in the next lemma.

Lemma 6.2. *Suppose that X^t is not locally constant. Then, for F as in case (i) and case (ii) above, we have*

$$\widetilde{\mathscr{A}_F} = \mathbb{R} \times \mathscr{A}_F.$$

Proof. Clearly, $\widetilde{\mathscr{A}_F}$ is a closed subgroup of $\mathbb{R} \times \mathscr{A}_F = \mathbb{R} \times \mathbb{Z}^d$. Therefore, it is sufficient to show that if χ is a character of $\mathbb{R} \times \mathbb{Z}^d$ which is trivial on $\widetilde{\mathscr{A}_F}$ then χ is trivial on $\mathbb{R} \times \mathbb{Z}^d$.

The characters of $\mathbb{R} \times \mathbb{Z}^d$ taken the form $\chi_{t,u}(x, y) = e^{2\pi i(tx + \langle u, y\rangle)}$, with $(t, u) \in \mathbb{R} \times \mathbb{R}^d/\mathbb{Z}^d$. If $\chi_{t,u}$ is trivial on $\widetilde{\mathscr{A}_F}$ then

$$\chi_{t,u}\left(\ell(\eta), \int_\eta F\right) = \exp 2\pi i \left(t\ell(\eta) + \int_\eta \langle u, F\rangle\right) = 1 \quad \forall \eta \in \mathscr{P}_{\sigma^r} \tag{6.1}$$

Passing to the shift map, this may be rewritten in the form

$$\exp 2\pi i(tr^n(x) + \langle u, f^n(x)\rangle) = 1 \tag{6.2}$$

whenever $\sigma^n x = x$ and $n \geq 1$. By Proposition 5.2 of [22], equation (6.2) may itself be rewritten as

$$2\pi tr + 2\pi\langle u, f\rangle + \psi \circ \sigma - \psi = \varphi, \tag{6.3}$$

where $\psi \in C(\Sigma(\Gamma), \mathbb{R})$ and $\varphi \in C(\Sigma(\Gamma), 2\pi\mathbb{Z})$. In particular, if (6.3) holds then $2\pi tr$ is cohomologous to a locally constant function. Applying the not locally constant condition, this forces $t = 0$.

Substituting $t = 0$ into (6.1),we have

$$\exp 2\pi i \left\langle u, \int_\eta F \right\rangle = 1 \quad \forall \eta \in \mathscr{P}_{\sigma^r}. \tag{6.4}$$

Since $\{[\gamma]_N : \gamma \in \mathscr{P}\}$ generates the torsion free part of $H_1(M_N, \mathbb{Z})$, we have that $u = 0 \in \mathbb{R}^d/\mathbb{Z}^d$. □

Thus we have shown that Proposition 6.1 may be applied in our cases. We may pass from $N_T^\rho(g_0 \otimes g)$ to a counting function by choosing g_0 to approximate the indicator function of the interval $(-\delta, 0]$, say, and choosing g to be the indicator function of an element of $\mathscr{A}_F$. We also use the following notation. Let $C_b(\mathbb{R}^+)$ denote the set set of bounded continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$ and let

$$C_{b,+}(\mathbb{R}^+) = \left\{ f \in C_b(\mathbb{R}^+) : \inf_{T>0} f(T) > 0 \right\}.$$

Here are our main results.

Theorem 6.3. *Let M be a smooth closed oriented Riemannian 3-manifold and let $X^t : M \to M$ be a transitive Anosov flow which is not locally constant. There is a strictly positive real analytic function $\mathfrak{h}_N : \mathrm{int}(\mathscr{C}_N) \to \mathbb{R}^+$ such that, for each $\rho \in \mathrm{int}(\mathscr{C}_N)$ and $\alpha \in H_1(M_N, \mathbb{Z})/\mathrm{torsion}$, we have*

$$|\{\gamma \in \mathscr{P}_N : T - \delta < \ell(\gamma) \le T,\ [\gamma]_N = \lfloor T\rho \rfloor + \alpha\}| \sim c_{\rho,\alpha}(T) \frac{e^{\mathfrak{h}_N(\rho)T}}{T^{1+(b+N)/2}},$$

as $T \to \infty$, where $c_{\rho,\alpha} \in C_{b,+}(\mathbb{R}^+)$.

If $0 \in \mathrm{int}(\mathscr{C}_N)$ then the next corollary gives a result for periodic orbits with fixed homology class in $H_1(M_N, \mathbb{Z})$. However, by work of Dehornoy, there are examples of geodesic flows where this condition fails, since all linking numbers of pairs of periodic orbits are negative [6].

Corollary 6.4. *If $0 \in \mathrm{int}(\mathscr{C}_N)$ then, for every $\alpha \in H_1(M_N, \mathbb{Z})/\mathrm{torsion}$,*

$$|\{\gamma \in \mathscr{P}_N : \ell(\gamma) \le T,\ [\gamma]_N = \alpha\}| \sim c(m) \frac{e^{\mathfrak{h}_N(0)T}}{T^{1+(b+N)/2}},$$

as $T \to \infty$, for some $c(m) > 0$.

Remark 6.5. One may replace the condition $T - \delta < \ell(\gamma) \le T$ with $\ell(\gamma) \le T$ by a simple argument. We can describe the functions $c_{\rho,\alpha}$ and constants $c(\alpha)$ more explicitly; see [3], [14], [15], [26], [27]. If γ is null-homologous in M then $[\gamma]_N$ allows us to read off the linking numbers of γ with K_i, $i = 1, \ldots, N$.

Following the arguments of [3], we can show that the periodic orbits considered in Theorem 6.3 become equidistributed with respect to $\pi_*(\mu_\rho)$, the projection of μ_ρ to M, as $T \to \infty$ (where μ_ρ is the equilibrium state defined at the end of section 5). We have the following result.

Theorem 6.6. *Let M be a smooth closed oriented Riemannian 3-manifold and let $X^t : M \to M$ be a transitive Anosov flow which is not locally constant. For each $\rho \in \mathrm{int}(\mathscr{C}_N)$ and $\alpha \in H_1(M_N, \mathbb{Z})/\mathrm{torsion}$, and for every* *continuous function* $\varphi : M \to \mathbb{R}$*, we have*

$$\lim_{T\to\infty} \frac{\displaystyle\sum_{\gamma\in\mathscr{P}_N \,:\, T-\delta<\ell(\gamma)\le T,\ [\gamma]_N=\lfloor T\rho\rfloor+\alpha} \frac{1}{\ell(\gamma)}\int_\gamma \varphi}{|\{\gamma \in \mathscr{P}_N \,:\, T-\delta < \ell(\gamma) \le T,\ [\gamma]_N = \lfloor T\rho\rfloor + \alpha\}|} = \int \varphi \, d\pi_*(\mu_\rho).$$

We end with a result for amenable covers of M_N. Let G be any quotient group of $\pi_1(M_N)$, with identity element 1_G. Then each periodic orbit $\gamma \in \mathscr{P}_N$ defines a conjugacy class $\langle\gamma\rangle_G$ in G. (The case where G is finite was considered at the start of the paper.) Suppose that G sits above $H_1(M_N, \mathbb{Z})$, i.e. that $G = \pi_1(M_N)/\Lambda$, where Λ is a normal subgroup of the commutator $[\pi_1(M_N), \pi_1(M_N)]$. Supposing that $0 \in \mathrm{int}(\mathscr{C}_N)$, the exponential growth rate of periodic orbits for which $\langle\gamma\rangle_G$ trivial is at most $\mathfrak{h}_N(0)$. The results of [8] may be adapted to show that equality holds if and only if G is amenable. More precisely, we have the following theorem.

Theorem 6.7. *Let* M *be a smooth closed oriented Riemannian* 3*-manifold and let* $X^t : M \to M$ *be a transitive Anosov flow which is not locally constant and such that* $0 \in \mathrm{int}(\mathscr{C}_N)$*. Let* $G = \pi_1(M_N)/\Lambda$*, where* Λ *is a normal subgroup of* $[\pi_1(M_N), \pi_1(M_N)]$*. Then*

$$\lim_{T\to\infty} \frac{1}{T} \log |\{\gamma \in \mathscr{P}_N \,:\, T-\delta < \ell(\gamma) \le T,\ \langle\gamma\rangle_G = \{1_G\}\}| \le \mathfrak{h}_N(0).$$

with equality if and only if G *is amenable.*

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