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Lengths, quasi-morphisms and statistics for free groups

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Dedicated to Professor Toshikazu Sunada on the occasion of his sixtieth birthday.

Abstract. We discuss natural classes of length functions and quasi-morphisms on free groups and obtain Gaussian limit laws.

0. Introduction

There is considerable interest in spaces of length functions defined on a free group F on $k \geq 2$ generators (or on its set of conjugacy classes). For example, the Culler-Vogtmann Outer space [**13**] has such a interpretation. The first aim of this note is to suggest a general notion of length function, which includes those which arise from isometric actions on negatively curved spaces and which are sufficiently regular to lead to nice asymptotic properties.

The second aim of the paper is to discuss the statistics of images group elements (or conjugacy classes) in $\mathbb{R}$ under natural mappings. A number of papers have studied this issue (and the related case of compact surface groups) for homomorphisms from F to $\mathbb{R}$ (or, more generally, $\mathbb{R}^d$, where $1 \leq d \leq k$.) A purely group theoretic point of view was taken in [**33**] and [**37**], while the papers [**26**],[**27**],[**32**] consider homomorphisms defined by periods of cusp forms on associated hyperbolic surfaces. In addition, many papers in ergodic theory contain related results, though the connection is not made explicit, notably [**3**],[**11**],[**23**]. In all these papers, suitably normalized images are shown to converge to a Gaussian distribution, as the number of elements considered increases according to some length function.

In this note we shall address the analogous question when the homomorphism is replaced by a more general function $\psi : F \to \mathbb{R}$. To obtain non-trivial results, it is necessary to impose some conditions on ψ. A natural class to consider is the set of quasi-morphisms $\psi : F \to \mathbb{R}$. These functions are of independent importance in many areas [**20**]. A function $\psi : F \to \mathbb{R}$ is called a quasi-morphism if $\psi(xy) - \psi(x) - \psi(y)$ is bounded for $(x, y) \in F \times F$. It is clear from the definition that if ψ is either a homomorphism or a bounded function then it is a quasi-morphism.

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We shall be interested in quasi-morphisms which satisfy an additional regularity property explained in section 1; we shall call these Hölder quasi-morphisms.

Below is an example of the type of statistical limit law that holds for Hölder quasi-morphisms. We use $|g|$ to denote the word length of g with respect to some fixed set of free generators for F.

THEOREM [**17**]. *Suppose that ψ is a Hölder quasi-morphism. Provided ψ is not bounded, the distribution of the normalized images $\{\psi(x)/\sqrt{|x|} : x \in F,\ |x| = n\}$ converges to a non-degenerate normal distribution with mean zero, as $n \to +\infty$.*

In this note, we have concentrated on free groups. However, using similar ideas, analogous results may be obtained for the fundamental groups of compact surfaces of genus at least 2 [**17**]. In recent work, Calegari and Fujiwara have obtained central limit theorems for quasi-morphisms of general Gromov hyperbolic groups (ordered by word length with respect to a finite set of generators) provided the quasi-morphisms satisfy a condition called bicombability [**9**]. (Bicombability is more restrictive than our Hölder condition but it includes, for example, the generalization of the Brooks quasi-morphisms introduced by Epstein and Fujiwara [**15**].)

We now give a brief outline of the contents of the paper. In section 1, we introduce a set of axioms for a useful class of length functions on a free group and, in section 2, discuss examples. In section 3, we discuss quasi-morphisms and state our main result on limit laws. In sections 4 and 5, we consider subshifts of finite type associated to free groups and the associated thermodynamic formalism. In section 6, we show how to prove a central limit theorem for conjugacy classes in a particular case and, in section 7, discuss how this may be extended to deal with group elements.

Some of this material was contained in the University of Manchester PhD thesis of Matthew Horsham.

1. Length Functions on Free Groups

Let F be a free group on $k \geq 2$ generators and let 1 denote the identity element in F. Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be a free generating set for F and write $\mathcal{A}^{-1} = \{a_1^{-1}, \ldots, a_k^{-1}\}$. Each $x \in F$, $x \neq 1$ has a unique representation as a reduced word in $\mathcal{A} \cup \mathcal{A}^{-1}$, i.e.,

$$x = x_0 x_1 \cdots x_{n-1},$$

where $x_i \in \mathcal{A} \cup \mathcal{A}^{-1}$, $i = 0, 1 \ldots, n-1$, and $x_{i+1} \neq x_i^{-1}$, $i = 0, 1, \ldots, n-2$. This is also the unique shortest representation of x as a word in the generators. We define the associated word length $|\cdot|$ on F by $|x| = n$ (and $|1| = 0$). Note that $|x^{-1}| = |x|$. We also define the Lyndon-Gromov product $(\cdot,\cdot) : F \times F \to \mathbb{R}^+$ by

$$(x, y) = (|x| + |y| - |x^{-1}y|)/2.$$

DEFINITION. We say that a function $L : F \to \mathbb{R}^+$ is a length function if

(L1) $L(1) = 0$;

(L2) there exists $A > 0$ such that, for all $x, y \in F$,

$$L(xy) \leq L(x) + L(y) + A;$$

(L3) there exists $C_1 > 0$ such that, for all $x \in F$,

$$L(x) \geq C_1 |x|.$$

(It is easy to check that this definition is independent of the choice of generating set $\mathcal{A}$.) Also note that (L2) gives the bound

$$L(x) \leq C_2|x|,$$

where $C_2 = (\sup_{a \in \mathcal{A} \cup \mathcal{A}^{-1}} L(a) + A) > 0$.

In order to prove results, we need an additional regularity condition on our length functions and we hope that the following one is natural. For any $a \in F$, write $\Delta_a L(x) = L(x) - L(ax)$.

Definition. We say that a length function $L : F \to \mathbb{R}^+$ is a Hölder length function if

(L4) for any $a \in F$, there exist $C_3, C_4 > 0$ such that, for all $x, y \in F$,

$$|\Delta_a L(x) - \Delta_a L(y)| \leq C_3 e^{-C_4(x,y)}.$$

(This definition is also independent of the choice of $\mathcal{A}$.)

It is easy to see that the word length $|\cdot|$ is itself a Hölder length function.

The condition (L2) allows us to define an associated homogeneous length function $l : F \to \mathbb{R}^+$ by

$$l(x) = \lim_{n \to +\infty} \frac{1}{n} L(x^n) = \inf_{n \geq 1} \frac{1}{n} L(x^n).$$

Where it is necessary to make the dependence on L clear, we shall write $l = \mathfrak{H}(L)$. Note that $l(x^n) = nl(x)$, for all $n \geq 0$. The condition (L2) ensures that $l(x) > 0$, for all $x \in F \backslash \{1\}$. (However, l itself fails to satisfy (L2).)

Let $\mathcal{C}(F)$ denote the set of non-trivial conjugacy classes in F. Each $w \in \mathcal{C}(F)$ contains a family of cyclically reduced words in $\mathcal{A} \cup \mathcal{A}^{-1}$, i.e., a reduced word $x_0 x_1 \cdots x_{n-1}$, such that $x_{n-1} \neq x_1^{-1}$, and its cyclic permutations. Then w contains no shorter words and we define $|w| = n = \min\{|x| : x \in w\}$.

It is easy to check that $l = \mathfrak{H}(L)$ is constant on conjugacy classes and so gives a well-defined function $l : \mathcal{C}(F) \to \mathbb{R}^+$. (In fact, for $w \in \mathcal{C}(F)$, $|w| = \mathfrak{H}(|\cdot|)(x)$, for any $x \in w$.)

Definition. We say that L or $l = \mathfrak{H}(L)$ is non-discrete if $\{l(w) : w \in \mathcal{C}(F)\}$ is not contained in a discrete subgroup of $\mathbb{R}$.

2. Examples of Hölder Length Functions

We shall now discuss some examples of Hölder length functions. Classically, in combinatorial group theory, one is interested in so-called based length functions which arise from isometric actions on simplicial $\mathbb{R}$-trees. Such functions were originally defined axiomatically by Lyndon [**24**], where they were $\mathbb{Z}$-valued, while the $\mathbb{R}$-valued case was considered by Chiswell [**10**]. Chiswell (see also [**18**]) discovered the connection with actions on trees. We now give the definition. Let Γ be a finite metric graph (i.e. a finite graph with positive lengths assigned to all the edges, making it into a metric space in the obvious way) such that $\pi_1(\Gamma) \cong F$. The universal cover of Γ, with the lifted metric, is a simplicial $\mathbb{R}$-tree $\mathcal{T}$. Choose a base point p in Γ and let $o \in \mathcal{T}$ be a choice of lift of p. Define $L : F \to \mathbb{R}^+$ by

$$L(x) = d_{\mathcal{T}}(o, ox),$$

then this is a based length function. Different choices of Γ and p may give rise to the same length function: we shall return to this issue later, when we discuss the Outer space associated to F.

A simplicial $\mathbb{R}$-trees are examples of a CAT(-1) spaces. Other examples are given by simply connected Riemannian manifolds with sectional curvatures ≤ -1. A rather more general way of obtaining length functions is the following. Let X be a CAT(-1) space and realize F as a convex co-compact group of isometries of X. Choose $o \in X$ and define $L : F \to \mathbb{R}$ by

$$L(x) = d_X(o, ox). \tag{2.1}$$

THEOREM 1 [**30**], [**31**]. *The function L defined by* (2.1) *is a Hölder length function.*

In some senses, homogeneous length functions are even more natural. Let us return to the metric graphs considered above and be a little more precise. Let $\mathcal{G}$ be a fixed graph with one vertex $*$ and k edges and identify F with $\pi_1(\mathcal{G})$ so that each a_i, $i = 1, \ldots, k$, corresponds to an (oriented) edge. Let Γ be a metric graph with fundamental group F such that each vertex has valency at least three together with a homotopy equivalence $g : \mathcal{G} \to \Gamma$; we call (Γ, g) a marked metric graph. Consider the set of all marked metric graphs whose edge lengths sum to one. We say that (Γ, g) and (Γ', g') are equivalent if there is an isometry $h : \Gamma \to \Gamma'$ such that $g \circ h$ is homotopic to g'. The Culler-Vogtmann Outer space $\mathbb{O}$ is defined to be the set of equivalence classes [**13**]. Its importance is that it is a space on which the outer automorphisms of F act in a natural way. (An alternative definition is the set of equivalence classes of marked metric graphs under the relation $(\Gamma, g) \sim (\Gamma', g')$ if there is a homothety $h : \Gamma \to \Gamma'$ such that $g \circ h$ is homotopic to g'.)

A point in $\mathbb{O}$ may be identified with a homogeneous length function $l : F \to \mathbb{R}^+$ in the following way. Identifying $x \in F$ with a homotopy class in $\pi_1(\mathcal{G})$, $l(x)$ may be defined to be the length of the shortest loop in Γ freely homotopic to $g(x)$. Equivalently, considering the universal cover $\mathcal{T}$, define $l : F \to \mathbb{R}$ by

$$l(x) = \inf_{o \in \mathcal{T}} d_{\mathcal{T}}(o, ox).$$

We have $d_{\mathcal{T}}(o, o(yxy^{-1})) = d_{\mathcal{T}}(oy, (oy)x)$, so it is easy to see that $l(x)$ depends only on the conjugacy class of x. Furthermore, l only depends on the point in $\mathbb{O}$ represented by (Γ, g). For each $x \in F$, there is a unique subset $A(x) \subset \mathcal{T}$, isometric to $\mathbb{R}$, called the axis of x, on which x acts by translation by $l(x)$. These functions $l : F \to \mathbb{R}$ are called hyperbolic length functions [**2**] or translation length functions [**12**]. Of course, we may obtain more examples, parametrized by $\mathbb{O} \times \{\lambda : \lambda > 0\}$, by dropping the requirement that the edge lengths sum to one.

Generalizing again to a CAT(-1) space X and a convex co-compact action $X \times F \to X$, if $L(x) = d_X(o, ox)$ then $l = \mathfrak{H}(L)$ satisfies

$$l(x) = \inf_{o \in X} d_X(o, ox) \tag{2.2}$$

and defines a function $l : \mathcal{C}(F) \to \mathbb{R}^+$. If X is a Riemannian manifold then, for $w \in \mathcal{C}(F)$, $l(w)$ is the length of the unique closed geodesic on X/F in the free homotopy class determined by w.

We shall define three sets of length functions $l : \mathcal{C}(F) \to \mathbb{R}^+$. Let $\mathcal{L}_0(F)$ denote the set of hyperbolic length functions defined above with out the restriction on the edge lengths sum. (Then $\mathbb{O} = P\mathcal{L}_0(F) = \mathcal{L}_0(F)/\sim$, where $l_1 \sim l_2$ if there exists $\lambda > 0$ such that $l_2 = \lambda l_1$.) Let $\mathcal{L}_1(F)$ denote the set of length functions obtained from convex co-compact actions on CAT(-1) spaces, as in (2.2). Finally, let

$$\mathcal{L}_2(F) = \{\mathfrak{H}(L) : L \text{ is a Hölder length function on } F\}.$$

Of course, these also have projectivized versions, $P\mathcal{L}_1(F)$ and $P\mathcal{L}_2(F)$. Clearly,

$$\mathcal{L}_0(F) \subset \mathcal{L}_1(F) \subset \mathcal{L}_2(F).$$

Furthermore, $\mathcal{L}_0(F)$ is a proper subset of $\mathcal{L}_1(F)$.

QUESTION. Do we have $\mathcal{L}_1(F) = \mathcal{L}_2(F)$?

3. Quasi-Morphisms

In this section we shall discuss a class of real-valued functions on a group which provide a reasonable generalization of homomorphisms.

A map $\psi : F \to \mathbb{R}$ is called a quasi-morphism if there exists $D \geq 0$ such that, for all $x, y \in F$,

$$|\psi(xy) - \psi(x) - \psi(y)| \leq D.$$

The most obvious examples of quasi-morphisms are homomorphisms and bounded maps. We shall say that a quasi-morphism ψ is a Hölder quasi-morphism if for any $a \in F$, there exist $C, c >$ such that, for all $x, y \in F$,

$$|\Delta_a \psi(x) - \Delta_a \psi(y)| \leq Ce^{-c(x,y)}.$$

(This definition is also independent of the choice of $\mathcal{A}$.) We say that ψ is homogeneous if $\psi(x^n) = n\psi(x)$, for all $x \in F$ and all $n \in \mathbb{Z}$.

LEMMA 3.1. *A homogeneous quasi-morphism is constant on conjugacy classes.*

PROOF. If ψ is a homogeneous quasi-morphism then

$$\psi(y^{-1}x^n y) = \psi((y^{-1}xy)^n) = n\psi(y^{-1}xy).$$

On the other hand, $\psi(y^{-1}x^n y) - \psi(x^n) = \psi(y^{-1}x^n y) - n\psi(x)$ is bounded as n increases. Dividing by n and letting $n \to +\infty$ gives $\psi(y^{-1}xy) = \psi(x)$, as required.

If ψ is a quasi-morphism then one may obtain a homogeneous quasi-morphism $\varphi = \mathfrak{H}(\psi)$ by the procedure of section 1, i.e.,

$$\varphi(x) = \lim_{n \to +\infty} \frac{1}{n} \psi(x^n).$$

With this definition, we also have

$$\varphi(x^{-1}) = -\varphi(x).$$

LEMMA 3.2. *There exists $C(\psi) \geq 0$ such that, for all $x \in F$, $|\psi(x) - \mathfrak{H}(\psi)(x)| \leq C(\psi)$. (In other words, $\psi = \mathfrak{H}(\psi) + b$, where $b : F \to \mathbb{R}$ is bounded.) As a consequence $\mathfrak{H}(\psi) = 0$ if and only if ψ is bounded.*

PROOF. See page 129 of [**16**].

Note that if ψ is a Hölder quasi-morphism, the associated homogeneous quasi-morphism $\varphi = \mathfrak{H}(\psi)$ is not necessarily Hölder.

Remark. Grigorchuk [**16**] uses the terminology quasicharacter and pseudocharacter for what we call quasi-morphism and homogeneous quasi-morphism, respectively.

Examples.
(i) As mentioned above if $\psi : F \to \mathbb{R}$ is either a homomorphism or a bounded map then ψ is a quasi-morphism. If ψ is a homomorphism the $\mathfrak{H}(\psi) = \psi$. If ψ is bounded then $\mathfrak{H}(\psi) = 0$. It is clear that a homomorphism is a Hölder quasi-morphism, however, a bounded map is in general not Hölder.
(ii) *Brooks examples.* Brooks showed that there were very many quasi-morphisms on F which were not homomorphisms. Let ξ be a reduced word in $\mathcal{A} \cup \mathcal{A}^{-1}$. Define $\psi_\xi : F \to \mathbb{Z}$ by setting $\psi_\xi(x)$ to be the difference between the number of times ξ and ξ^{-1} occur as subwords of x when x is written as a reduced word in $\mathcal{A} \cup \mathcal{A}^{-1}$. Then ψ_ξ is a quasi-morphism [**6**]. A similar construction for surface groups is given in [**7**].
(iii) *Barge-Ghys examples.* Realize F as a co-compact group of isometries of a simply connected Riemannian manifold X with sectional curvatures ≤ -1 and choose $o \in X$. Let ω be a (not necessarily closed) smooth 1-form on X/F. Define $\psi_\omega : F \to \mathbb{R}$ by $\psi_\omega(x) = \int_o^{ox} \tilde{\omega}$, where $\tilde{\omega}$ is the lift of ω to X and where the integral is taken over the geodesic joining o to ox. Then

$$\psi_\omega(xy) - \psi_\omega(x) - \psi_\omega(y) = \int_o^{oxy} \tilde{\omega} - \int_o^{ox} \tilde{\omega} - \int_o^{oy} \tilde{\omega} = \int_\triangle d\tilde{\omega},$$

where $\triangle$ is the geodesic triangle joining o, oxy and oy. Since the area of geodesic triangles in X is uniformly bounded, we have that ψ_ω is a quasi-morphism [**4**].

Theorem 2. *The Brooks quasi-morphisms and the Barge-Ghys quasi-morphisms are examples of Hölder quasi-morphisms.*

Proof. For the Brooks quasi-morphisms this is immediate. For the Barge-Ghys quasi-morphisms it follows from the analysis in [**28**], where a closely related result is proved.

As for lengths, quasi-morphisms fall into two classes.

Definition. We say that ψ or $\varphi = \mathfrak{H}(\psi)$ is non-discrete if $\{\varphi(w) : w \in \mathcal{C}(F)\}$ is not contained in a discrete subgroup of $\mathbb{R}$.

Remark. The set of quasi-morphisms on a group Γ is closely related to its bounded cohomology [**5**],[**16**],[**28**]. Let $H_b^2(\Gamma, \mathbb{R})$ denote the second bounded cohomology group on Γ and let $\rho_2^\Gamma : H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ denote the natural map. Then $\ker \rho_2^\Gamma$ is isomorphic to the set of quasi-morphisms on Γ modulo homomorphisms and bounded maps [**16**].

Free groups and, more generally, groups which are hyperbolic in the sense of Gromov admit many quasi-morphisms; this corresponds to the fact that $\ker \rho_2^\Gamma$ is infinite dimensional [**6**],[**7**],[**15**]. In contrast, higher rank groups often do not admit quasi-morphisms. For example, if Γ is an irreducible co-compact lattice in a Lie group of rank ≥ 2 then ρ_2^Γ is injective [**8**], so the only quasi-morphisms of Γ are sums of $v + b$, where $v : \Gamma \to \mathbb{R}$ is a homomorphism and $b : \Gamma \to \mathbb{R}$ is bounded.

Our main result is the following pair of central limit theorems. For convenience, we shall restrict to the case where L (or, equivalently, l) and ψ (or, equivalently, φ)

are non-discrete. The proof for conjugacy classes will be sketched in section 6 and, in section 7, we briefly indicate the modifications required to study elements of F. (For the proof of the theorem stated in the introduction, which is more elementary, see [**17**].)

THEOREM 3. *Let $L : F \to \mathbb{R}^+$ be a non-discrete Hölder length and let $\psi : F \to \mathbb{R}$ be a non-discrete Hölder quasi-morphism. Provided ψ is not bounded, there exists $\sigma = \sigma(L, \psi) > 0$ such that, for $\epsilon > 0$,*

$$\lim_{T\to+\infty} \frac{\#\{x \in F \,:\, T-\epsilon < L(x) \le T,\ \psi(x)/\sqrt{L(x)} \le y\}}{\#\{x \in F \,:\, T-\epsilon < L(x) \le T\}} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{y} e^{-t^2/2\sigma^2}\,dt$$

and, writing $l = \mathfrak{H}(L)$ and $\varphi = \mathfrak{H}(\psi)$,

$$\lim_{T\to+\infty} \frac{\#\{w \in \mathcal{C}(F) \,:\, T-\epsilon < l(w) \le T,\ \varphi(w)/\sqrt{l(w)} \le y\}}{\#\{w \in \mathcal{C}(F) \,:\, T-\epsilon < l(w) \le T\}} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{y} e^{-t^2/2\sigma^2}\,dt.$$

REMARK. In fact, σ only depends on l and φ.

4. Subshifts of finite type

Naturally associated to F and the generators $\mathcal{A}$ is the shift space Σ^+ consisting of all infinite reduced words in $\mathcal{A} \cup \mathcal{A}^{-1}$. More precisely,

$$\Sigma^+ = \{x = (x_n)_{n=0}^{\infty} \in (\mathcal{A} \cup \mathcal{A}^{-1})^{\mathbb{Z}^+} \,:\, x_{n+1} \neq x_n^{-1}\ \forall n \in \mathbb{Z}^+\}.$$

We define the (one-sided) subshift of finite type $\sigma : \Sigma^+ \to \Sigma^+$ by $(\sigma x)_n = x_{n+1}$. If we define a metric d on Σ^+ by

$$d(x, y) = \sum_{n=0}^{\infty} (1 - \delta_{x_n, y_n}) 2^{-n} \tag{4.1}$$

then Σ^+ is compact and σ is continuous. The topological entropy of $\sigma : \Sigma^+ \to \Sigma^+$ is given by $h(\sigma) = \log(2k-1)$.

A point $x \in \Sigma^+$ is a periodic point if $\sigma^n x = x$, for some $n \ge 1$ and, for each $n \ge 1$, we shall write

$$\mathrm{Fix}_n = \{x \in \Sigma^+ \,:\, \sigma^n x = x\}.$$

If $x \in \mathrm{Fix}_n$ then the set $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ is a periodic orbit; it is a prime periodic orbit if there is no $m < n$ such that $\sigma^m x = x$. Recall from section 1 that a conjugacy class $w \in \mathcal{C}(F)$ contains a cyclically reduced word $x_0 x_1 \cdots x_{n-1}$ in $\mathcal{A} \cup \mathcal{A}^{-1}$. By concatenating this word we obtain a periodic point $x \in \mathrm{Fix}_n$ and the other points in the periodic orbit correspond exactly to the cyclic permutations of the word. It is clear that this gives a natural bijection between $\mathcal{C}(F)$ and the set of periodic points of $\sigma : \Sigma^+ \to \Sigma^+$, such that, if $x, \sigma x, \ldots, \sigma^{n-1}x$ ($\sigma^n x = x$) corresponds to $w \in \mathcal{C}(F)$ then $|w| = n$. We say that a conjugacy class is primitive if none of its elements are non-trivial powers of elements of F. A conjugacy class is primitive if and only if the corresponding periodic orbit is prime. We write $\mathcal{P}(F)$ for the set of primitive conjugacy classes in F.

In order to represent elements of F as elements of a shift space, it is convenient to augment Σ^+ by adding an extra "dummy" symbol 0. Introduce a square matrix

A, with rows indexed by $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{0\}$, such that

$$A(i,j) = \begin{cases} 1 & \text{if } i,j \in \mathcal{A} \cup \mathcal{A}^{-1} \text{ and } j \neq i^{-1} \\ 0 & \text{if } i,j \in \mathcal{A} \cup \mathcal{A}^{-1} \text{ and } j = i^{-1} \\ 1 & \text{if } i \in \mathcal{A} \cup \mathcal{A}^{-1} \cup \{0\} \text{ and } j = 0 \\ 0 & \text{if } i = 0 \text{ and } j \in \mathcal{A} \cup \mathcal{A}^{-1}. \end{cases}$$

We may then define the shift space

$$\Sigma_A = \{x = (x_n)_{n=0}^{\infty} \in (\mathcal{A} \cup \mathcal{A}^{-1} \cup \{0\})^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}^+\},$$

with the shift map $\sigma : \Sigma_A \to \Sigma_A$ defined as before. The formula (4.1) defines a metric on Σ_A, making it compact and σ continuous. If we define B to be the submatrix of A obtained by deleting the row and column indexed by 0 then we may write $\Sigma^+ = \Sigma_B \subset \Sigma_A$. Furthermore, if we write Σ_A^0 for the set of sequences in Σ_A ending in an infinite string of 0s then $\Sigma_A^0 = \Sigma_A \backslash \Sigma_B$ and Σ_A^0 is dense in Σ_A. There is a natural identification between non-trivial elements in F and Σ_A^0 given by $\iota : F \backslash \{1\} \to \Sigma_A^0 : x_0 x_1 \cdots x_{n-1} \mapsto (x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots)$.

It is easy to see that enlarging Σ^+ to Σ_A only introduces one extra periodic point, an infinite string of 0s, which we denote $\dot{0}$. When we write Fix_n we shall continue to mean the periodic points in Σ^+.

We say that a non-negative matrix M is aperiodic if there exists $N > 0$ such that all the entries of M^N are positive. The matrix A is not aperiodic but the matrix B is aperiodic (this corresponds to $\sigma : \Sigma^+ \to \Sigma^+$ being topologically mixing). Since B is aperiodic, the Perron-Frobenius Theorem ensures that it has a simple eigenvalue $\lambda > 1$ such that $|\mu| < \lambda$ for all other eigenvalues μ and the topological entropy of the shift $\sigma : \Sigma_A \to \Sigma_A$ is given by $h(\sigma) = \log \lambda$ [**25**]. In fact, $\lambda = 2k - 1$. It is easy to see that A has the same spectrum as B except for an extra 1.

There is an extensive theory related to Hölder continuous functions on subshifts of finite type and we shall show that we may associate such a function to any Hölder length function $L : F \to \mathbb{R}^+$. First define $r : \Sigma_A^0 \to \mathbb{R}$ by $r(\dot{0}) = 0$ and

$$r(x_0, x_1, \ldots, x_{n-1}, \dot{0}) = L(x_0 x_1 \cdots x_{n-1}) - L(x_1 \cdots x_{n-1}). \qquad (4.2)$$

LEMMA 4.1 [**22**],[**30**],[**31**]. *If $L : F \to \mathbb{R}^+$ is a Hölder length function then $r : \Sigma_A^0 \to \mathbb{R}$ is Hölder continuous and has a Hölder continuous extension to a function $r : \Sigma_A \to \mathbb{R}$.*

The final statement is standard given that Σ_A^0 is dense in Σ_A.

The importance of the function r is given by the following lemma, where we use the notation

$$r^n(x) = r(x) + r(\sigma x) + \cdots + r(\sigma^{n-1} x).$$

LEMMA 4.2. *Let $r : \Sigma_A \to \mathbb{R}$ be the extension of the function defined by equation* (4.2)*.*

(i) *For any reduced word $x_0 x_1 \cdots x_{n-1}$,*

$$r^n(x_0, x_1, \ldots, x_{n-1}, \dot{0}) = L(x_0 x_1 \cdots x_{n-1}).$$

(ii) *If $x_0 x_1 \cdots x_{n-1}$ is a cyclically reduced word in a conjugacy class $w \in \mathcal{C}(F)$ then $r^n(x) = l(w)$, where $l = \mathfrak{H}(L)$ and $x \in \mathrm{Fix}_n$ is the associated periodic point.*

(iii) *There exists $N > 0$ such that $r^N : \Sigma_A \to \mathbb{R}$ is strictly positive. (In particular, if ν is a σ-invariant probability measure on Σ_A then $\int r d\nu > 0$.*

PROOF. Part (i) follow directly from the definition. To prove (ii), for $m \geq 1$, let $x^{(m)}$ denote the m-fold concatenation of the string $(x_0, x_1, \ldots, x_{n-1})$. Then, since r is Hölder continuous,

$$|r^{mn}(x) - r^{mn}(x^{(m)}\dot{0})| \leq |r|_\alpha \sum_{j=1}^{mn} 2^{-j\alpha} = |r|_\alpha \frac{1 - 2^{-(mn+1)\alpha}}{1 - 2^{-\alpha}} \leq \frac{|r|_\alpha}{1 - 2^{-\alpha}}.$$

Here $\alpha > 0$ denotes the Hölder exponent for r and $|r|_\alpha$ the best choice of Hölder constant. Noting that $r^{mn}(x) = mr^n(x)$, this gives us

$$\begin{aligned} r^n(x) &= \lim_{m \to +\infty} \frac{1}{m} r^{mn}(x^{(m)}\dot{0}) \\ &= \lim_{m \to +\infty} \frac{1}{m} L((x_0 x_1 \cdots x_{n-1})^m) = l(w). \end{aligned}$$

By part (ii), we have that $r^n(x) > 0$ whenever $x \in \mathrm{Fix}_n$ and it follows that r is cohomologous to a strictly positive function, i.e., there exists a strictly positive Hölder continuous function $r_+ : \Sigma_A \to R$ and a continuous function $u : \Sigma_A \to R$ such that $r = r_+ + u \circ \sigma - u$ [**35**]. Thus, $r^N = r_+^N + u \circ \sigma^N - u \geq (\inf r_+)N - 2||u||_\infty$, which is positive provided N is sufficiently large. This proves (iii).

Given a Hölder quasi-morphism $\psi : F \to \mathbb{R}$, we may also associate a Hölder continuous function $f : \Sigma_A \to \mathbb{R}$. As in (4.2), we define $f : \Sigma_A^0 \to \mathbb{R}$ by

$$f(x_0, x_1, \ldots, x_{n-1}, \dot{0}) = \psi(x_0 x_1 \cdots x_{n-1}) - \psi(x_1 \cdots x_{n-1}). \tag{4.3}$$

The following is the analogue of Lemmas 4.1 and 4.2.

LEMMA 4.3. *If $\psi : F \to \mathbb{R}^+$ is a Hölder quasi-morphism then $\psi : \Sigma_A^0 \to \mathbb{R}$ is Hölder continuous and has a Hölder continuous extension to a function $r : \Sigma_A \to \mathbb{R}$. Furthermore,*

(i) *for any reduced word $x_0 x_1 \cdots x_{n-1}$,*

$$f^n(x_0, x_1, \ldots, x_{n-1}, \dot{0}) = \psi(x_0 x_1 \cdots x_{n-1}).$$

(ii) *If $x_0 x_1 \cdots x_{n-1}$ is a cyclically reduced word in a conjugacy class $w \in \mathcal{C}(F)$ then $f^n(x) = \varphi(w)$, where $\varphi = \mathfrak{H}(\psi)$ and $x \in \mathrm{Fix}_n$ is the associated periodic point.*

5. Thermodynamic Formalism

We shall now review some of the thermodynamic formalism associated to our shift maps. It will be sufficient to do this for $\sigma : \Sigma^+ \to \Sigma^+$.

Let $\mathcal{M}(\Sigma^+)$ denote the set of σ-invariant probability measure on Σ^+. For $f \in C(\Sigma^+, \mathbb{R})$ we define the pressure $P(f)$ by

$$P(f) = \sup_{\nu \in \mathcal{M}(\Sigma^+)} h(\nu) + \int f d\nu,$$

where $h(\nu)$ denotes the measure theoretic entropy of σ with respect to ν. If f is Hölder continuous then the above supremum is attained by a unique measure $\mu_f \in \mathcal{M}(\Sigma_A)$, called the equilibrium state for f.

Define $\delta > 0$ by $P(-\delta r) = 0$. Given the correspondence between conjugacy classes in $\mathcal{C}(F)$ and periodic orbits for $\sigma : \Sigma^+ \to \Sigma^+$ and part (ii) of Lemma 4.2, the first part of the following lemma follows from standard results for suspended flows (or semi-flows) over subshifts of finite type [**25**]. (Alternatively, it may be proved directly along the lines of section 6.) The second part may be proved as in [**29**].

Proposition 5.1. *Suppose that $L : F \to \mathbb{R}^+$ is a non-discrete Hölder length function and that $l = \mathcal{H}(L)$. Then, for $\epsilon > 0$,*

$$\#\{w \in \mathcal{C}(F) \,:\, T - \epsilon < l(w) \leq T\} \sim (1 - e^{-\delta\epsilon})\frac{e^{\delta T}}{\delta T}, \quad \textit{as } T \to +\infty$$

and

$$\#\{x \in F \,:\, T - \epsilon < L(x) \leq T\} \sim (1 - e^{-\delta\epsilon})e^{\delta T}, \quad \textit{as } T \to +\infty.$$

Remark. More precisely, quoting results on periodic orbits gives

$$\#\{w \in \mathcal{P}(F) \,:\, l(w) \leq T\} \sim \frac{e^{\delta T}}{\delta T}, \quad \text{as } T \to +\infty$$

but replacing $\mathcal{P}(F)$ by $\mathcal{C}(F)$ introduces a discrepancy which is at worst $O(Te^{\delta T/2})$ and it is then easy to deduce the asymptotic in the range $T - \epsilon < l(w) \leq T$.

Lemma 5.1. *Let $\psi : F \to \mathbb{R}$ be a Hölder quasi-morphism and let $f : \Sigma^+ \to \mathbb{R}$ be the associated Hölder continuous function (restricted to Σ^+). Then*

$$\int f \, d\mu_{-\delta r} = 0.$$

Proof. For $w \in \mathcal{C}(F)$, write $w^{-1} = \{x^{-1} : x \in w\}$. This is a fixed point free involution of the set $\{w \in \mathcal{C}(F) : |w| = n\}$ and we have $l(w^{-1}) = l(w)$ and $\varphi(w^{-1}) = -\varphi(w)$.

The equilibrium state $\mu_{-\delta r}$ is the weak* limit of weighted averages over periodic points in the following way: for $g \in C(\Sigma^+)$,

$$\int g \, d\mu_{-\delta r} = \lim_{n \to +\infty} \frac{1}{n} \frac{\sum_{x \in \mathrm{Fix}_n} g^n(x) e^{-\delta r^n(x)}}{\sum_{x \in \mathrm{Fix}_n} e^{-\delta r^n(x)}}.$$

If we put $g = f$ then the Right Hand Side above (without the limit) becomes

$$\frac{1}{n} \frac{\sum_{|w|=n} \varphi(w) e^{-\delta l(w)}}{\sum_{|w|=n} e^{-\delta l(w)}},$$

which, in view of the above involution, is equal to zero.

6. L-functions and Limit Theorems for Conjugacy Classes

In this section, we shall prove Theorem 3 for conjugacy classes in the case where both l and φ are non-discrete. (At the end of the section, we shall briefly describe how to handle the simpler case of discrete φ.) We shall do this via a local limit theorem which describes the distribution of $\varphi(w) - \rho l(w)$, where ρ is a real parameter. The results will be uniform as ρ varies in a small compact neighbourhood of zero; ultimately, this will allow us to have ρ depend on T.

We need an appropriate set for ρ to lie in. We shall write

$$I_\varphi = \overline{\left\{ \frac{\varphi(w)}{l(w)} : w \in \mathcal{C}(F) \right\}} = \left\{ \frac{\int f d\nu}{\int r d\nu} : \nu \in \mathcal{M}(\Sigma^+) \right\};$$

I_φ is a closed interval containing zero in its interior. (Note that, by Lemma 4.2(iii), $\int r d\nu > 0$, for $\nu \in \mathcal{M}(\Sigma^+)$. See [**25**], for example, for the relationship between I_φ and the suspended semi-flow over Σ^+ defined by $r^N > 0$.)

We will need to use a pair of thermodynamic functions. We define a function $\mathfrak{p} : \mathbb{R} \to \mathbb{R}$ implicitly by the equation $P(-\mathfrak{p}(t)r + tf) = 0$. Since

$$\frac{d}{d\tau} P(-cr + \tau f)\bigg|_{\tau = t} = \int r \, d\mu_{-cr+tf} > 0,$$

we deduce from the Implicit Function Theorem that $\mathfrak{p}$ is real analytic. We also have that $\mathrm{int}(I_\varphi) = \{\mathfrak{p}'(t) : t \in \mathbb{R}\}$.

We introduce another function $\mathfrak{h} : I_\varphi \to \mathbb{R}$ by

$$\mathfrak{h}(\rho) = \sup \left\{ \frac{h(\nu)}{\int r d\nu} : \nu \in \mathcal{M}(\Sigma^+) \text{ and } \frac{\int f d\nu}{\int r d\nu} = \rho \right\}.$$

The functions $\mathfrak{h}$ and $\mathfrak{p}$ are related by the identities $-\mathfrak{h}'(\mathfrak{p}'(t)) = t$ and

$$\mathfrak{h}(\rho) = \mathfrak{p}((\mathfrak{p}')^{-1}(\rho)) - (\mathfrak{p}')^{-1}(\rho)\rho$$

(i.e., $-\mathfrak{h}$ is the Legendre transform of $\mathfrak{p}$ [**34**]). We write $\rho(\xi) = (\mathfrak{h}')^{-1}(\xi)$. Then $\rho(\xi)$ depends analytically on ξ,

$$\mathfrak{h}(\rho(\xi)) = \mathfrak{p}(\xi) - \xi\rho(\xi) \tag{6.1}$$

and $\mathfrak{h}''(\rho(\xi)) = -\mathfrak{p}''(\xi)^{-1}$. From now on, to simplify notation, we shall write $\rho = \rho(\xi)$.

Lemma 6.1. *$\mathfrak{h}(\rho)$ has a maximum at $\rho = 0$ and $\mathfrak{h}(0) = \delta$. In particular, $\mathfrak{h}'(0) = 0$ and $\rho(0) = 0$.*

Proof. Recall from Lemma 5.1 that

$$\int f \, d\mu_{-\delta r} = 0.$$

Both statements in the lemma then follow from

$$\frac{h(\mu_{-\delta r})}{\int r \, d\mu_{-\delta r}} = \delta \quad \text{and} \quad \frac{h(\nu)}{\int r \, d\nu} < \delta \ (\nu \neq \mu_{-\delta r})$$

which is equivalent to

$$h(\mu_{-\delta r}) - \delta \int r d\mu_{-\delta r} = 0 \quad \text{and} \quad h(\nu) - \delta \int r d\nu < 0 \ (\nu \neq \mu_{-\delta r}).$$

The latter follows from $P(-\delta r) = 0$.

The conjugacy class case of Theorem 3 may be derived from the following local limit theorem.

PROPOSITION 6.1. *For $\rho \in \mathrm{int}(I_\varphi)$ and $\kappa > 0$, we have*

$$\#\{w \in \mathcal{C}(F) : T - \epsilon < l(w) \le T,\ \varphi(w) - \rho l(w) \in (-\kappa, \kappa)\}$$
$$\sim (1 - e^{-\mathfrak{h}(\rho)\epsilon}) C(\rho) \left(\int_{-\kappa}^{\kappa} e^{-\xi t} dt \right) \frac{e^{\mathfrak{h}(\rho)T}}{T^{3/2}},$$

as $T \to +\infty$, where

$$C(\rho) = \sqrt{\frac{-\mathfrak{h}''(\rho)}{2\pi}} \frac{1}{\mathfrak{h}(\rho)}.$$

Furthermore, the convergence is uniform for ρ in any small compact neighbourhood of $0 \in \mathrm{int}(I_\psi)$.

REMARK. In fact, uniform convergence holds for ρ in any compact subset of $\mathrm{int}(I_\psi)$. The interval $(-\kappa, \kappa)$ may be replaced by an arbitrary interval (a, b).

By recasting this result in terms of a suspended semi-flow over Σ^+ (with the slight additional complication that the roof function is $r^N > 0$, rather than r), this follows from results of Lalley [**21**] and Babillot and Ledrappier [**3**]. However, we shall employ an L-function approach as in, for example, [**19**]. In particular, we shall use the version of the Agmon-Delange Tauberian theorem proved in [**19**] and show that a slightly more careful analysis allows this method to give the same uniform results obtained in [**3**], [**21**].

Proposition 6.1 may be proved by studying an appropriate family of generating functions, which, in turn, may be studied via a family of L-functions. For an integrable function $v : \mathbb{R} \to \mathbb{R}^+$, write

$$\widetilde{\eta}_\xi(s) = -\sum_{w \in \mathcal{P}(F)} v(\varphi(w) - \rho l(w)) l(w) e^{-sl(w) + \xi(\varphi(w) - \rho l(w))}.$$

Of course, we would like v to be some approximation to the indicator function of the interval $(-\kappa, \kappa)$ but, as we shall be taking Fourier transforms, we need to proceed more carefully. In fact, we shall take v to be real analytic, so that its Fourier transform is compactly supported. Later on, a standard unsmoothing argument will be used to recover the desired result for an indicator function. A key part of our analysis will be to show that $\widetilde{\eta}_\xi(s)$ has a square root singularity at $s = \mathfrak{h}(\rho)$.

We define an L-function associated to our data by

$$L_\xi(s, t) = \prod_{w \in \mathcal{P}(F)} \left(1 - e^{-sl(w) + (\xi + it)(\varphi(w) - \rho l(w))}\right)^{-1}.$$

This has the representation in terms of periodic points for $\sigma : \Sigma^+ \to \Sigma^+$:

$$L_\xi(s, t) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} e^{-sr^n(x) + (\xi + it)(f^n(x) - \rho r^n(x))}.$$

It is then standard to study this function via the following family of operators. We shall write $C^\alpha(\Sigma^+)$ for the space of (complex-valued) α-Hölder continuous functions on Σ^+, equipped with the norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{x \ne y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

Given a function $f \in C^\alpha(\Sigma^+)$ we shall define the associated transfer operator $\mathcal{L}_f : C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ by

$$\mathcal{L}_f g(x) = \sum_{\sigma y = x} g(y) e^{f(y)}.$$

The following result is well-known.

LEMMA 6.2 [**25**].
(i) *If $f \in C^\alpha(\Sigma^+)$ is real-valued then $\mathcal{L}_f : C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ has a simple isolated maximal eigenvalue $e^{P(f)}$ with a strictly positive associated eigenfunction h_f and a unique probability measure m_f on Σ^+ such that $\mathcal{L}_f^* m = e^{P(f)} m$. Furthermore, the rest of the spectrum is contained in a disk of radius strictly smaller than $e^{P(f)}$. If h is normalized so that $\int h dm_f = 1$ then $\mu_f = h_f m_f$.*
(ii) *If $f \in C^\alpha(\Sigma^+)$ is complex-valued then the spectral radius of $\mathcal{L}_f : C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ is less than or equal to $e^{P(\Re f)}$.*
(iii) *The spectral radius of $\mathcal{L}_f : C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ is strictly less than $e^{P(\Re f)}$ unless $\Im f = v \circ \sigma - v + M + a$, where $v \in C(\Sigma^+, \mathbb{R})$, $M \in C(\Sigma^+, 2\pi\mathbb{Z})$ and $a \in \mathbb{R}$ is a constant. If such an identity holds then $\mathcal{L}_f$ has a simple isolated maximal eigenvalue $e^{P(\Re f) + ia}$ and the rest of the spectrum is contained in a disk of radius strictly smaller than $e^{P(\Re f)}$.*

If $\mathcal{L}_f$ has an eigenvalue λ of modulus $e^{P(\Re f)}$ (in particular if $f \in C^\alpha(\Sigma^+)$ is real-valued) then it has a neighbourhood $\mathcal{N}(f) \subset C^\alpha(\Sigma^+)$ such that, for $g \in \mathcal{N}(f)$, $\mathcal{L}_g$ has a simple isolated eigenvalue $\lambda(g)$, depending analytically on g and such that $\lambda(f) = \lambda$. For $g \in \mathcal{N}(f)$, we define its pressure by $e^{P(g)} = \lambda(g)$. There are corresponding eigenfunctions h_g and eigenfunctionals ν_g, which also depend analytically on g.

If $z, w \in \mathbb{C}$, with $|\Im z|$ and $|\Im w|$ sufficiently small, then $P(-wr + zf)$ is defined. The function $\mathfrak{p}(z)$ is then defined implicitly by $P(-\mathfrak{p}(z)r + zf) = 0$ and is analytic.

Lemma 6.2 enables one to study a generalized zeta function $\zeta : C^\alpha(\Sigma^+) \to \mathbb{C}$ given by

$$\zeta(g) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} e^{g^n(x)},$$

which converges for $P(\Re g) < 0$ to a non-zero analytic function. The essential point is that this has an analytic and non-zero extension to a neighbourhood of $\{g : P(\Re g) \le 0\}$ except where $P(g) = 0$ [**25**]. With this definition,

$$L_\xi(s,t) = \zeta(-sr + (\xi + it)(f - \rho r)) = \zeta(-(s + \xi\rho + it\rho)r + (\xi + it)f).$$

This is non-zero and analytic for

$$P(-(\Re s + \xi\rho)r + \xi f) < 0,$$

i.e., for $\Re s + \xi\rho > \mathfrak{p}(\xi)$. By equation (6.1), this becomes $\Re s > \mathfrak{h}(\rho)$. We have the following result.

PROPOSITION 6.2 (CF. [**36**]). *The function $L_\xi(s,t)$ is analytic and non-zero in the set*

$$\{(s,t) \in \mathbb{C} \times \mathbb{R} : \Re s > \mathfrak{h}(\rho)\}$$

and has an analytic and non-zero extension to a neighbourhood of

$$\{(s,t) \in \mathbb{C} \times \mathbb{R} : \Re s = \mathfrak{h}(\rho), t \neq 0\} \cup \{(s,t) \in \mathbb{C} \times \mathbb{R} : \Re s = \mathfrak{h}(\rho), \Im s \neq 0\}.$$

Furthermore, for $\xi \in \mathbb{R}$, s *close to* $\mathfrak{h}(\rho)$ *and* $|t|$ *small,* $L_\xi(s,t)$ *takes the form* $a(s,\xi+it)/(s-s_\xi(t))$*, where* $s_\xi(t) = \mathfrak{p}(\xi+it) - (\xi+it)\rho$*, for some non-zero analytic function* $a(s,\xi+it)$*.*

Proof. The only thing we have to check is that if $ar + bf$ is of the form $v \circ \sigma - v + M$, where $v \in C(\Sigma^+, \mathbb{R})$ and $M \in C(\Sigma^+, 2\pi\mathbb{Z})$, there $a = b = 0$. Considering sums around periodic orbits, this is equivalent to

$$\exp i(al(w) + b\varphi(w)) = 1 \quad \forall w \in \mathcal{C}(F).$$

Comparing with w^{-1}, we obtain $\exp ib\varphi(w) = \exp -ib\varphi(w)$, so either $b = 0$ or $b\varphi(w) \in 2\pi\mathbb{Z}$, for all $w \in \mathcal{C}(F)$. However, the latter cannot occur since φ is non-discrete. Hence, we have $\exp ial(w) = 1$, for all $w \in \mathcal{C}(F)$. Since l is non-discrete, this forces $a = 0$, as well.

Lemma 6.3. $s_\xi'(0) = 0$, $\Re s_\xi''(0) < 0$ *and* $\Im s_\xi''(0) = 0$*. (In particular,* $|\Im s_\xi(t)| = O(|t|^3)$*, as* $t \to 0$*.)*

Proof. From the definition of $s_\xi(t)$, we have

$$s_\xi'(0) = i\mathfrak{p}'(\xi) - i\rho = 0$$

and

$$s_\xi''(0) = -\mathfrak{p}''(\xi) < 0,$$

the latter being real and negative.

The proof of the theorem now follows [**36**] very closely, specifically employing the analysis and Tauberian theorem from Katsuda & Sunada [**19**].

Denote the logarithmic derivative of $L_\xi(s,t)$ by

$$\eta_\xi(s,t) = \frac{\partial}{\partial s} \log L_\xi(s,t) = -\sum_{w \in \mathcal{P}(F)} \sum_{n=1}^{\infty} l(w) e^{n(-sl(w)+(\xi+it)(\varphi(w)-\rho l(w)))}.$$

This function is non-zero and analytic in the set $\{s : \Re s \geq \mathfrak{h}(\rho)\} \times \mathbb{R} - \{(\mathfrak{h}(\rho), 0)\}$ and in a neighbourhood around the pole can be written $-1/(s - s_\xi(t)) + a_1(s,\xi,t)$ for analytic function a_1. In fact, one may ignore the terms with $n \geq 2$ in the above summation without affecting these properties, so we shall abuse notation and write

$$\eta_\xi(s,t) = -\sum_{w \in \mathcal{P}(F)} l(w) e^{-sl(w)+(\xi+it)(\varphi(w)-\rho l(w))}.$$

Using the Fourier Inversion Formula, we have

$$\begin{aligned} \widetilde{\eta}_\xi(s) &= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{v}(-t) \left(\sum_{w \in \mathcal{P}(F)} l(w) e^{-sl(w)+\xi(\varphi(w)-\rho l(w))} e^{it(\varphi(w)-\rho l(w))} \right) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{v}(-t) \eta_\xi(s,t)\, dt. \end{aligned}$$

In view of the above analysis, the principal part of $\widetilde{\eta}_\xi(s)$ comes from integrating over a neighbourhood of 0. More precisely, for any $\epsilon > 0$,

$$\widetilde{\eta}_\xi(s) = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \frac{\widehat{v}(-t)}{s - s_\xi(t)}\,dt + a_2(s,\xi),$$

where $a_2(s,\xi)$ is analytic for $\{s : \Re s \geq \mathfrak{h}(\rho)\}$.

In view of Lemma 6.3, $\Re s_\xi'(0) = 0$ and $\Re s_\xi''(0) < 0$, so we may use the Morse Lemma to make a suitable smooth change of co-ordinates and write

$$\widetilde{\eta}_\xi(s) = \frac{1}{\pi\sqrt{2\mathfrak{p}''(\xi)}}\int_{-\epsilon}^{\epsilon} \frac{1+b(\theta)}{s - \mathfrak{h}(\rho) + \theta^2 + iQ(\theta)}\,d\theta + a_3(s,\xi), \tag{6.1}$$

where $b(\theta)$ is a smooth function with $f(0) = 0$, $Q(\theta) = \Im s_\xi(t(\theta)) = O(|\theta|^3)$ and $a_3(s,\xi)$ is analytic for $\{s : \Re s \geq \mathfrak{h}(\rho)\}$. The extra factor $(2/\mathfrak{p}''(\xi))^{1/2}$ appears as the Jacobian of the change of variables at the origin.

It will turn out that $\widetilde{\eta}_\xi(s)$ has a square root singularity at $s = \mathfrak{h}(\rho)$ and, in order to make further progress, we shall need to apply a Tauberian theorem that is valid for this type of singularity. Specifically, we shall use the version of the Agmon-Delange Tauberian theorem ([**1**],[**14**]) proved in [**19**]. To state this, we need to consider a family of complex functions $F_\xi(s)$ which satisfy the following conditions.

(T1) $F_\xi(s)$ is analytic for $\Re s > 1$.

(T2) The limit

$$\lim_{\epsilon\to 0+}\left(F_\xi(1+\epsilon+it) - \frac{A}{\sqrt{\epsilon+it}}\right)$$

exists for almost every point $t \in \mathbb{R}$ and is in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.

(T3) There exists a locally integrable function $h(t)$ such that

$$\sup_\xi \sup_{\epsilon>0}\left|F_\xi(1+\epsilon+it) - \frac{A}{\sqrt{\epsilon+it}}\right| \leq h(t).$$

LEMMA 6.4 [**19**]. *Let $\phi_\xi(T)$ be a family of increasing functions with $\phi_\xi(T) = 0$ for $0 \leq T \leq T_0$, for some $T_0 > 0$ (independent of ξ) and let*

$$F_\xi(s) = \int_0^\infty T^{-\frac{1}{2}} e^{-sT} d\phi_\xi(T).$$

Suppose that $F_\xi(s)$ satisfies (T1), (T2) and (T3). Then, uniformly for ξ in any compact interval, $\phi_\xi(T) \sim A\pi^{-\frac{1}{2}} e^T$.

The uniformity follows from a careful examination of the proof in [**19**].

The integral in (6.1) is in exactly the form analysed by Katsuda and Sunada [**19**]. They prove the following proposition which states that the function has the properties necessary to allow the Tauberian theorem to be applied. Again, the uniformity follows from a careful examination of the proof in [**19**].

LEMMA 6.5 [**19**]. *The functions $\sqrt{\mathfrak{p}''(\xi)\mathfrak{h}(\rho)}\widetilde{\eta}_\xi(\mathfrak{h}(\rho)s)$ satisfy (T1), (T2) and (T3) with*

$$A = \frac{\hat{v}(0)\sqrt{\pi}}{\sqrt{2\pi}}.$$

We may write
$$\widetilde{\eta}_\xi(s) = -\int_0^\infty T^{-\frac{1}{2}} e^{-sT} dS_v(T),$$
where
$$S_\xi(T) = \sum_{l(w)\le T} l(w)^{3/2} e^{\xi(\varphi(w)-\rho l(w))} v(\varphi(w) - \rho l(w)).$$
Applying Lemmas 6.4 and 6.5 (with a suitable renormalization), we have
$$S_\xi(T) \sim C(\rho)\hat{v}(0) e^{\mathfrak{p}(\xi)T},$$
where
$$C(\rho) = \frac{1}{\sqrt{2\pi\mathfrak{p}''(\xi))}\mathfrak{h}(\rho)} = \sqrt{\frac{-\mathfrak{h}''(\rho)}{2\pi}}\frac{1}{\mathfrak{h}(\rho)},$$
uniformly for ξ (or, equivalently, ρ) in a compact interval. (The condition $\phi_\xi(x) = 0$ for $0 \le x \le T_0$, for some $T_0 > 0$, will hold for ξ in a compact interval.)

We are now ready to complete the proof of Proposition 6.1.

Proof of Proposition 6.1. Suppose that ρ lies in a sufficiently small neighbourhood of zero that $\mathfrak{h}(\rho) > \delta/2$. (This is not strictly necessary but it makes the discussion easier.) The asymptotic
$$\#\{w \in \mathcal{P}(F) : l(w) \le T,\ \varphi(w) - \rho l(w) \in (-\kappa,\kappa)\} \sim C(\rho)\left(\int_{-\kappa}^{\kappa} e^{-\xi t} dt\right)\frac{e^{\mathfrak{h}(\rho)T}}{T^{3/2}},$$
as $T \to +\infty$ now follows from fairly standard unsmoothing and partial summation arguments and this may be done to preserve the uniformity (cf. [**21**],[**38**]). Furthermore, replacing $\mathcal{P}(F)$ by $\mathcal{C}(F)$ introduces a discrepancy which is no worse than $O(Te^{\delta T/2})$, with the implied constant independent of ρ. Finally, it is easy to deduce the asymptotic for $T - \epsilon < l(w) \le T$.

The Central Limit Theorem for conjugacy classes will now follow from the next result.

Proposition 6.3. *For $c \in \mathbb{R}$, $\epsilon > 0$ and $\kappa > 0$, we have*
$$\frac{\#\{w \in \mathcal{C}(F) : T - \epsilon < l(w) \le T,\ \varphi(w) - c\sqrt{l(w)} \in (-\kappa,\kappa)\}}{\#\{w \in \mathcal{C}(F) : T - \epsilon < l(w) \le T\}} \sim \frac{1}{\sqrt{2\pi}\sigma}\frac{2\kappa}{\sqrt{T}} e^{-c^2/2\sigma^2},$$
as $T \to +\infty$, where $\sigma^2 = \mathfrak{p}''(0) = -1/\mathfrak{h}''(0)$.

Proof. We shall apply Proposition 6.1 with $\rho = c/\sqrt{T}$, so that
$$\varphi(w) - \rho l(w) = \varphi(w) - \frac{cl(w)}{\sqrt{T}} = \varphi(w) - c\sqrt{l(w)} + O\left(\frac{1}{\sqrt{T}}\right).$$
(This is why we need uniformity in ρ.) As $T \to +\infty$, $\mathfrak{h}(\rho) \to \delta$ and (since $\xi \to 0$)
$$\int_{-\kappa}^{\kappa} e^{-\xi t} dt \to 2\kappa$$
and we have
$$\mathfrak{h}(\rho)T = \mathfrak{h}\left(\frac{c}{\sqrt{T}}\right)T = \delta T - \frac{2}{\sigma^2} + O\left(\frac{1}{\sqrt{T}}\right).$$

Comparing Proposition 6.1 with the asymptotic

$$\#\{w \in \mathcal{C}(F) : T - \epsilon < l(w) \le T\} \sim (1 - e^{-\delta\epsilon}) \frac{e^{\delta T}}{\delta T}$$

from Proposition 5.1 gives the result.

7. Limit Theorems for Group Elements

In this section we shall briefly sketch how the method of the previous section may be adapted to prove a central limit for elements of F and thus complete the proof of Theorem 3.

The key result is the following.

PROPOSITION 7.1. *For $\rho \in \mathrm{int}(I_\varphi)$ and $\kappa > 0$, we have*

$$\#\{x \in F : T - \epsilon < L(x) \le T,\ \psi(x) - \rho L(x) \in (-\kappa, \kappa)\}$$
$$\sim (1 - e^{-\mathfrak{h}(\rho)\epsilon}) \sqrt{\frac{\mathfrak{h}''(\rho)}{2\pi}} \left(\int_{-\kappa}^{\kappa} e^{-\xi t} dt \right) \frac{e^{\mathfrak{h}(\rho)T}}{T^{1/2}},$$

as $T \to +\infty$. Furthermore, the convergence is uniform for ρ in a small compact neighbourhood of $0 \in \mathrm{int}(I_\varphi)$.

REMARK. As in Proposition 6.1, uniform convergence holds for ρ in any compact subset of $\mathrm{int}(I_\psi)$ and the interval $(-\kappa, \kappa)$ may be replaced by an arbitrary interval (a, b).

This may be proved in a similar manner to Proposition 6.1. We shall consider a generating function

$$\widetilde{\omega}_\xi(s) = \sum_{x \in F} v(\psi(x) - \rho L(x)) e^{-sL(x) + \xi(\psi(x) - \rho L(x))},$$

where $v$has a compactly supported Fourier transform. This may be studied via the functions

$$\omega_\xi(s, t) = \sum_{x \in F} e^{-sL(x) + (\xi + it)(\psi(x) - \rho L(x))}.$$

In turn, to study these we use a family of extended transfer operators associated to the larger shift space Σ_A. More precisely, for $f \in C^\alpha(\Sigma_A)$, we define the transfer operator $\mathcal{L}_f : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$ by

$$\mathcal{L}_f g(x) = \sum_{\substack{\sigma y = x \\ y \ne \dot{0}}} g(y) e^{f(y)}.$$

This has the same spectral properties as $\mathcal{L}_{f|\Sigma^+}$ and, furthermore, by Lemmas 4.2 and 4.3, we may write

$$\omega_\xi(s, t) = \sum_{n=0}^{\infty} (\mathcal{L}^n_{-sr + (\xi + it)(f - \rho r)} 1)(\dot{0}).$$

One may then proceed in a similar manner to section 6 with, in particular, $\widetilde{\omega}_\xi(s)$ having a square root singularity at $s = \mathfrak{h}(\rho)$.

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