THE MANHATTAN CURVE AND THE CORRELATION OF LENGTH SPECTRA ON HYPERBOLIC SURFACES

Richard Sharp

University of Manchester

0. Introduction

Let Γ be a co-compact Fuchsian group (with no elliptic elements) and let Σ denote the associated hyperbolic surface $\mathbb{H}^2/\Gamma$. Then Γ is isomorphic to the fundamental group of Σ and each non-trivial conjugacy class in Γ contains a unique closed geodesic on Σ. If we write $l[\gamma]$ for the length of the closed geodesic in the conjugacy class $[\gamma]$ then it was proved in [6] (though it was already implicit in [10]) that, for a fixed $\epsilon > 0$, we have the asymptotic formula

$$\mathrm{Card}\{[\gamma] : l[\gamma] \in (T, T+\epsilon)\} \sim (e^\epsilon - 1)\frac{e^T}{T}, \text{ as } T \to +\infty. \tag{0.1}$$

(Here, $f(T) \sim g(T)$ means that $\lim_{T\to+\infty} f(T)/g(T) = 1$.)

Now suppose that Γ_1 and Γ_2 are two isomorphic co-compact Fuchsian groups. It is natural to consider conjugacy classes for which the corresponding closed geodesics on $\mathbb{H}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$ have approximately the same length. In [9], it was shown that if Γ_1 and Γ_2 are not conjugate in $PSL(2,\mathbb{R})$ then there exist constants $C = C(\epsilon) > 0$ and $\alpha \in (0,1)$ such that

$$\mathrm{Card}\{[\gamma] : l_1[\gamma], l_2[\gamma] \in (T, T+\epsilon)\} \sim C\frac{e^{\alpha T}}{T^{3/2}}, \text{ as } T \to +\infty,$$

where $l_i[\gamma]$ denotes the length of the closed geodesic on $\mathbb{H}^2/\Gamma_i$ lying in the conjugacy class $[\gamma]$. In [9], α was called the correlation number and is independent of ϵ.

The purpose of this short note is to identify the exponent $\alpha = \alpha(\Gamma_1, \Gamma_2)$ as the value

$$\alpha = a + b,$$

where (a, b) is the point on the curve $\mathcal{C}(\Gamma_1, \Gamma_2)$ for which the normal has slope 1 and where $\mathcal{C}(\Gamma_1, \Gamma_2)$ is the Manhattan curve associated to (Γ_1, Γ_2), defined in [3].

We now recall the definition of $\mathcal{C}(\Gamma_1, \Gamma_2)$. This is the convex curve which bounds the set

$$\Delta = \{(a, b) \in \mathbb{R}^2 : \sum_{[\gamma]} e^{-al_1[\gamma] - bl_2[\gamma]} < +\infty\}.$$

(One may easily check that Δ is a convex set.) The formula (0.1) shows that $(1,0),(0,1) \in \mathcal{C}(\Gamma_1,\Gamma_2)$.

Remark. An analogous definition may be made for (non-compact) convex co-compact subgroups of $SO(n,1)$, $n \geq 2$.

The principal properties of $\mathcal{C}(\Gamma_1,\Gamma_2)$ are summarized in the following result.

Proposition 1 ([3]).
- (i) *$\mathcal{C}(\Gamma_1,\Gamma_2)$ is a straight line connecting $(1,0)$ and $(0,1)$ if and only if Γ_1 and Γ_2 are conjugate in $PSL(2,\mathbb{R})$.*
- (ii) *$\mathcal{C}(\Gamma_1,\Gamma_2)$ is a C^1 curve. Its asymptotes have normals with slopes $\mathrm{dil}_-(\Gamma_1,\Gamma_2)$ at $-\infty$ and $\mathrm{dil}_+(\Gamma_1,\Gamma_2)$ at $+\infty$. At $(1,0)$ and $(0,1)$ its normals have slope $i(\Gamma_1,\Gamma_2)$ and $1/i(\Gamma_2,\Gamma_1)$, respectively.*

The quantities $\mathrm{dil}_+(\Gamma_1,\Gamma_2)$, $\mathrm{dil}_-(\Gamma_1,\Gamma_2)$ and $i(\Gamma_1,\Gamma_2)$ are, respectively, the maximal and minimal geodesic stretches and the intersection of the surfaces $\mathbb{H}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$. They are defined as follows:

$$\mathrm{dil}_+(\Gamma_1,\Gamma_2) = \sup_{[\gamma]} \frac{l_2[\gamma]}{l_1[\gamma]},$$

$$\mathrm{dil}_-(\Gamma_1,\Gamma_2) = \inf_{[\gamma]} \frac{l_2[\gamma]}{l_1[\gamma]}$$

and

$$i(\Gamma_1,\Gamma_2) = \lim_{n\to+\infty} \frac{l_2[\gamma_n]}{l_1[\gamma_n]},$$

where $\{[\gamma_n]\}_{n=1}^{\infty}$ is a sequence of conjugacy classes for which the associated closed geodesics γ_n become equidistributed on $\mathbb{H}^2/\Gamma_1$ with respect to area, i.e., for every continuous function $f : \mathbb{H}^2/\Gamma_1 \to \mathbb{R}$, $\lim_{n\to\infty} \int_{\gamma_n} f/l_1(\gamma_n) = \int f dArea$. (For background on this material, see [1], [4], [5], [11], [12].) By a theorem of Thurston, $i(\Gamma_1,\Gamma_2) \geq 1$ and the inequality is strict unless Γ_1 and Γ_2 are conjugate in $PSL(2,\mathbb{R})$.

We shall now state our main result.

Theorem 1. *The Manhattan curve $\mathcal{C}(\Gamma_1,\Gamma_2)$ is real analytic. The correlation number $\alpha = \alpha(\Gamma_1,\Gamma_2)$ satisfies*

$$\alpha = a + b,$$

where (a,b) is the point on $\mathcal{C}(\Gamma_1,\Gamma_2)$ for which the normal has slope equal to 1.

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1. Thermodynamic Formalism

In this section we briefly review the thermodynamic formalism associated to the geodesic flow over a hyperbolic surface. A reference for this material is given by [7] (see also [2], [8]). Our motivation for doing this is so that we can write $\mathcal{C}(\Gamma_1,\Gamma_2)$ as the graph of a certain function defined in terms of pressure.

Let Σ denote the surface $\mathbb{H}^2/\Gamma_1$ and set $M = T_1\Sigma$, the unit-tangent bundle. The geodesic flow $\phi_t : M \to M$ is defined by $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$, where $\gamma(t)$ is the unique geodesic on Σ (parametrized by arc-length) such that $\gamma(0) = x$, $\dot{\gamma}(0) = v$. This is an example of an Anosov flow and its ergodic properties are well understood.

There is an obvious one-to-one correspondence between closed geodesics on Σ (and hence non-trivial conjugacy classes in $\pi_1\Sigma$) and periodic ϕ-orbits on M. If the conjugacy class $[\gamma]$ corresponds to the periodic orbit τ then $l_1[\gamma] = \lambda(\tau)$, where $\lambda(\tau)$ is the least period of τ. For a continuous function $f : M \to \mathbb{R}$, define the pressure

$$P(f) = \sup\{h(\mu) + \int f d\mu : \mu \text{ is a } \phi\text{-invariant probability measure}\},$$

where $h(\mu)$ denotes the entropy of ϕ with respect to μ. If f is Hölder continuous then this supremum is attained at a unique ergodic measure μ_f, called the equilibrium state of f. The measure μ_0 is the Liouville measure for ϕ; locally it is the product of hyperbolic area on Σ and (normalized) arc-length on S^1.

Next we recall the idea of cohomology for the flow ϕ_t. A function $u : M \to \mathbb{R}$ is continuously ϕ-differentiable if the limit

$$u'(x) := \lim_{t \to 0+} \frac{1}{t}(u(\phi_t x) - u(x))$$

exists for every $x \in M$ and is continuous. Two continuous functions $f, g : M \to \mathbb{R}$ are said to be cohomologous (with respect to ϕ_t) if

$$F - G = u'$$

for some continuously ϕ-differentiable function u.

Let $f : X \to \mathbb{R}$ be Hölder continuous and suppose that f is not cohomologous to a constant function. Then the function $P(tf)$ is real analytic and

$$P'(tf) := \frac{d}{dt}P(tf) = \int f d\mu_{tf}. \tag{1.1}$$

2. The Correlation Number

In this section we shall represent the length spectrum on $\Sigma_2 = \mathbb{H}^2/\Gamma_2$ in terms of a single real valued function on M and characterize the correlation number α in terms of this function. A simple calculation based on Thermodynamic Formalism will then allow us to prove Theorem 1.

In [9], we defined a strictly positive Hölder continuous function $\psi : M \to \mathbb{R}$ with the property that

$$\int_\tau \psi = l_2[\gamma]$$

if τ is the periodic ϕ-orbit lying over the closed geodesic in $[\gamma]$.

If a sequence of periodic ϕ-orbits τ_n become equidistributed with respect to μ_0 (i.e., $\lim_{n\to\infty} \int_{\tau_n} f/\lambda(\tau_n) = \int f d\mu_0$ for every continuous function $f : M \to \mathbb{R}$) then the corresponding sequence of closed geodesics becomes equidistributed with respect to area. Therefore we have that

$$\int \psi d\mu_0 = \lim_{n\to\infty} \frac{\int_{\tau_n} \psi}{\lambda(\tau_n)} = \lim_{n\to\infty} \frac{l_2[\gamma_n]}{l_1[\gamma_n]} = i(\Gamma_1, \Gamma_2).$$

The following result was also proved.

Lemma 1 ([9])**.** *The function ψ is not cohomologous to a constant function.*

Remark. In fact, a stronger result about the cohomological properties of $a + b\psi$ ($a, b \in \mathbb{R}$) was given in [9].

In view of correspondence between conjugacy classes and periodic orbits we may write

$$\sum_{[\gamma]} e^{-al_1[\gamma]-bl_2[\gamma]} = \sum_{\tau} e^{\int_\tau (-a-b\psi)}.$$

It is a standard result that for any Hölder continuous function $f : M \to \mathbb{R}$

$$\lim_{T\to\infty} \frac{1}{T} \log \sum_{\lambda(\tau)\le T} e^{\int_\tau f} = P(f).$$

Thus we deduce that $\sum_{[\gamma]} e^{-al_1[\gamma]-bl_2[\gamma]}$ converges if $P(-a - b\psi) < 0$ and diverges if $P(-a - b\psi) > 0$ and we may identify $\mathcal{C}(\Gamma_1, \Gamma_2)$ with the set

$$\{(a, b) : P(-a - b\psi) = 0\} = \{(a, b) : P(-b\psi) = a\}.$$

If we define a function $q(t)$ implicitly by

$$P(-q(t)\psi) = t \tag{2.1}$$

then $\mathcal{C}(\Gamma_1, \Gamma_2)$ is the graph of q. By Lemma 1, $P(-s\psi)$ is real analytic. Since $P'(-s\psi) = -\int \psi d\mu_{-s\psi} \neq 0$, we may apply the Implicit Function Theorem to show that $q(t)$ is real analytic.

We are interested in the point on $\mathcal{C}(\Gamma_1, \Gamma_2)$ where the normal has slope equal to 1; this is the point $(t, q(t))$ for which $dq/dt = -1$. We may use (1.1) to relate the derivative to the function ψ. We have that

$$\begin{aligned} 1 &= \frac{d}{dt} P(-q(t)\psi) \\ &= \left(-\int \psi d\mu_{-q(t)\psi}\right) \frac{dq}{dt}. \end{aligned}$$

Thus we obtain

$$\frac{dq}{dt} = \frac{-1}{\int \psi d\mu_{-q(t)\psi}}. \tag{2.2}$$

We shall now complete the proof by obtaining the required formula for the correlation number. In [9], α was characterized by the formula $\alpha = h(\mu_{b\psi})$, where b is chosen so that $\int \psi d\mu_{b\psi} = 1$. By the definition of $\mu_{b\psi}$ we then have

$$\alpha = P(b\psi) - \int b\psi d\mu_{b\psi} = P(b\Psi) - b.$$

Setting $b = -q(t)$ and using (2.1) we obtain

$$\alpha = t + q(t),$$

where t is chosen so that $\int \psi d\mu_{-q(t)\psi} = 1$. By (2.2), this occurs precisely when the normal to $\mathcal{C}(\Gamma_1, \Gamma_2)$ has slope equal to 1, as required.

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Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL