LIVSIC THEOREMS, MAXIMIZING MEASURES AND THE STABLE NORM

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Abstract. In this article we consider two results motivated by Livsic's well known theorem that, for a hyperbolic system, a Hölder continuous function is determined, up to a coboundary, by its values around closed orbits. The first result relates to negative values around orbits and the second result to values for finitely many orbits. We also present some new results on maximizing measures and the boundary of the unit ball in the stable norm for a surface.

0. Introduction

In the study of hyperbolic dynamical systems there is a class of results which relate the local behaviour of the system on a countable number of closed orbits to the global behaviour of the system. These results are generally called Livsic type theorems.

The classical Livsic theorem for transitive Anosov flows was originally established by Livsic [13] in 1971, and subsequently rediscovered by Guillemin and Kazhdan in their work on isospectral rigidity [10]. Let us recall the statement. If $\phi_t : X \to X$ is a transitive Anosov flow and $f : X \to \mathbb{R}$ is a Hölder continuous function we let $\lambda_f(\tau) = \int_0^{\lambda(\tau)} f(\phi_t x_\tau)dt$ (for any $x_\tau \in \tau$) denote the integral of f around ϕ-closed orbit τ of least period $\lambda(\tau)$.

Livsic Theorem. *Let $\phi_t : X \to X$ be a transitive Anosov flow and let $f : X \to \mathbb{R}$ be Hölder continuous functions such that $\lambda_f(\tau) = 0$, for every ϕ-closed orbit τ. Then there exists a Hölder continuous function $V : X \to \mathbb{R}$ which is differentiable in the flow direction and satisfies $\frac{dV(\phi_t x)}{dt}|_{t=0} = f(x)$.*

Our main result will be to establish a version of the Livsic Theorem in the case where the integrals $\lambda_f(\tau)$ are supposed to be non-positive. We recall the necessary background material on Anosov flows in Section 1 and prove the result in Section 2.

Various versions of the Livsic Theorem for non-positive weightings $\lambda_f(\tau)$ have been studied by many authors, including Bousch, Contreras, Conze, Guivarc'h, Jenkinson, Lopes, Savchenko and Thieullen [1],[2],[5],[6],[14],[24]. There are close connections between these results and a variety of topics, including maximizing measures and the stable norm on homology. We consider the nature of the boundary of the unit ball in the stable norm in Section 3.

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In Section 4 we study the extent to which a function is specified by a knowledge of the weights over a finite number of closed orbits.

While completing this paper, we received a preprint by Lopes and Thieullen containing a result similar to Theorem 1 [15].

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1. Anosov and geodesic flows

In this section we collect together some basic definitions and results that will be needed in the rest of the paper.

A C^1 flow $\phi_t : X \to X$ on a compact manifold is called *Anosov* if there is a continuous splitting of the unit tangent bundle $TX = E^0 \oplus E^s \oplus E^u$, where E^0 is the one-dimensional bundle tangent to the flow direction, and there exist $C, \lambda > 0$ such that $||D\phi_t|E^s|| \leq Ce^{-\lambda t}$ and $||D\phi_{-t}|E^u|| \leq Ce^{-\lambda t}$, for $t \geq 0$. We say that the flow is *transitive* if there is a dense orbit.

Following an argument of Mather, one can assume that the Riemannian metric on X has been chosen so that we may take $C = 1$.

Let V be a compact manifold with negative sectional curvatures and let $\phi_t : SV \to SV$ be the geodesic flow on the unit tangent bundle. This is a classical example of a transitive Anosov flow.

The following technical lemma is easily shown.

Lemma 1.1. *There exist constants $\Theta > 0$ and $C, \lambda > 0$ such that:*

(i) *for x, y in the same stable manifold $d(\phi_t x, \phi_t y) \leq Ce^{-\lambda t} d(x, y)$ for $t \geq 0$;*
(ii) *for y, y' in the same unstable manifold $d(\phi_{-t} y, \phi_{-t} y') \leq Ce^{-\lambda t} d(y, y')$ for $t \geq 0$; and*
(iii) *for arbitrary points x, x' we have $d(\phi_t x, \phi_t x') \leq Ce^{\Theta T} d(x, x')$ for $T \geq 0$.*

We recall some well known constructions for Anosov flows.

Local product structure. For $\epsilon_0 > 0$ sufficiently small, we define the local stable and unstable manifolds by

$$W^{ss}_{\epsilon_0}(x) = \{y \in M : d(\phi_t x, \phi_t y) \leq \epsilon \quad \forall t \geq 0\} \text{ and}$$
$$W^{su}_{\epsilon_0}(x) = \{y \in M : d(\phi_{-t} x, \phi_{-t} y) \leq \epsilon \quad \forall t \geq 0\}.$$

These are C^1 embedded disks which satisfy $T_x W^{ss}_{\epsilon_0}(x) = E^s$ and $T_x W^{su}_{\epsilon_0}(x) = E^u$ [3, p.432]. Moreover, the embedding varies in a (Hölder) continuous fashion with x. Typically, in practice, one fixes $\epsilon_0 > 0$ sufficiently small and subsequently works on a smaller scale. We can choose $0 < \epsilon < \epsilon_0$ such that whenever $x_1, x_2 \in M$ satisfy $d(x_1, x_2) < \epsilon$ then there exists $s = s(x_1, x_2) \in \mathbb{R}$ and $z := \langle x_1, x_2 \rangle \in M$ such that $z \in W^{ss}_{\epsilon_0}(x_1)$ and $\phi_s z \in W^{su}_{\epsilon_0}(x_2)$ (i.e., z lies on the same local unstable manifold as x_1, but the point z has to be pushed forward by the flow to $\phi_s(z)$ to lie on the same strong stable manifold as x_2). Moreover, by continuity of the embeddings of the local stable and unstable manifolds $s(x_1, x_2) \to 0$ as $d(x_1, x_2) \to 0$. The locally defined maps $\langle \cdot, \cdot \rangle$ and $s(\cdot)$ are referred to as canonical coordinates [3, p.432].

Markov sections and parallelepipeds. We need to consider a choice of Markov sections $\mathcal{T} = \{T_1, \ldots, T_k\}$ for the Anosov flow, whose existence is given by [3, Theorem 2.5] and [22, Theorem 2.1]. Each section T_i is a closed set which lies in the interior of codimension-one disks D_i transverse to the flow [3, pp.434-435]. It is also assumed that the diameter of each section T_i is smaller than $\epsilon_0 > 0$ then there exists $\delta > 0$ such that $\cup_{i=1}^k \phi_{[-\delta,\delta]} T_i = X$. For each $1 \le i \le k$, we let $\mathrm{pr}_{T_i} : \phi_{[-\delta,\delta]} T_i \to T_i$ be the natural projection along ϕ-orbits. The sections T_i can be chosen to be "rectangles" in the sense that for $x, y \in T_i$ we have that $\mathrm{pr}_{T_i}(\langle x, y\rangle) \in T_i$ [3, p.432]. Let $\Pi : \cup_{i=1}^k T_i \to \cup_{i=1}^k T_i$ denote the Poincaré map, i.e., the discrete map which takes a point on a section to a subsequent point along its ϕ-orbit on the union of the sections [3, p.435]. A simple formulation of the Markov property is that given $x, y \in \mathrm{int}(T_i)$ then there exists $z \in \mathrm{int}(T_i)$ such that $\Pi^n(x), \Pi^n(z) \in \mathrm{int}(T_{i_n})$ and $\Pi^{-n}(x), \Pi^{-n}(z) \in \mathrm{int}(T_{i_{-n}})$, for $n \ge 0$, i.e., the ϕ-orbit of z passes through the same sections as x flowing forwards and passes through the same sections as y flowing backwards cf. [3, p.437]. We refer the reader to [3] for full details.

We can assume that each of the sections $\mathcal{T}$ are Hölder continuous and foliated by pieces of local strong stable manifolds [22, p.95]. (This is easily achieved by adjusting the sections in the flow direction, which does not effect the Markov property Since the foliation by strong stable manifolds is only Hölder the resulting sections are Hölder. A similar construction appears in [23, p.240] for the case of analytic foliations which leads to analytic sections.) We can define a natural projection $\pi_i : T_i \to U_i$ along the strong stable manifolds, where we can assume that U_i lies in a single piece of local unstable manifold.

It is convenient to replace the original rectangles $\{T_i\}_{i=1}^k$ by their refinements $\{T_i \cap \Pi^{-1} T_j\}_{i,j=1}^k$. The advantage is that the Poincaré maps $\Pi|\mathrm{int}_{D_i} T_i$ are then continuous. From the definition of the Poincaré map we can write $\Pi(x) = \phi_{r(x)}(x)$, where $r : \cup_{i=1}^k \mathrm{int}_{D_i} T_i \to \mathbb{R}^+$ is the return time. In particular, the function r has a continuous extension from the interiors to $\cup_{i=1}^k T_i$.

A cover for X. For technical reasons, it is useful to choose slightly larger rectangles $\widehat{T}_i \subset \mathrm{int} D_i$ with $T_i \subset \mathrm{int}_{D_i}(\widehat{T}_i)$, for $i = 1, \ldots, n$. For $\epsilon > 0$ sufficiently small, we call the sets

$$P_i = \overline{\{\phi_t(x) : x \in \mathrm{int}_{D_i}(\widehat{T}_i), -\epsilon \le t \le r(x) + \epsilon\}}, \text{ for } i = 1. \ldots, n,$$

Fattened parallelepipeds. These contain the usual parallelepipeds which partition the manifold, i.e., $\overline{\{\phi_t(x) : x \in T_i, 0 \le t \le r(x)\}}$, for $1 \le i \le k$ [22, p.95]. We can also assume without loss of generality that

(a) $\mathrm{diam}(P_i) < \epsilon_0$, and
(b) for $x \in P_i$ and $\phi_S x \in P_l$ we have $\phi_S W^{ss}(x, P_i) \subset W^{ss}(\phi_S x, P_l)$,

where $W^{ss}(x, P_i) = W^{ss}_{\epsilon_0}(x) \cap P_i$.

The following standard result is useful.

Lemma 1.2 (Anosov closing lemma). *Let $f : X \to \mathbb{R}$ be a α-Hölder continuous function. There exists $\epsilon_1 > 0$ and $C > 0$ such that if $x, \phi_T x \in X$ satisfy $d(x, \phi_T x) < \epsilon_1$ then there exists a closed orbit τ such that $|\lambda_f(\tau) - \int_0^T f(\phi_t x) dt| \le C d(x, \phi_T x)^\alpha$.*

Finally, we recall a well known fact. Let $f, g : X \to \mathbb{R}$ be Hölder continuous functions, then the following are equivalent:

(a) $\lambda_f(\tau) = \lambda_g(\tau)$ for every ϕ-closed orbit τ; and
(b) $\int f d\mu = \int g d\mu$ for every ϕ-invariant probability measure μ.

2. The Non-Positive Livsic Theorem

The following result describes Hölder continuous functions which have negative (or positive) integrals around all periodic orbits.

Theorem 1 (Non-Positive Livsic Theorem). *Let $\phi_t : X \to X$ be a transitive Anosov flow. Let $f : X \to \mathbb{R}$ be a Hölder continuous function such that $\lambda_f(\tau) \leq 0$ for every ϕ-closed orbit τ. Then there exists a Hölder continuous function $V : X \to \mathbb{R}$ (possibly with a smaller Hölder exponent) such that $\int_0^T f(\phi_t x)dt \leq V(\phi_T x) - V(x)$.*

The analogous result for Anosov diffeomorphisms was proved by Bousch [1] and Lopes and Thieullen [14]. Previously, there were similar results for expanding maps in [24] and [6]. We present a proof of the Non-Positive Livsic Theorem for Anosov flows which is inspired by the elegant proof in [14] of the corresponding result for diffeomorphisms. A recent preprint by Lopes and Thieullen [15] also contains a version of the Non-Positive Livsic Theorem for flows. However, their result is stronger in that the function V can be chosen to be differentiable along flow lines and, in consequence, the proof is somewhat more intricate.

Part of the motivation for proving Theorem 1 for flows comes from a comparison with the results and conjectures of Mather and Mañé on Lagrangian actions and flows [19], [16]. Suppose that f satisfies the hypotheses of Theorem 1 and that $\sup_\mu \int f d\mu = 0$. Following the analogy with Lagrangian flows, let us call a ϕ-invariant probability measure μ which satisfies $\int f d\mu = 0$ a *maximizing measure*. It is easy to deduce the following corollary to Theorem 1.

Corollary. *A maximizing measure μ has support in the set*

$$\left\{ x : \int_0^T f(\phi_t x)dt = V(\phi_T x) - V(x), \forall T \geq 0 \right\}.$$

Theorem 1 has a useful application to $\mathbb{R}^n$-extensions of Anosov flows. Let $\phi_t : X \to X$ be a transitive Anosov flow. Let $f : X \to \mathbb{R}^n$ be a Hölder continuous function. If τ is a closed orbit for ϕ of least period $\lambda(\tau)$ then we can weight it by f and write $\lambda_f(\tau) = \int_0^{\lambda(\tau)} f(\phi_u x)du \in \mathbb{R}^n$.

Definition. We denote $\mathcal{S} = \{\lambda_f(\tau) : \tau \text{ a closed orbit } \}$. We say that the function f is *separating* if there is no codimension one hyperplane for which $\mathcal{S}$ lies wholly in one closed half-space.

The terminology arose in [21] since every half-plane separates $\mathcal{S}$ into two non-empty components.

We can denote $F(x,t) = \int_0^t f(\phi_t x, u)du$ and define a skew product flow

$$\Phi_t : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$$
$$\Phi_t(x,z) = (\phi_t x, z + F(x,t)).$$

We say that Φ is transitive if there exists a dense orbit. The following generalizes a result of Niţică and Pollicott [21] for Anosov diffeomorphisms.

Corollary. *If Φ is transitive then f is separating.*

Proof. Assume for a contradiction that f is not separating. Let $W \subset \mathbb{R}^n$ be the associated hyperplane and let $v \in \mathbb{R}^n$ be a vector normal to W. Thus for every τ we can assume that $\langle \lambda_f(\tau), v\rangle \geq 0$, say. Assuming Theorem 1, we can show that $\langle f, v\rangle$ is cohomologous to a positive function. In particular, we conclude that Φ_t cannot be transitive since there exists $M > 0$ such that all points (x', z') in the Φ-orbit of (x, z) satisfy $\langle z', v\rangle \geq \langle z, v\rangle - M$. $\square$

Remark. In contrast to the case of Anosov diffeomorphisms, the converse statement is not necessarily true. For example, we can consider the geodesic flow on a negatively curved manifold and closed one-forms ω_i which correspond to a basis for the first cohomology group, of dimension n. We associate a function $f(x) = (f_1(x), \dots , f_n(x))$, say, where $f_i(x) = \omega_i(\mathcal{X}(x))$, $i = 1, \dots , n$, where $\mathcal{X}(x)$ denotes the vector field generating ϕ. We can identify $\mathcal{S}$ with a uniform lattice in $\mathbb{R}^d$ and thus it must be separating. However, the skew product Φ is not transitive since otherwise one can use arguments in [21] to show that $\mathcal{S}$ is dense.

Proof of Theorem 1. We begin with the following definitions. Assume that $f : X \to \mathbb{R}$ is a α-Hölder continuous function with $\lambda_f(\tau) \leq 0$ for every closed orbit τ.

Definition. Given $x \in X$ we can define

$$S_T f(x) := \int_{-T}^{0} f(\phi_t x)dt, \text{ for } T > 0.$$

If $x, y \in X$ lie on the same stable manifold then we can define

$$\Delta_f(x, y) = \int_0^\infty [f(\phi_t y) - f(\phi_t x)]dt.$$

Lemma 2.1.

(1) *There exists $A \geq 0$ such that for all $T \geq 0$ and $x \in X$ we have $S_T f(x) \leq A$;*
(2) *There exists $B \geq 0$ such that providing $y, y' \in T_i$ are on the same unstable manifold then $|S_T f(y) - S_T f(y')| \leq Bd(y, y')^\alpha$, for all $T \geq 0$;*
(3) *The function $V_i : P_i \to \mathbb{R}^+$ $(i = 1, \dots , k)$ defined by*

$$V_i(x) = \sup \{S_T f(y) + \Delta_f(x, y) \,:\, T > 0, y \in W^{ss}(x, P_i)\}$$

is Hölder continuous on P_i;
(4) *If $x \in P_i$ and $\phi_S x \in P_l$ then $\int_0^S f(\phi_t x)dt \leq V_l(\phi_S x) - V_i(x)$.*

Proof. For part (1), observe that we can use the Anosov closing lemma for the flow to approximate orbit segments by closed orbits τ so that $S_T f(x)$ differs from the integral $\lambda_f(\tau) \leq 0$ by a uniform constant $A > 0$, say.

For part (2), we observe that since f is Hölder continuous, we can find $C > 0$ such that

$$\begin{aligned} |S_T f(y) - S_T f(y')| &\leq \int_0^T |f(\phi_{-t} y) - f(\phi_{-t} y')|dt \\ &\leq C \int_0^T (e^{-\lambda t})^\alpha d(y, y')^\alpha dt \\ &\leq Bd(y, y')^\alpha, \end{aligned}$$

say, by applying Lemma 1.1 (ii).

For part (3), we first consider the case that $x, x' \in P_i$ lie on the same strong stable manifold. Observe that if $y \in W^{ss}(x, P_i) = W^{ss}(x', P_i)$ then there exists $C_1 > 0$ such that $|\Delta_f(x,y) - \Delta_f(x',y)| \le C_1 d(x,x')^\alpha$. For any $T > 0$ we can write

$$\begin{aligned} S_T f(y) + \Delta_f(x,y) &\le S_T f(y) + \Delta_f(x',y) + C_1 d(x,x')^\alpha \\ &\le V_i(x) + C_1 d(x,x')^\alpha. \end{aligned}$$

Taking the supremum over y and T we see that $V_i(x) \le V_i(x') + C_1 d(x,x')^\alpha$. Interchanging x and x' we see that $V_i(x') \le V_i(x) + C_1 d(x,x')^\alpha$ and so deduce that $|V_i(x) - V_i(x')| \le C_1 d(x,x')^\alpha$.

We next consider the case that $x, x' = \phi_u x \in P_i$ lie on the same orbit segment. For any $y \in W^{ss}(x, P_i)$ we can associate $y' := \phi_u y \in W^{ss}(x', P_i)$. For $T > 0$ we can then write

$$\begin{aligned} S_T f(x) + \Delta_f(x,y) &= S_T f(x') + \Delta_f(x',y') + \int_{-T}^{-T+u} f(\phi_t x) dt - \int_0^u f(\phi_t x) dt \\ &\quad + \int_0^u [f(\phi_t y) - f(\phi_t x)] dt \\ &\le S_T f(x') + \Delta_f(x',y') + 4||f||_\infty u \\ &\le V_i(x') + C_2 d(x,x'), \end{aligned}$$

for some $C_2 > 0$. Taking a supremum over y and T we have that $V_i(x) \le V_i(x') + C_2 d(x,x')$. Interchanging x and x' we have that $V_i(x') \le V_i(x) + C_2 d(x,x')$, and so we can deduce that $|V_i(x) - V_i(x')| \le C_2 d(x,x')$.

Finally, consider $x, x' \in P_i$ on the same strong unstable manifold. If y lies on the same stable manifold as x then we have, by definition, that $y' := \langle x', y \rangle$ lies on the same stable manifold as x'. Moreover, $\phi_s y'$is in the same unstable manifold as y, where $s := s(x', y)$. For each $R \ge 0$ and $T \ge 0$ we have a bound

$$\begin{aligned} S_T f(y) + \Delta_f(x,y) &= S_T f(y') + \Delta_f(x',y') \\ &\quad + \int_{-T}^{R} [f(\phi_t y) - f(\phi_t y')] dt - \int_0^R [f(\phi_t x) - f(\phi_t x')] dt \\ &\quad + \Delta_f(\phi_R x, \phi_R y) - \Delta_f(\phi_R x', \phi_R y'). \end{aligned} \tag{2.1}$$

Given that the stable and unstable foliations are Hölder we can find $C_3, C_4 > 0$ and $0 < \gamma < 1$ to estimate $|s| \le C_3 d(x,x')^\gamma$ and $d(y,y') \le C_4 d(x,x')^\gamma$. (The Hölder dependence follows from the strong stable and strong unstable foliations being Hölder continuous. In particular, the bound on s follows from the joint non-integrability of the foliation [8]). Let us fix $0 < \beta < \alpha\gamma$ and then choose $R > 0$ such that $e^{\Theta \alpha R} = d(x,x')^{-\beta}$. (Without loss of generality, we can assume that the parallelograms have been chosen sufficiently small.) We can now bound the fourth term in (2.1) by

$$\left| \int_0^R [f(\phi_t x) - f(\phi_t x')] dt \right| \le C_5 e^{\Theta \alpha R} ||f||_\alpha d(x,x')^\alpha = C_5 ||f||_\alpha d(x,x')^{(\alpha - \beta)},$$

for some $C_5 > 0$.

We can also bound the third term in (2.1) as

$$\begin{aligned}\left|\int_{-T}^{R}[f(\phi_t y) - f(\phi_t y')]dt\right| &\leq \left|\int_{-T}^{R}[f(\phi_t y) - f(\phi_{t+s} y')]dt\right| + 2||f||_\infty s \\ &\leq C_5 e^{\lambda\Theta\alpha}||f||_\alpha d(y, \phi_s y')^\alpha + 2||f||_\infty s \\ &= C_5 e^{\lambda\Theta\alpha}||f||_\alpha (d(y, y') + d(y', \phi_s y'))^\alpha + 2||f||_\infty s \\ &= C_5 e^{\lambda\Theta\alpha}||f||_\alpha (C_3 d(x, x')^\gamma + C_4 d(x, x')^\gamma)^\alpha \\ &\qquad + 2||f||_\infty C_3 d(x, x')^\gamma \\ &\leq C_6(||f||_\alpha + ||f||_\infty) d(x, x')^{(\alpha\gamma-\beta)},\end{aligned}$$

for some $C_6 > 0$. We can bound the last two terms in (2.1) by

$$\begin{aligned}\Delta_f(\phi_R x, \phi_R y) + \Delta_f(\phi_R x', \phi_R y') &\leq C_7 ||f||_\alpha e^{-\lambda\alpha R}\epsilon_0^\alpha \\ &= C_7 ||f||_\alpha \epsilon_0^\alpha d(x, x')^{\lambda\alpha/\Theta},\end{aligned}$$

for some $C_7 > 0$. If we write $\tau = \min\{\alpha\lambda/\Theta, (\alpha\gamma - \beta)\}$ then we can collect together these estimates and bound

$$\begin{aligned}S_T f(y) + \Delta_f(x, y) &\leq S_T f(y') + \Delta_f(x', y') + C_8 d(x, x')^\tau \\ &\leq V_i(x') + C_8 d(x, x')^\tau,\end{aligned}$$

for $C_8 > 0$. Taking the supremum over T and y on the Left Hand Side gives $V_i(x) \leq V_i(x') + C_8 d(x, x')^\tau$. Interchanging x and x' gives $V_i(x') \leq V_i(x) + C_8 d(x, x')^\tau$ and we deduce that $|V_i(x') - V_i(x')| \leq C_8 d(x, x')^\tau$. We can combine these three cases to give Hölder continuity of V_i. More precisely, we can join two points in P_i by paths consisting of pieces of stable and unstable manifolds and pieces of orbit segments such that lengths of each is less than a constant multiple of their separation. The result then follows by the triangle inequality.

For part (4), for $x \in P_i$ and $\phi_S x \in P_l$ we have $\phi_S W^{ss}(x, P_i) \subset W^{ss}(\phi_S x, P_j)$. The definitions give that for any $y \in W^{ss}(x, P_i)$ and $T > 0$ we have that

$$\begin{aligned}&S_T f(y) + \Delta_f(x, y) \\ &= S_T f(y) + \left(\Delta_f(\phi_S x, \phi_S y) + \int_0^S [f(\phi_t y) - f(\phi_t x)]dt\right) \\ &= \left(\int_{-(S+T)}^0 f(\phi_t(\phi_S y))dt + \Delta_f(\phi_S x, \phi_S y)\right) - \int_0^S f(\phi_t x)dt \\ &= S_{T+S} f(\phi_S y) + \Delta_f(\phi_S x, \phi_S y) - \int_0^S f(\phi_t x)dt \\ &\leq V_l(\phi_S x) - \int_0^S f(\phi_t x)dt\end{aligned}$$

By taking a supremum over T and y on the Left Hand Side we have that $V_i(x) \leq V_l(\phi_S x) - \int_0^S f(\phi_t x)dt$, as required. $\square$

The final step is to patch together interiors of fattened parallelepipeds to form a cover $\mathcal{U} = \{U_1, \ldots, U_k\}$ where $U_i = \mathrm{int}(P_i)$. Let $\Phi_i : X \to [0,1]$, for $i = 1, \ldots, k$ be smooth functions which are a partition of unity subordinate to $\mathcal{U}$. We can define $V(x) = \sum_{i=1}^k \Phi_i(x) V_i(x)$.

For any indices i, j we have the inequality

$$V_j(\phi_T x)\Phi_i(x)\Phi_j(\phi_T x) \geq \left[\int_0^T f(\phi_t x)dt + V_i(x)\right] \Phi_i(x)\Phi_j(\phi_T x). \tag{2.2}$$

In particular, since the Left Hand Side of (2.2) is positive, we need only consider the case where the Right Hand Side is non-zero, which requires that $x \in U_i$ and $\phi_T x \in U_j$. In this case, (2.2) follows from Lemma 2.1, part (4). Summing over $1 \leq i, j \leq k$ gives

$$\begin{aligned} \sum_{i,j} \Phi_i(x)\Phi_j(\phi_T x)V_i(x) - \sum_{i,j} \Phi_i(x)\Phi_j(\phi_T x)V_j(\phi_T x) \\ \geq \sum_{i,j} \Phi_i(x)\Phi_j(\phi_T x) \int_0^T f(\phi_t x)dt, \end{aligned}$$

or equivalently

$$\sum_i \Phi_i(x)V_i(x) - \sum_j \Phi_j(\phi_T x)V_j(\phi_T x) \geq \int_0^T f(\phi_t x)dt,$$

i.e., $V(x) - V(\phi_T x) \geq \int_0^T f(\phi_t x)dt$.

3. The stable norm on homology

In this section we shall consider a simple geometric problem which is closely related to that of understanding maximizing measures for geodesic flows. Let V be a compact Riemann surface of genus $\mathfrak{g} \geq 2$ and let $\phi_t : SV \to SV$ denote the geodesic flow on the unit tangent bundle SV. The fundamental group of V has the standard one-relator presentation

$$\left\langle a_1, \ldots, a_{\mathfrak{g}}, b_1, \ldots, b_{\mathfrak{g}} : \prod_{i=1}^{\mathfrak{g}} [a_i, b_i] = 1 \right\rangle,$$

[27, p.141]. We can identify the real first homology group $H_1(V, \mathbb{R})$ with $\mathbb{R}^{2\mathfrak{g}}$ and $H_1(V, \mathbb{Z})$ with $\mathbb{Z}^{2\mathfrak{g}}$ by taking the basis corresponding to these generators.

Definition. The *stable norm* on $H_1(V, \mathbb{R})$ is defined by

$$||h|| = \inf \left\{ \sum_{i=1}^k |r_i| \mathrm{length}(\gamma_i) : h = \sum_{i=1}^k r_i \gamma_i \right\}$$

where the infimum ranges over all representations of the homology class h as a sum $h = \sum_{i=1}^k r_i \gamma_i$, where $r_i \in \mathbb{R}$ and $\gamma_i \in H_1(V, \mathbb{Z})$ are Lipschitz 1-cycles [8, pp. 50-51], [17, p.2]. We denote the unit ball in the stable norm by $\mathcal{B}$.

The set $\mathcal{B}$ is compact and convex. The next lemma relates $\mathcal{B}$ to the integrals of a particular family of functions on SV with respect to probability which are invariant under the geodesic flow [17, pp.13-15]. These functions are defined via harmonic 1-forms on V and are thus naturally associated to cohomology classes in $H^1(V, \mathbb{R})$. More precisely, let $\omega_1, \ldots, \omega_{2\mathfrak{g}}$ be a basis of harmonic 1-forms for $H^1(V, \mathbb{R})$ dual to the basis for $H_1(V, \mathbb{R})$ and define $f_i : SV \to \mathbb{R}$ by $f_i(v) = \langle \omega_i, v \rangle$, for $i = 1, \ldots, 2\mathfrak{g}$.

Lemma 3.1 [17]**.**

$$\mathcal{B} = \left\{ \left(\int f_1 d\mu, \dots , \int f_{2\mathfrak{g}} d\mu \right) \ : \mu \ \textit{a } \phi\textit{-invariant probability measure} \right\}. \quad (3.1)$$

The set $\mathcal{B}$ also has the following interpretation in terms of closed geodesics. Given a closed geodesic γ we can denote its length by $\lambda(\gamma)$ and its homology class by $[\gamma]$. Let μ_γ denote the ϕ-invariant probability measure defined by

$$\int g d\mu_\gamma = \frac{1}{\lambda(\gamma)} \int_0^{\lambda(\gamma)} g(\phi_t v) dt,$$

where $v \in SV$ is any unit vector tangent to γ. Then $\int f_i d\mu_\gamma = \lambda(\gamma)^{-1} \int_\gamma \omega_i = \lambda(\gamma)^{-1} \langle \omega_i, [\gamma] \rangle$, $i = 1, \dots , 2\mathfrak{g}$, so that

$$\left(\int f_1 d\mu_\gamma, \dots , \int f_{2\mathfrak{g}} d\mu_\gamma \right) = \frac{[\gamma]}{\lambda(\gamma)}.$$

Since the measures corresponding to periodic orbits are weak* dense, it is easy to see that

$$\mathcal{B} = \overline{\{ [\gamma]/\lambda(\gamma) \ : \ \gamma \text{ a closed geodesic } \}}. \quad (3.2)$$

An interesting feature of $\mathcal{B}$ is the lack of smoothness of the boundary $\partial\mathcal{B}$ at points of rational direction. (A point $x \in \mathbb{R}^{2\mathfrak{g}}$ has rational direction if there exists $c > 0$ such that $cx \in \mathbb{Z}^{2\mathfrak{g}}$.) This was first observed in the case of the hyperbolic punctured torus, for which $\mathcal{B} \subset \mathbb{R}^2$, by McShane and Rivin [20]. In particular, they showed that ∂B has a corner at each point of rational direction. For higher genus surfaces this situation is more complicated. However, Massart showed that for a compact surface of genus $g \geq 2$, ∂B is not differentiable. More precisely, he showed that at each point of rational direction, ∂B contains a flat of dimension $\mathfrak{g} - 1$ and ∂B is only differentiable tangent to the flat [18]. Below we shall give a simple argument to show that ∂B has a corner at certain rational points.

A standard cone in $\mathbb{R}^{2\mathfrak{g}}$ takes the form

$$\left\{ (x_1, \dots , x_{2g}) \in \mathbb{R}^{2\mathfrak{g}} \ : \ x_1 \geq 0 \text{ and } \sum_{i=2}^{2\mathfrak{g}} x_i^2 \leq \lambda^2 x_1^2 \right\},$$

for some $\lambda > 0$, and $(0, \dots , 0)$ is the vertex of the cone. We say the points $(x_1, \dots , x_{2\mathfrak{g}}) \in \mathbb{R}^{2g}$ with $\sum_{i=2}^{2\mathfrak{g}} x_i^2 < \lambda^2 x_1^2$ are in the interior. If $\lambda = \tan\theta$ with $0 < \theta < \pi/2$ then we call θ the angle of the vertex.

Definition. We define a cone in $\mathbb{R}^{2\mathfrak{g}}$ to be an isometric image of the standard cone in $\mathbb{R}^{2\mathfrak{g}}$.

Proposition 3.2. *For each simple closed geodesic γ with $[\gamma]/\lambda(\gamma) \in \partial\mathcal{B}$ we can find a cone containing $\mathcal{B}$ whose vertex is $[\gamma]/\lambda(\gamma)$.*

Our approach is essentially constructive and, in principle, can be used to estimate the angles of some corners.

To prove Proposition 3.2, we shall require the following technical lemma, which summarizes results on the geometric coding of the geodesic flow.

Lemma 3.3. *There exists a piecewise linear fractional expanding Markov map $f : I \to I$, where I is a disjoint union of arcs $I_1, \ldots, I_k$ in the unit circle, a piecewise C^ω positive function $r : I \to \mathbb{R}$ and a piecewise constant function $g : I \to \mathbb{Z}^{2\mathfrak{g}}$, such that:*

1. *periodic orbits $\{x, fx, \ldots, f^{n-1}x\}$, with $f^n x = x$ correspond to closed geodesics γ on V;*
2. *the sum $r^n(x) = r(x) + r(fx) + \cdots + r(f^{n-1}x)$ is equal to the length $\lambda(\gamma)$ of the corresponding closed geodesic; and*
3. *the sum $g^n(x) = g(x) + g(fx) + \cdots + g(f^{n-1}x) \in \mathbb{Z}^{2\mathfrak{g}}$ is the homology class of γ.*

Moreover,

(a) *for any given simple closed geodesic γ we can arrange this coding so that γ corresponds to a fixed point $f(\eta) = \eta$ and the associated homology class $[\gamma]$ is given by $g(\eta) = e_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^{2\mathfrak{g}}$, and*

(b) *for $1 \leq j \leq k$, $g(I_j)$ is of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$.*

Proof. Parts (1)-(3) are due to Series [25, pp.106-107,120,122] (cf. also [4],[26]). The original construction of Series involves choosing a standard $4g$-sided fundamental domain whose boundary consists of geodesic arcs. There is a natural side pairing and the corresponding linear fractional transformation on the boundary leads to the transformation f. In particular, each side of the fundamental domain is identified with a standard generator (or its inverse) from the fundamental group.

There are $2\mathfrak{g}$ side pairings (plus their inverses) which correspond to generators for the fundamental group $\pi_1(V)$. Under abelianization, these correspond to independent vectors in $H_1(V, \mathbb{Z}) = \mathbb{Z}^{2\mathfrak{g}}$. The function g can be constructed as a function which is constant on each of the associated intervals I_i. In particular, if is defined to be the image in homology of the corresponding generator (or inverse) with respect to some fixed homology basis. Then part (3) immediately follows.

Moreover, we can choose this fundamental domain so that γ lifts to a geodesic arc connecting two identified sides. (This follows easily from the observation that given any simple closed geodesic we can find a surface homeomorphism that maps it to a meridian curve [27, pp.196-197]). Since we are at liberty to change the homology basis as proves convenient, we can take the images of the side elements basis vectors to be the basis vectors. This immediately gives part (a) and (b). $\square$

Lemma 3.4. *Let I_i be the interval containing η. The function r in Lemma 3.3 can be chosen so that the restriction $r : I_i \to \mathbb{R}$ is minimized at the fixed point η, i.e., $\lambda(\gamma) = r(\eta) = \inf\{r(y) : y \in I_i\}$.*

Proof. Let us assume that the linear fractional map $f : I_i \to \mathbb{C}$ restricted to I_i takes the form

$$f(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \text{ where } a, b \in \mathbb{C} \text{ satisfy } |a|^2 - |b|^2 = 1.$$

The fixed point η corresponds to the end point of the lift of a closed geodesic to the Poincaré disk which is preserved by this linear fractional transformation. By conjugating by another linear fractional transformation, if necessary, we can assume that this geodesic lies on the real axis with $\eta = 1$. In particular, this implies that $a, b \in \mathbb{R}$. It is a standard fact that after adding a coboundary, if necessary, the roof function $r : I_i \to \mathbb{R}$ can be assumed to be of the special form $r(x) = \log|f'(x)| = 2\log|\bar{b}x + \bar{a}|$ [12, p.37]. Finally, since $-\bar{a}/\bar{b}$ lies on the real axis we immediately see that $|\bar{b}x + \bar{a}| = |\bar{b}||x - (-\bar{a}/\bar{b})|$ is minimized at $x = \eta$. $\square$

Proof of Proposition 3.2. Let us denote $\rho = \inf\{r(y) : y \in \bigcup_{j\neq i} I_j\} > 0$ and choose $\theta = \cot^{-1}(\rho/\lambda(\gamma))$.

We shall consider a point in $\mathcal{B} - \{[\gamma]/\lambda(\gamma)\}$. It suffices to consider only points of the form $[\gamma']/\lambda(\gamma') \in \mathcal{B}$, where γ' is a primitive closed geodesic, since such points are dense in $\mathcal{B}$. Assume that γ' corresponds to a periodic point $f^n x = x$ and let I_j, $j \neq i$, be an interval intersecting the orbit of x. We let $n_i = \#\{0 \leq l \leq n-1 : f^l x \in I_i\}$, $n_j = \#\{0 \leq l \leq n-1 : f^l x \in I_j\}$ and $e_j = g(I_j)$.

We define $p_\theta : I \to \mathbb{R}$ by

$$p_\theta(y) = \begin{cases} \sin\theta & \text{if } y \in I_i \\ \cos\theta & \text{if } y \in I_j \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we deduce that $p_\theta^n(x) = \sin(\theta)n_i + \cos(\theta)n_j$ and $r^n(x) \geq n_i\lambda(\gamma) + \rho n_j$. We can now write

$$\begin{aligned} \langle e_1\sin\theta + e_j\cos\theta, \frac{[\gamma']}{\lambda(\gamma')}\rangle &= \frac{p_\theta^n(x)}{r^n(x)} \\ &\leq \frac{\sin(\theta)n_i + \cos(\theta)n_j}{n_i\lambda(\gamma) + \rho n_j} \\ &= \frac{\sin\theta}{\lambda(\gamma)}\left(\frac{n_i + \cot(\theta)n_j}{n_i + \left(\frac{\rho}{\lambda(\gamma)}\right)n_j}\right) \\ &= \frac{\sin\theta}{\lambda(\gamma)} = \left\langle e_1\sin\theta + e_j\cos\theta, \frac{[\gamma]}{\lambda(\gamma)}\right\rangle. \end{aligned} \tag{3.3}$$

In particular, we conclude that $[\gamma']/\lambda(\gamma')$ lies in the half space

$$\mathcal{H}_j = \frac{[\gamma]}{\lambda(\gamma)} + \{w \in \mathbb{R}^{2\mathfrak{g}} : \langle e_1\sin\theta + e_j\cos\theta, w\rangle \leq 0\}.$$

We can repeat this argument for each $j \neq i$ and deduce that $\mathcal{B}$ is contained in $\bigcap_{j\neq i}\mathcal{H}_j$. (If the orbit of x does not intersect I_j then the argument goes through with $n_j = 0$.) Finally, this set is contained in a cone with vertex $[\gamma]/\lambda(\gamma)$ and angle equal to $\tan^{-1}(\sqrt{2\mathfrak{g}-1}\lambda(\gamma)/\rho)$. $\square$

4. Finite Livsic Theorems

In this final section we shall consider a class of Livsic-type theorems for Anosov systems involving only finitely many periodic orbits. Finite Livsic theorems for the very special case of contact Anosov flows $\phi_t : X \to X$ on a three-dimensional compact manifold were considered by S. Katok [11]. More precisely, she considers C^2 functions $f : X \to \mathbb{R}$ whose integrals over all periodic orbits of length at most L vanish. She then shows that, given any sufficiently small $\lambda > 0$, there exist h

and F in $C^{1+\lambda}(X)$ such that $f = \mathcal{D}F + h$, where $\mathcal{D}$ denotes the derivative in the direction of the flow, and $\|h\|_1 \le C(\lambda)L^{-\lambda/(3-\lambda)}$, where $C(\lambda)$ is independent of f.

In contrast, we shall consider the case of general Anosov diffeomorphisms. In this simpler context, without the encumbrance of having to deal with the flow direction, one can still prove more modest result by completely elementary methods. These results, valid for Hölder continuous functions, do not give a global bound as in [11] but rather give estimates over the remaining periodic orbits. This, in turn, leads to an estimate on the integrals of f with respect to invariant probability measures.

Theorem 2. *Let $T : X \to X$ be a transitive Anosov diffeomorphism. There exist constants $C > 0$ and $\beta > 0$ such that, for any α-Hölder continuous function $f : X \to \mathbb{R}$ we have that*

$$|f^n(x)| \le C||f||_\alpha n L^{-\alpha\beta} \text{ for all periodic orbits } T^n x = x,$$

where $L = \sup\{N : f^n(x) = 0$ whenever $T^n x = x$ and $0 \le n \le N-1\}$.

One can easily construct examples to show that the above bound is not sharp. For example, let T be the usual Arnold cat map and choose f to vanish on all periodic points of period at most L. Then one sees that $||f||_\infty$ becomes exponentially small as L increases.

Since measures supported on periodic orbits are weak* dense in the space of T-invariant probability measures, the theorem implies the follow estimate on the integrals of f.

Corollary. *For any T-invariant probability measure μ, we have that*

$$\left| \int f d\mu \right| \le C||f||_\alpha L^{-\alpha\beta}.$$

For $0 < \alpha \le 1$, let $C^\alpha(X)$ be the space of α-Hölder continuous functions and let $\mathcal{C} \subset C^\alpha(X)$ be the closed subspace of coboundaries. Theorem 2 follows easily from the next proposition.

Proposition 4.1. *There exist constants $C_0 > 0$ and $\beta > 0$ such that for any α-Hölder function $f : X \to \mathbb{R}$ we have that*

$$||f - \mathcal{C}||_\infty \le C_0||f||_\alpha L^{-\alpha\beta},$$

where $L = \sup\{N : f^n(x) = 0$ whenever $T^n x = x$ and $n \le N\}$.

Given $x \in X$ and $N > 0$, we let $\mathcal{O}(x, T, N) = \{T^n(x) : 0 \le n \le N-1\}$ denote the associated orbit segment of length N. We begin by defining a function $u : \mathcal{O}(x, T, N) \to \mathbb{R}$ by

$$u(T^n x) = \sum_{i=0}^{n-1} f(T^i x), \text{ for } 1 \le n \le N.$$

The next lemma gives us a Hölder estimate for u on $\mathcal{O}(x, T, N)$ provided $N \le L$.

Lemma 4.2. *There exists $C_1 > 0$ such that for all $x \in X$ the function $u : \mathcal{O}(x, T, N) \to \mathbb{R}$ satisfies $|u(T^{n_2}x) - u(T^{n_1}x)| \le C_1 ||f||_\alpha d(T^{n_2}x, T^{n_1}x)^\alpha$, for $0 \le n_1 \le n_2 \le N - 1 \le L - 1$.*

Proof. By the Anosov Closing Lemma there exists $\epsilon > 0$ and $C_1 > 0$ such that whenever $d(T^{n_1}x, T^{n_2}x) < \epsilon$ we can choose a periodic orbit $T^{(n_2-n_1)}z = z$ such that $|\sum_{n=n_1}^{n_2-1} f(T^n x) - f^{(n_2-n_1)}(z)| \le C_1 ||f||_\alpha d(T^{n_2}x, T^{n_1}x)^\alpha$. In particular, since $f^{(n_2-n_1)}(z) = 0$ we see that

$$|u(T^{n_2}x) - u(T^{n_1}x)| = \left| \sum_{n=n_1}^{n_2-1} f(T^n x) \right| \le C_1 ||f||_\alpha d(T^{n_2}x, T^{n_1}x)^\alpha,$$

as required. □

We can easily extend u as a Hölder function to X without increasing the Hölder norm by defining $u(z) = \inf\{u(T^n x) + ||u||_\alpha d(z, T^n x)^\alpha : 0 \le n \le N - 1\}$ (cf [7, p.202]). For each $x \in X$ there exists $\eta > 0$ such that $\mathcal{O}(x, T, N)$ is η-dense in X, i.e., given $z \in X$ we can choose $T^n x \in \mathcal{O}(x, T, N)$ with $d(z, T^n x) < \eta$. We can then estimate

$$\begin{aligned} |u(Tz) - u(z) - f(z)| \le & |u(Tz) - u(T^{n+1}x)| + |u(T^{n+1}x) - u(T^n x) - f(T^n x)| \\ & + |u(T^n x) - u(z)| + |f(T^n x) - f(z)| \\ \le & C_1 ||f||_\alpha (||DT||_\infty)^\alpha \eta^\alpha + 0 + C_1 ||f||_\alpha \eta^\alpha + ||f||_\alpha \eta^\alpha. \end{aligned} \tag{4.1}$$

The next lemma gives an estimate on the size of η.

Lemma 4.3. *There exists $C_2, \beta > 0$ such that for every $N > 0$ we can choose $x \in X$ such that $\mathcal{O}(x, T, N)$ is η-dense for $\eta = C_2 N^{-\beta}$.*

Proof. This is easily seen using a Markov partition $\mathcal{R} = \{R_1, \dots, R_k\}$ for the diffeomorphism. Since transitive Anosov diffeomorphisms are topologically mixing, the associated transition matrix A is aperiodic, i.e., there exists $n_0 \ge 1$ such that $A^{n_0} > 0$. For any given $n \ge 1$ we can consider all cylinders

$$\mathcal{C}_n = \left\{ \bigcap_{j=-[n/2]}^{[n/2]} T^j R_{i_j} : R_{i_j} \in \mathcal{R} \right\}$$

of length n. The number of such cylinders can be bounded by $B_1 e^{hn}$, say, where $h > 0$ is the topological entropy of T and $B_1 > 0$ is a constant. The diameter of the cylinders decreases exponentially fast, i.e., there exist constants $B_2 > 0$ and $\lambda > 0$ such that $\mathrm{diam}(\bigcap_{j=-[n/2]}^{[n/2]} T^j R_{i_j}) \le B_2 e^{-\lambda n}$, say.

Let us choose n such that $B_1 e^{hn}(n + n_0) \le N < B_1 e^{h(n+1)}(n + 1 + n_0)$. We would like to choose $x \in X$ so that the orbit $\mathcal{O}(x, T, N)$ passes through each of the n-cylinders. We can easily construct such an orbit symbolically by concatenating all of the n-cylinders in the subshift of finite type, linked by words of length n_0 between them. It is easy to see that we choose x such that $\eta \le B_2 e^{-\lambda n}$ and, provided $\beta < \lambda/h$, there exists $B_3 > 0$ such that $B_2 e^{-\lambda n} \le B_3 N^{-\beta}$. The lemma immediately follows. □

To apply Lemma 4.2 we can set $N = L$. We then see from (4.1) that there exists $C_0 > 0$ such that

$$\begin{aligned}|u(Tz) - u(z) - f(z)| &\leq C_0||f||_\alpha \eta^\alpha \\ &= C_0||f||_\alpha L^{-\alpha\beta}.\end{aligned}$$

This completes the proof of Proposition 4.1.

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