CORRELATIONS OF LENGTH SPECTRA FOR NEGATIVELY CURVED MANIFOLDS

MARK POLLICOTT AND RICHARD SHARP

Abstract. In this paper we obtain asymptotic estimates for pairs of closed geodesics on negatively curved manifolds, the differences of whose lengths lie in a prescribed family of shrinking intervals, were the geodesics are ordered with respect to a discrete length. In certain cases, this discrete length can be taken to be the word length with respect to a set of generators for the fundamental group.

1. Introduction

It is a classical problem to study the closed geodesics on hyperbolic Riemann surfaces of higher genus (and more general negatively curved manifolds). Indeed, following on from Selberg's classical 1956 paper on trace formulae [29], Huber showed that there was an asymptotic formula, often called a Prime Geodesic Theorem, for the number of closed geodesics in terms of a bound on their length [10]. However, considerably less well understood are results on the differences in their lengths. In [24], the authors studied asymptotics for pairs of closed geodesics on a compact negatively curved surface, the difference of whose lengths lay in a prescribed (and possibly shrinking) interval. In this setting, we ordered the geodesics according to their word length with respect to a fixed generating set for the fundamental group. Similar results were obtained recently by Petkov and Stoyanov for certain open billiards, where the ordering is given by the number of reflections [18].

In this article we will consider extensions of these results to higher dimensions. We start by considering a special case of independent interest. If Γ is a Schottky group then every non-trivial (primitive) conjugacy class contains a (prime) closed geodesic γ. We will use $l(\gamma)$ to denote the length of γ and $|\gamma|$ to denote the word length of the associated conjugacy class (with respect to the Schottky generators S). Given a sequence $\epsilon_n > 0$, let $I_n(z)$ denote the interval of length $\epsilon_n > 0$ centred at $z \in \mathbb{R}$ and let $\pi_S(n, I_n(z))$ be the number of pairs of conjugacy classes in Γ whose word lengths are both less than n but for which the difference in lengths of the associated closed geodesics lies in $I_n(z)$.

Theorem 1.1. *Let $\Gamma = \langle a_1, \ldots, a_p\rangle$ be a Schottky group in Isom$(\mathbb{H}^N)$ with no parabolic elements and let $|\cdot|$ denote the word length with respect to $S = \{a_1^{\pm 1}, \ldots, a_p^{\pm 1}\}$. Then there exists $\eta > 0$ such that, for any sequence $\epsilon_n > 0$ satisfying $\epsilon_n^{-1} = O(e^{\eta n})$, we have that*

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{\sigma n^{5/2}}{\epsilon_n (2p-1)^{2n}} \pi_S(n, I_n(z)) - \frac{(2p-1)^2}{(2\pi)^{1/2}(2p-2)^2} e^{-z^2/2\sigma^2 n} \right| = 0,$$

where $\sigma > 0$ is defined by

$$\sigma^2 = \lim_{n\to+\infty} \frac{n}{(2p-1)^{2n}} \sum_{|\gamma|=|\gamma'|=n} (l(\gamma) - l(\gamma'))^2.$$

In particular, this implies that

$$\pi_S(n, I_n(z)) \sim \frac{1}{(2\pi)^{1/2}(2p-2)^2\sigma} \frac{\epsilon_n (2p-1)^{2n+2}}{n^{5/2}} \text{ as } n \to +\infty.$$

In the following section, we will discuss a generalization of this result to arbitrary compact negatively curved manifolds, for which the sectional curvatures are 1/4-pinched. This includes all compact quotients of $\mathbb{H}^N$. The price to be paid for this generality is that, apart from in special cases, the word length $|\cdot|$ needs to be replaced with a less natural ad hoc "discrete length".

We will now outline the contents of the paper. In the next section we will state our main results. In section 3, we discuss Anosov flows and their periodic orbits. In section 4, we explain how to study flow orbits by means of a Markov partition and an induced expanding map. In section 5, we introduce a product dynamical system that also allows us to study pairs of periodic orbits. In sections 6 and 7, we carry out the technical analysis required to prove Theorem 2.2 in section 8. Finally, in section 9, we prove Theorem 1.1.

2. Statement of main results

We shall discuss results analogous to Theorem 1.1 for higher dimensional compact manifolds. Here it would again be natural to use the word length with respect to some set of generators for the fundamental group but we cannot link this to a sufficiently well behaved symbolic dynamics to make the approach work. In fact, we will use a more ad hoc length (associated to Markov partitions for the geodesic flow) which retains the following important property of word length. The word length $|\cdot|$ is comparable to the geometric length $l(\cdot)$ induced by the given Riemannian metric, in the sense that there exist constants $0 < C_1 < C_2$ such that

$$C_1 l(\gamma) \leq |\gamma| \leq C_2 l(\gamma),$$

for all closed geodesics γ.

We can formulate a more general result as follows. Let V be a compact manifold equipped with a smooth Riemannian metric with negative sectional curvatures. We shall write $\mathcal{P}$ for the set of prime closed geodesics on V.

Definition 2.1. We say that a function $\mathfrak{n} : \mathcal{P} \to \mathbb{Z}$ is l-comparable if there exist constants $0 < C_1 < C_2$ such that

$$C_1 l(\gamma) \leq \mathfrak{n}(\gamma) \leq C_2 l(\gamma),$$

for all $\gamma \in \mathcal{P}$. We say that $\mathfrak{n}$ is strongly l-comparable if it is l-comparable and if there exists $\xi \in \mathbb{R}$ such that, for any $\epsilon > 0$,

$$\frac{\#\left\{\gamma \in \mathcal{P} : \mathfrak{n}(\gamma) \leq N, \ \left|\frac{l(\gamma)}{\mathfrak{n}(\gamma)} - \xi\right| > \epsilon\right\}}{\#\{\gamma \in \mathcal{P} : \mathfrak{n}(\gamma) \leq N\}}$$

tends to zero exponentially fast, as $N \to +\infty$.

Given a sequence $\epsilon_n > 0$ and $z \in \mathbb{R}$, we shall write

$$I_n(z) = \left[z - \frac{\epsilon_n}{2}, z + \frac{\epsilon_n}{2}\right]$$

and

$$\pi(n, I_n(z)) = \#\{(\gamma, \gamma') \in \mathcal{P} \times \mathcal{P} : \mathfrak{n}(\gamma), \mathfrak{n}(\gamma') \leq n, \ l(\gamma) - l(\gamma') \in I_n(z)\}.$$

Theorem 2.2. *Let V be a compact smooth Riemannian manifold whose sectional curvatures lie in an interval $[-\kappa, -\kappa/4]$, for some $\kappa > 0$. Then there exists a strongly l-comparable function $\mathfrak{n} : \mathcal{P} \to \mathbb{Z}$ and a number $\eta > 0$ such that, for any sequence $\epsilon_n > 0$ satisfying $\epsilon_n^{-1} = O(e^{\eta n})$, we have that*

$$\lim_{n \to +\infty} \sup_{z \in \mathbb{R}} \left| \frac{\sigma n^{5/2}}{\epsilon_n \lambda^{2n}} \pi(n, I_n(z)) - \frac{\lambda^2}{(2\pi)^{1/2}(\lambda - 1)^2} e^{-z^2/2\sigma^2 n} \right| = 0,$$

where $\lambda > 1$ and $\sigma > 0$ are defined by

$$\lambda = \lim_{n\to+\infty} \left(\#\{\gamma \in \mathcal{P} : \mathfrak{n}(\gamma) \leq n\}\right)^{1/n} \tag{2.3}$$

and

$$\sigma^2 = \lim_{n\to+\infty} \frac{1}{(\#\{\gamma \in \mathcal{P} : \mathfrak{n}(\gamma) = n\})^2} \sum_{\mathfrak{n}(\gamma)=\mathfrak{n}(\gamma')=n} \frac{(l(\gamma) - l(\gamma'))^2}{n}. \tag{2.4}$$

Remark 2.5. A particular case of the theorem is obtained by taking a constant sequence $\epsilon_n = \epsilon > 0$. In this case, the proof is considerably easier and does not require the transfer operator estimates in section 6 (cf. the proof of Theorem 1 in [24]).

Corollary 2.6. *For all $\delta > 0$ and $z \in \mathbb{R}$,*

$$\liminf_{n\to+\infty} \frac{n^{5/2}}{\epsilon_n \lambda^{2n}} \#\{(\gamma, \gamma') \in \mathcal{P} \times \mathcal{P} : l(\gamma), l(\gamma') \leq (\xi + \delta)n,\ l(\gamma) - l(\gamma') \in I_n(z)\} > 0,$$

where ξ is as in Definition 2.1.

In certain special cases we can give a more satisfactory result, where $\mathfrak{n}$ is derived from the word length with respect to an appropriate set of generators of $\pi_1 V$. Recall that a Kleinian group is a discrete group of isometries of the hyperbolic space $\mathbb{H}^N$, $N \geq 2$. A Kleinian group Γ is said to satisfy the even corners condition if Γ admits a fundamental domain R which is a finite sided polyhedron (possibly with infinite volume) such that $\bigcup_{g\in\Gamma} g\partial R$ is a union of hyperplanes. (This definition was introduced by Bowen and Series [6] when $N = 2$. We are interested in the case $N \geq 3$ which was studied by Bourdon [2].) The polyhedron R may have finite or infinite volume but we are be interested in the finite volume case and, in particular, when $\mathbb{H}^N/\Gamma$ is compact. Such examples exist for $N = 3, 4$ [33] but not for $N \geq 5$ [3], [16].

For such a Kleinian group, let S be the set of generators associated to R and let $|\cdot| : \Gamma \to \mathbb{Z}^+$ denote the usual word length with respect to S, i.e. $|g|$ denotes the smallest number of elements of $S \cup S^{-1}$ required to write g. As above, let $\mathcal{P}$ denote the set of prime closed geodesics on $\mathbb{H}^N/\Gamma$. There is a natural one-to-one correspondence between $\gamma \in \mathcal{P}$ and non-trivial conjugacy classes $c(\gamma)$ in Γ. We define $|\cdot| : \mathcal{P} \to \mathbb{Z}^+$ by

$$|\gamma| = \min\{|g| : g \in c(\gamma)\}.$$

If we write

$$\pi_S(n, I_n(z)) = \#\{(\gamma, \gamma') \in \mathcal{P} \times \mathcal{P} : |\gamma|, |\gamma'| \leq n,\ l(\gamma) - l(\gamma') \in I_n(z)\}$$

then we have the following result.

Theorem 2.7. *Let Γ be a co-compact Kleinian group which admits a fundamental domain satisfying the even corners condition and let S be the associated generators. Then there exists a number $\eta > 0$ such that, for any sequence $\epsilon_n > 0$ satisfying $\epsilon_n^{-1} = O(e^{\eta n})$, we have that*

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{\sigma n^{5/2}}{\epsilon_n \lambda^{2n}} \pi_S(n, I_n(z)) - \frac{\lambda^2}{(2\pi)^{1/2}(\lambda - 1)^2} e^{-z^2/2\sigma^2 n} \right| = 0,$$

where $\lambda > 1$ and $\sigma > 0$ are defined as in Theorem 2.2.

The key point in this setting is that the even corners condition ensures that the action of Γ on the boundary of $\mathbb{H}^N$, which we may identify with the unit sphere $\mathbb{S}^{N-1}$, may be modelled by a Markov expanding map which can play the role of τ is section 3.

3. The Geodesic Flow and Periodic Orbits

Let V be a compact smooth Riemannian manifold with negative sectional curvatures and let $M = SV$ denote the unit-tangent bundle, i.e.,

$$M = \{(x, v) \in TV : \|v\|_x = 1\},$$

where $\|\cdot\|_x$ is the norm induced by the Riemannian structure on T_xV. The geodesic flow $\phi_t : M \to M$ is defined as follows. Given $(x, v) \in M$, there is a unique unit-speed geodesic $\gamma : \mathbb{R} \to V$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. We then define $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$.

A C^1 flow ϕ_t on M is called an Anosov flow if there is a continuous splitting of the tangent bundle

$$TM = E^0 \oplus E^s \oplus E^u,$$

where E^0 is the line bundle tangent to the flow and where there exists constants $C, c > 0$ such that

(i) $\|D\phi_t v\| \le Ce^{-ct}\|v\|$, for all $v \in E^s$ and $t > 0$;
(ii) $\|D\phi_{-t} v\| \le Ce^{-ct}\|v\|$, for all $v \in E^u$ and $t > 0$.

We say that ϕ_t is transitive if it has a dense orbit and mixing if $U \cap \phi_t(V) \neq \varnothing$ for all non-empty open $U, V \subset M$ and all sufficiently large t. The geodesic flow on $M = SV$ is a mixing Anosov flow.

There is a natural one-to-one correspondence between periodic orbits for ϕ_t and closed geodesics on V, with the least period being equal to the length of the closed geodesic. Our notation will not distinguish between the two sets of objects. The mixing of the geodesic flow is equivalent to the fact that the set of lengths $\{l(\gamma) : \gamma \in \mathcal{P}\}$ is not contained in a discrete subgroup of $\mathbb{R}$.

We use the lengths $l(\gamma)$ to define a function of a complex variable $\zeta_\phi(s)$, the zeta function, by

$$\zeta_\phi(s) = \prod_{\gamma \in \mathcal{P}} \left(1 - e^{-sl(\gamma)}\right)^{-1},$$

whenever the product converges. In fact, the product converges for $\mathrm{Re}(s) > h$, where $h > 0$ denotes the topological entropy of the geodesic flow, and defines a non-zero analytic function in this half-plane. Furthermore, the mixing property implies that, apart from a simple pole at $s = h$, $\zeta_\phi(s)$ has a non-zero analytic extension to a neighbourhood of $\mathrm{Re}(s) \ge h$ [17]. When V is 1/4-pinched, the following stronger result holds.

Lemma 3.1. *Apart from a simple pole at $s = h$, $\zeta_\phi(s)$ has an analytic extension to $\mathrm{Re}(s) > h - \epsilon$, for some $\epsilon > 0$.*

Proof. This was proved for surfaces in [23] but holds in higher dimensions subject to 1/4-pinching. The key ingredient in the proof is Dolgopyat's bounds on iterates of transfer operators. (An extension of Dolgopyat's bounds to a wider class of flows and potentials is given in [31], which is shown to cover the above case in [32]). □

4. Markov Sections and an Expanding Map

The dynamics of the geodesic flow may be captured by a finite number of local cross sections, as described below. For $x \in M$ and $\epsilon > 0$, we define the *local strong stable manifold* $W^{ss}_\epsilon(x)$ by

$$W^{ss}_\epsilon(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \le \epsilon \ \forall t \ge 0\}$$

and the *local strong unstable manifold* $W^{su}_\epsilon(x)$ by

$$W^{su}_\epsilon(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \le \epsilon \ \forall t \le 0\}.$$

One may choose (arbitrarily small) local cross sections $\mathcal{T} = \{T_1, \ldots, T_k\}$ transverse to the flow on M such that every orbit hits $\coprod_{i=1}^k T_i$ infinitely often for both positive and negative times and with bounded times between hits. Furthermore, the elements of $\mathcal{T}$ fit together nicely under the action of the Poincaré map $\Pi : \coprod_{i=1}^k T_i \to \coprod_{i=1}^k T_i$ (the Markov property):

(i) if $x \in \mathrm{int}(T_i)$ and $\Pi(x) \in \mathrm{int}(T_j)$ then $\Pi(W_\epsilon^{ss}(x) \cap T_i) \subset W_\epsilon^{ss}(\Pi(x)) \cap T_j$;
(ii) if $x \in \mathrm{int}(T_i)$ and $\Pi^{-1}(x) \in \mathrm{int}(T_l)$ then $\Pi^{-1}(W_\epsilon^{su}(x) \cap T_i) \subset W_\epsilon^{su}(\Pi^{-1}(x)) \cap T_l$.

A C^1 function $r : \coprod_{i=1}^k T_i \to \mathbb{R}^+$ is determined by the times between hitting the cross sections:

$$r(x) = \inf \left\{ t > 0 : \phi_t(x) \in \coprod_{i=1}^k T_i \right\}.$$

The Poincaré map between sections gives rise to a C^1 expanding map obtained in the following way. Each local cross section T_i is foliated by local stable manifolds $S_i(x) = W_\epsilon^{ss}(x) \cap T_i$ and contains a piece of local unstable unstable manifold U_i. After collapsing along stable manifolds, the Poincaré map $\Pi : \coprod_{i=1}^k T_i \to \coprod_{i=1}^k T_i$ induces a Markov expanding map $\tau : \coprod_{i=1}^k U_i \to \coprod_{i=1}^k U_i$. Furthermore, the 1/4-pinching condition on the Riemannian metric ensures that $S_i(x)$ has C^1 dependence on x and so the map τ is C^1. A desirable feature of this construction is that r is constant on each $S_i(x)$ and so defines a C^1 function, which, abusing notation, we denote by $r : \coprod_{i=1}^k U_i \to \mathbb{R}^+$.

The Markov property of the map τ enables it to be coded by a subshift of finite type. We define a $k \times k$ matrix A by

$$A(i,j) = \begin{cases} 1 & \text{if } \mathrm{int}(U_i) \cap \tau^{-1}(\mathrm{int}(U_j)) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

We then define

$$\Sigma_A^+ = \{(x_n)_{n=0}^\infty \in \{1, \ldots, k\}^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1 \ \forall n \geq 0\}$$

and the (one-sided) subshift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by $(\sigma x)_n = x_{n+1}$. We give Σ_A^+ the metric $d(x,y) = 2^{-n(x,y)}$, where $n(x,y) = \min\{m : x_m \neq y_m\}$ (with the convention that $n(x,x) = -\infty$). This metric makes Σ_A^+ into a compact space and σ into a continuous map.

In the above setting, the matrix A is aperiodic (i.e. there exists $n \geq 1$ such that A^n has all its entries positive). By the Perron-Frobenius Theorem, A has a positive eigenvalue λ, with all the other eigenvalues having strictly smaller modulus. Furthermore, $\lambda > 1$ and is related to the topological entropy $h(\sigma)$ of $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by $h(\sigma) = \log \lambda$. The number of periodic points of period n is given by

$$\#\mathrm{Fix}_n(\sigma) = \mathrm{trace}(A^n) = \lambda^n + O((\theta_0 \lambda)^n),$$

where $0 < \theta_0 < 1$.

Given any σ-invariant probability measure ν on Σ_A, we may define its entropy $h_\sigma(\nu)$. This always satisfies $h_\sigma(\nu) \leq h(\sigma)$ and there is a unique σ-invariant probability measure μ_0, called the measure of maximal entropy, for which $h(\mu_0) = h(\sigma)$.

Lemma 4.1. *Define* $\pi : \Sigma_A^+ \to \coprod_{i=1}^k U_i$ *by*

$$\pi((x_n)_{n=0}^\infty) \in \bigcap_{n=0}^\infty \tau^{-n}(U_{x_n}).$$

Then π *is a well-defined Hölder surjection such that* $\tau \circ \pi = \pi \circ \sigma$. *Furthermore,* π *is one-to-one on a residual set and almost everywhere with respect to every to fully supported ergodic measures.*

In particular, the topological entropy of τ satsfies $h(\tau) = h(\sigma) = \log\lambda$. The topological entropy gives the exponential growth rate of periodic points for τ. More precisely, if we write

$$\mathrm{Fix}_n(\tau) = \left\{ x \in \coprod_{i=1}^k U_i \,:\, \tau^n x = x \right\}$$

then there exists $0 < \theta_1 < 1$ such that

$$\#\mathrm{Fix}_n(\tau) = \lambda^n + O((\theta_1\lambda)^n).$$

Every periodic orbit γ for $\phi_t : M \to M$ corresponds to a (possibly non-unique) periodic orbit $\{x, \tau x, \ldots \tau^{n-1}x\}$ (with $\tau^n x = x$) for $\tau : \coprod_{i=1}^k U_i \to \coprod_{i=1}^k U_i$. This non-uniqueness is caused by orbits passing through the boundaries of the cross sections. If $\{x, \tau x, \ldots \tau^{n-1}x\}$ is unique then we define

$$\mathfrak{n}(\gamma) = n,$$

i.e., the period x. If there is more than one τ-orbit corresponding to γ then we choose $\mathfrak{n}(\gamma)$ to be equal to the smallest period of these orbits. We also have the identity

$$l(\gamma) = r^n(x) := r(x) + r(\tau x) + \cdots + r(\tau^{n-1}x),$$

where $\{x, \tau x, \ldots \tau^{n-1}x\}$ is any τ-orbit corresponding to γ.

The next result shows that the overcounting described above does not cause a problem for our analysis. It follows because the extra symbolic orbits associated to those passing through the boundaries of the sections may be exactly accounted for by a finite number of auxiliary subshifts of finite type, each with entropy smaller than $h(\sigma)$ [5].

Lemma 4.2. *There exists* $0 < \theta_2 < 1$ *such that*

$$\#\{\gamma \in \mathcal{P} \,:\, \mathfrak{n}(\gamma) = n\} = \frac{\#\mathrm{Fix}_n(\tau)}{n} + O((\theta_2\lambda)^n).$$

The next lemma is for technical convenience.

Lemma 4.3. *The cross sections can be chosen so that* $\tau : \coprod_{i=1}^k U_i \to \coprod_{i=1}^k U_i$ *(or, equivalently,* $\sigma : \Sigma_A^+ \to \Sigma_A^+$*) has a fixed point. In particular, there will exist* $\gamma \in \mathcal{P}$ *such that* $\mathfrak{n}(\gamma) = 1$.

Proof. The result follows by refining the cross sections $\{T_i\}$. □

Lemma 4.4. *The function* $\mathfrak{n} : \mathcal{P} \to \mathbb{Z}$ *is strongly* l*-comparable.*

Proof. That $\mathfrak{n} : \mathcal{P} \to \mathbb{Z}$ is an l-comparable function follows immediately from the fact that the function r is bounded above and below away from zero. In particular, if $\mathfrak{n}(\gamma) = n$ and $l(\gamma) = r^n(x)$ then

$$\left(\max\left\{ r(y) \,:\, y \in \coprod_{i=1}^k U_i \right\}\right)^{-1} l(\gamma) \le \mathfrak{n}(\gamma) \le \left(\min\left\{ r(y) \,:\, y \in \coprod_{i=1}^k U_i \right\}\right)^{-1} l(\gamma).$$

That $\mathfrak{n}$ is strongly l-comparable follows from large deviation properties for periodic points of $\sigma : \Sigma_A^+ \to \Sigma_A^+$ [12], [13]. In particular, $\xi = \int r \, d\mu_0$, where μ_0 is the measure of maximal entropy for σ. □

Given a continuous function $f : \coprod_{i=1}^k U_i \to \mathbb{R}$, we define its pressure $P(f)$ by

$$P(f) = \sup\left\{ h(\nu) + \int f \, d\nu \,:\, \nu \text{ is a } \sigma\text{-invariant probability measure} \right\}.$$

Pressure also satisfies the identity

$$P(f) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\tau^n x = x} e^{f^n(x)}.$$

Define a transfer operator $L_{-sr} : C^1(\coprod_{i=1}^k U_i, \mathbb{C}) \to C^1(\coprod_{i=1}^k U_i, \mathbb{C})$ by

$$L_{-sr} w(x) = \sum_{\tau y = x} e^{-sr(x)} w(y).$$

If $s \in \mathbb{R}$ then L_{-sr} has a simple eigenvalue equal to $e^{P(sr)}$ and the rest of the spectrum is contained in a disc of strictly smaller radius. Perturbation theory ensures that this simple eigenvalue, which we still denote by $e^{P(sr)}$, extends analytically for $s \in \mathbb{C}$ with $|\mathrm{Im}(s)|$ sufficiently small.

The following lemma gives the connection between the transfer operators and the sum over periodic orbits.

Lemma 4.5. *There exists* $0 < \theta_3 < 1$ *such that, for any* $x_0 \in \coprod_{i=1}^k U_i$,

$$\sum_{\tau^n x = x} e^{itr^n(x)} = (L_{itr}^n 1)(x_0)\left(1 + O(\max\{1, |t|\} n \theta_3^n)\right).$$

Proof. This result appears in [27]. □

A key ingedient in our analysis will be the following Dolgopyat-type result.

Lemma 4.6. [9], [31] *Given* $\epsilon > 0$, *there exists* $C > 0$ *and* $0 < \theta_4 < 1$ *such that for* $|t| \geq \epsilon$ *and* $p[\log|t|] \leq n \leq (p+1)[\log|t|]$, *where* $p \geq 1$,

$$\|L_{itr}^n 1\|_\infty \leq C \lambda^n \theta_4^{p[\log|t|]/2}.$$

5. A Product System

In order to study pairs of closed geodesics, we shall consider a direct product dynamical system. Let $\widetilde{U} = \coprod_{i=1}^k U_i \times \coprod_{i=1}^k U_i$ and define $\widetilde{\tau} : \widetilde{U} \to \widetilde{U}$ by

$$\widetilde{\tau}(x, y) = (\tau x, \tau y).$$

Clearly, $\widetilde{\tau} : \widetilde{U} \to \widetilde{U}$ is semi-conjugate to the subshift of finite type $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, where $\widetilde{\Sigma} = \Sigma_A^+ \times \Sigma_A^+$ and $\widetilde{\sigma}(x, y) = (\sigma x, \sigma y)$. We shall denote the semi-conjugacy by $\widetilde{\pi} : \widetilde{\Sigma} \to \widetilde{U}$.

The topological entropy $h(\widetilde{\tau})$ of $\widetilde{\tau} : \widetilde{U} \to \widetilde{U}$ is equal to $2\log\lambda$ and its measure of maximal entropy $\widetilde{\mu}$ satisfies $\widetilde{\mu} = \mu \times \mu$, where μ is the measure of maximal entropy for $\tau : \coprod_{i=1}^k U_i \to \coprod_{i=1}^k U_i$.

Following the approach in [24], we define a function $R : \widetilde{U} \to \mathbb{R}$ by

$$R(x, y) = r(x) - r(y).$$

From this definition, we have

$$\int R(x, y)\, d\widetilde{\mu}(x, y) = \int r(x)\, d\mu(x) - \int r(y)\, d\mu(y) = 0. \tag{5.1}$$

Lemma 5.2. *The function* $R \circ \widetilde{\pi} : \widetilde{\Sigma} \to \mathbb{R}$ *is not cohomologous to the sum of a function taking values in a discrete subgroup of* $\mathbb{R}$ *and a constant function, i.e., there are no continuous functions* $\Psi : \widetilde{\Sigma} \to \mathbb{R}$ *and* $M : \widetilde{\Sigma} \to a\mathbb{Z}$ *such that* $R = \Psi \circ \widetilde{\sigma} - \Psi + M + c$, *where* c *is a real constant.*

Proof. Assume for a contradiction that $R \circ \widetilde{\pi}$ is cohomologous to such a $M + c$, as above, then $R^n(x, y) - nc = r^n(x) - r^n(y) - nc = M^n(x, y)$ whenever $\sigma^n x = x$ and $\sigma^n y = y$. By Lemma 4.3, we can find a fixed point $\sigma y = y$ and a corresponding $\gamma_0 \in \mathcal{P}$ with $\mathfrak{n}(\gamma_0) = 1$. Then, for any periodic point $\sigma^n x = x$ and corresponding $\gamma \in \mathcal{P}$, we have $l(\gamma) - n(l(\gamma_0) + c) = r^n(x) - nr(y) - nc = M^n(x, y)$. In other words,

$r : \Sigma \to \mathbb{R}$ is cohomologous to $r(y) + c + M(\cdot, y)$, i.e., the sum of a constant and a function valued in $a\mathbb{Z}$. In particular, this forces r to be a locally constant function. For such functions, the zeta function

$$\zeta(-sr) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} e^{-sr^n(x)}$$

has poles arbitrarily close to $\mathrm{Re}(s) = h(\sigma^r) = h(\phi)$. Since $\zeta(-sr)/\zeta_\phi(s)$ is non-zero and analytic for $\mathrm{Re}(s) > h(\phi) - \epsilon$, for some $\epsilon > 0$, which is impossible by Lemma 3.1. This contradiction completes the proof. □

For $s \in \mathbb{R}$, we may define the pressure function

$$P(sR) = \sup \left\{ h_{\widetilde{\tau}}(\nu) + s \int R \, d\nu \, : \, \nu \text{ is a } \widetilde{\tau}\text{-invariant probability measure} \right\},$$

with an analogous definition for $P(s(R \circ \pi))$. In fact, it follows from the semi-conjugacy between $\widetilde{\tau}$ and $\widetilde{\sigma}$ that $P(sR) = P(s(R \circ \pi))$.

Lemma 5.3. *Writing $\mathfrak{p}(s) = P(sR)$, we have $\mathfrak{p}(0) = 2\log\lambda$,*

$$\mathfrak{p}'(0) = 0 \quad \text{and} \quad \mathfrak{p}''(0) = \sigma^2 > 0,$$

where σ^2 is defined by equation (2.4).

Proof. The first statement is immediate from the definition. By a standard result,

$$\mathfrak{p}'(0) = \int R(x, y) \, d\widetilde{\mu}(x, y) = 0.$$

Since $R \circ \pi$ is not cohomologous to a constant, we have $\mathfrak{p}''(0) > 0$.

To obtain the formula for $\mathfrak{p}''(0)$, consider the series

$$\xi(z, s) = 1 + \sum_{n=1}^{\infty} z^n \sum_{\widetilde{\tau}^n(x,y)=(x,y)} e^{sR^n(x,y)}.$$

This converges to an analytic function for $|z| < \lambda^{-2}$ and $|s|$ small (depending on z). Furthermore,

$$\xi(z, s) = \frac{1}{1 - ze^{\mathfrak{p}(s)}} + \xi_1(z, s),$$

where $\xi_1(z, s)$ is analytic for $|z| < \lambda^{-2+\kappa}$, for some $\kappa > 0$, and $|s|$ small (depending on z). (We do not give details but the arguments are similar to those in chapters 5 and 6 of [17].) We have

$$\begin{aligned} \frac{\partial^2}{\partial s^2} \xi(z, s) \bigg|_{s=0} &= \sum_{n=1}^{\infty} z^n \sum_{\widetilde{\tau}^n(x,y)=(x,y)} (R^n(x, y))^2 \\ &= \frac{z\mathfrak{p}''(0)\lambda^2}{(1 - z\lambda^2)^2} + \xi_2(z), \end{aligned}$$

where we have used that $\mathfrak{p}'(0) = 0$ and where $\xi_2(z)$ is analytic for $|z| < \lambda^{-2+\kappa}$. Since (for $|z| < \lambda^{-2}$)

$$\frac{z\mathfrak{p}''(0)\lambda^2}{(1 - z\lambda^2)^2} = \mathfrak{p}''(0) \sum_{n=1}^{\infty} n\lambda^{2n+2} z^{n+1},$$

we get

$$\left| \sum_{\widetilde{\tau}^n(x,y)=(x,y)} (R^n(x, y))^2 - \mathfrak{p}''(0)(n - 1)\lambda^{2n} \right| = O(\lambda^{2-\kappa}).$$

Hence, as $\#\mathrm{Fix}_n(\widetilde{\tau})$ is asymptotic to λ^{2n},

$$\lim_{n\to+\infty} \frac{1}{\#\mathrm{Fix}_n(\widetilde{\tau})} \sum_{\widetilde{\tau}^n(x,y)=(x,y)} \frac{(R^n(x,y))^2}{n} = \mathfrak{p}''(0).$$

Since, as $n \to +\infty$,

$$\#\mathrm{Fix}_n(\tau) \sim n\#\{\gamma \in \mathcal{P} : \mathfrak{n}(\gamma) = n\}$$

and

$$\sum_{x,y\in\mathrm{Fix}_n(\tau)} (r^n(x) - r^n(y))^2 \sim n^2 \sum_{\mathfrak{n}(\gamma)=\mathfrak{n}(\gamma')=n} (l(\gamma) - l(\gamma'))^2,$$

we finally obtain

$$\mathfrak{p}''(0) = \lim_{n\to+\infty} \frac{1}{(\#\{\gamma \in \mathcal{P} : \mathfrak{n}(\gamma) = n\})^2} \sum_{\mathfrak{n}(\gamma)=\mathfrak{n}(\gamma')=n} \frac{(l(\gamma) - l(\gamma'))^2}{n} = \sigma^2.$$

□

Let $L_{itR} : C^1(\widetilde{U}, \mathbb{C}) \to C^1(\widetilde{U}, \mathbb{C})$ be the transfer operator defined by

$$L_{itR}w(x,y) = \sum_{\widetilde{\tau}(x',y')=(x,y)} e^{itR(x',y')} w(x',y').$$

Then $e^{\mathfrak{p}(s)} = e^{P(sR)}$ is a simple eigenvalue of L_{sR} such that the rest of the spectrum is contained in a disk of strictly smaller radius. We may extend $e^{P(sR)}$ to an analytic function of $s \in \mathbb{C}$, provided $|\mathrm{Im}(s)|$ is sufficiently small, by defining $e^{P(sr)}$ to be the maximal simple eigenvalue of L_{sR} guaranteed by perturbation theory [11].

The following lemma will be useful later.

Lemma 5.4. *Suppose that $|t|$ is sufficiently small that $\mathfrak{p}(it)$, $P(itr)$ and $P(-itr)$ are defined. Then $\mathfrak{p}(it)$ is real valued and $e^{(it)} = e^{P(itr)+P(-itr)}$. Furthermore,*

$$\sum_{n,m=1}^{N} e^{nP(itr)} e^{mP(-itr)} = \frac{e^{(N+1)\mathfrak{p}(it)}}{(e^{P(itr)} - 1)(e^{P(-itr)} - 1)} (1 + O(\rho^N))$$

for some $0 < \rho < 1$.

Proof. For the first part we observe by the variational principle (cf. [17]) that for $s \in \mathbb{R}$,

$$\begin{aligned} e^{\mathfrak{p}(s)} &= \lim_{n\to+\infty} \left(\sum_{\widetilde{\tau}^n(x,y)=(x,y)} e^{sR^n(x,y)} \right)^{1/n} \\ &= \lim_{n\to+\infty} \left(\sum_{\tau^n x=x} e^{sr^n(x,y)} \right)^{1/n} \left(\sum_{\tau^n y=y} e^{-sr^n(x,y)} \right)^{1/n} \\ &= e^{P(sr)+P(-sr)}. \end{aligned}$$

The identity then follows by the uniqueness of the analytic extension. Furthermore, since $R^n(x,y) = -R^n(y,x)$, we have that $\sum_{\widetilde{\tau}^n(x,y)=(x,y)} e^{itR^n(x,y)}$ is real valued and thus so is $\mathfrak{p}(it)$. For the second part, we can write

$$\begin{aligned} \sum_{n,m=1}^{N} e^{nP(itr)} e^{mP(-itr)} &= \left(\frac{e^{(N+1)P(itr)} - 1}{e^{P(itr)} - 1} \right) \left(\frac{e^{(N+1)P(-itr)} - 1}{e^{P(-itr)} - 1} \right) \\ &= \frac{e^{(N+1)[P(itr)+P(-itr)]}}{(e^{P(itr)} - 1)(e^{P(-itr)} - 1)} (1 + O(\rho^N)) \\ &= \frac{e^{(N+1)\mathfrak{p}(it)}}{(e^{P(itr)} - 1)(e^{P(-itr)} - 1)} (1 + O(\rho^N)), \end{aligned}$$

for some $0 < \rho < 1$, as required. $\square$

From Lemma 5.3, we have

$$\left.\frac{d\mathfrak{p}(it)}{dt}\right|_{t=0} = 0 \quad \text{and} \quad \left.\frac{d^2\mathfrak{p}(it)}{dt^2}\right|_{t=0} = -\sigma^2 < 0.$$

Lemma 5.5. *There exists $\epsilon > 0$ such that*

(i) *for $t \in (-\epsilon, \epsilon)$, we may write*

$$e^{\mathfrak{p}(it)} = \lambda^2 \left(1 - \frac{\sigma^2 t^2}{2} + O(|t|^3)\right),$$

with the implied constant uniform on any bounded interval;

(ii) *there is a smooth change of co-ordinates $v = v(t)$ on $(-\epsilon, \epsilon)$, such that $e^{\mathfrak{p}(it)} = \lambda^2(1 - v^2)$;*

(iii) *for any $t \in \mathbb{R}$,*

$$\lim_{N \to +\infty} e^{N\mathfrak{p}(it/\sigma\sqrt{N})}\lambda^{-2N} = e^{-t^2/2}$$

and, for sufficiently large N,

$$e^{N\mathfrak{p}(it/\sigma\sqrt{N})}\lambda^{-2N} \leq e^{-t^2/4} \quad \text{and} \quad \left|e^{N\mathfrak{p}(it/\sigma\sqrt{N})}\lambda^{-2N} - e^{-t^2/2}\right| \leq e^{-t^2/4}.$$

6. Some estimates

Before giving the proof of Theorem 2.2, we need to prove some preliminary asymptotic estimates. Their importance will become apparent in the next section.

Define

$$\mathcal{S}_N(t) = \sum_{n,m=1}^{N} \sum_{(\tau^n x, \tau^m y) = (x,y)} e^{it(r^n(x) - r^m(y))}.$$

When analysing this function, the essential philosophy is that, for t close to zero, $\mathcal{S}_N(t)$ is approximated by $e^{\mathfrak{p}(it)N}$, where $e^{\mathfrak{p}(it)}$ is well behaved (as in Lemma 5.5), while, for $|t|$ large, $\mathcal{S}_N(t)$ may be bounded in terms of $\|L_{itR}^N\|$ (using Lemmas 4.5 and 4.6).

Let $\epsilon > 0$ be the value given by Lemma 5.5.

Lemma 6.1. *There exists $0 < \theta_5 < 1$ such that, for $|t| < \epsilon$,*

$$\mathcal{S}_N(t) = e^{N\mathfrak{p}(it)} + O\left((\theta_5\lambda^2)^N\right).$$

Proof. The result follows from the estimate

$$\sum_{\tau^n x = x} e^{\pm itr^n(x)} = e^{nP(\pm itr)} + O\left((\theta_5^{1/2}\lambda)^n\right),$$

for some $0 < \theta_5 < 1$, which may be derived from [20]. $\square$

Lemma 6.2. *There exists $0 < \theta < 1$, $\alpha \geq 1$ such that, for $|t| \geq \epsilon$,*

$$|\mathcal{S}_N(t)| = O(\lambda^{2N}\theta^N|t|^\alpha).$$

Proof. By Lemma 4.5, for any $(x_0, y_0) \in \widetilde{U}$,

$$\mathcal{S}_N(t) = \sum_{n,m=1}^{N} \left((L_{itr}^n 1)(x_0)(1 + O(|t|n\theta_3^n))\right)\left((L_{-itr}^m 1)(y_0)(1 + O(|t|n\theta_3^n))\right)$$

Applying Lemma 4.6, the required estimate holds with $\theta = \max\{\theta_3.\theta_4\}$. $\square$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^k compactly supported function. (The value of k will be chosen later.) We define

$$A_1(N,z) = \left| \int_{-\epsilon\sigma\sqrt{N}}^{\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ \lambda^{-2N} \mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right) \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right) - \frac{1}{\sqrt{2\pi}} \frac{\left(\int \chi(x)dx\right)\lambda^2}{(\lambda-1)^2} e^{-t^2/2} \right\} dt \right|,$$

$$A_2(N,z) = \left| \int_{|t|\geq\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ \lambda^{-2N} \left(\mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right) \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right)\right\} dt \right|$$

and

$$A_3(N,z) = \left| \int_{|t|\geq\epsilon\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ \frac{\left(\int \chi(x)dx\right)\lambda^2}{(\lambda-1)^2} e^{-t^2/2} \right\} dt \right|.$$

We refer the reader to [24] for the proof of the following lemma.

Lemma 6.3. *We have that*

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} A_1(N,z) = 0 \quad \text{and} \quad \lim_{N\to+\infty} \sup_{z\in\mathbb{R}} A_3(N,z) = 0.$$

The proof of the analogous result for $A_2(N,z)$ requires the estimates on transfer operators discussed above. The following preliminary estimate will be useful.

Lemma 6.4. *If χ is C^k and compactly supported then $\widehat{\chi}(u) = O(|u|^{-k})$.*

Proof. This standard result follows with integration by parts on the definition of the Fourier transform. □

We are now in a position to bound $A_2(N,z)$. It is here that we use the condition $\epsilon_N^{-1} = O(e^{\eta N})$ for a suitably small $\eta > 0$.

Lemma 6.5. *Provided $\eta > 0$ is sufficiently small, $\sup_{z\in\mathbb{R}} A_2(N,z) \to 0$ as $N \to +\infty$.*

Proof. Using Lemma 6.2, there exists $C > 0$ such that we can bound

$$A_2(N,z) \leq C\theta^N \int_{|t|\geq\epsilon\sigma\sqrt{N}} \left| \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right| \left(\frac{|t|}{\sigma\sqrt{N}}\right)^\alpha dt.$$

In particular, for $\beta > 0$ we can bound

$$A_2(N,z) \leq \frac{C\theta^N \|\widehat{\chi}\|_\infty}{(\sigma\sqrt{N})^\alpha} \int_{\epsilon\sigma\sqrt{N}}^{e^{\beta N}} |t|^\alpha dt + \int_{e^{\beta N}}^\infty \left| \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right| \left(\frac{|t|}{\sigma\sqrt{N}}\right)^\alpha dt.$$

The first term is of order $O(\theta^N e^{\beta N(\alpha+1)})$. This tends to zero (uniformly in z) as $N \to +\infty$ provided we choose $\beta > 0$ sufficiently small that $\theta e^{\beta(\alpha+1)} < 1$. For the second term we can use Lemma 6.4 to bound the integral

$$\begin{aligned}\int_{e^{\beta N}}^\infty \left| \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right| \left(\frac{|t|}{\sigma\sqrt{N}}\right)^\alpha dt &= O\left(\epsilon_N^{-k} N^{(k-\alpha)/2} \int_{e^{\beta N}}^\infty \frac{1}{t^{k-\alpha}} dt\right) \\ &= O\left(\frac{\epsilon_N^{-k} N^{(k-\alpha)/2}}{e^{(k-1-\alpha)\beta N}}\right),\end{aligned}$$

which will tend to zero as $N \to +\infty$, provided we take $\eta < \beta$ and then take k sufficiently large that $(k-1-\alpha)/k > \eta/\beta$. □

7. Proof of Theorem 2.2

In this section $\chi : \mathbb{R} \to \mathbb{R}$ will denote a smooth integrable non-negative function. (Ultimately, χ will be used to approximate the indicator function of the interval $[-1/2, 1/2]$.) In order to obtain results for the shrinking intervals $I_N(z) = [z - \epsilon_N/2, z + \epsilon_N/2]$, we shall consider a sequence of rescaled functions $\chi_N^{(z)}$, defined by $\chi_N^{(z)}(x) = \chi(\epsilon_N^{-1}(x-z))$.

Define

$$\psi_N(\chi) = \sum_{n,m=1}^{N} \sum_{(\tau^n x, \tau^m y)=(x,y)} \chi(r^n(x) - r^m(y)).$$

We can write

$$\widehat{\chi}_N^{(z)}(u) = e^{izu}\epsilon_N \widehat{\chi}(\epsilon_N u).$$

We need to consider

$$A(N,z) := \left| \frac{\sigma\sqrt{N}}{\epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)}) - \frac{\lambda^2 \int \chi(x)dx}{\sqrt{2\pi}(\lambda-1)^2} e^{-z^2/2\sigma^2 N} \right|,$$

where $\psi_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N} \sum_{(\tau^n x, \tau^m y)=(x,y)} \chi_N^{(z)}(r^n(x) - r^m(y))$.

Proposition 7.1.

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} A(N,z) = 0.$$

We begin with the following observation.

Lemma 7.2. *We can write*

$$e^{-z^2/2\sigma^2 N} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iuz/\sigma\sqrt{N}} e^{-u^2/2} du.$$

Using Fourier inversion, we can write

$$\begin{aligned}
\frac{\sigma\sqrt{N}}{\epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)}) &= \frac{1}{2\pi} \frac{\sigma\sqrt{N}}{\epsilon_N \lambda^{2N}} \int_{-\infty}^{\infty} \mathcal{S}_N(u) \widehat{\chi}_N^{(z)}(u) du \\
&= \frac{1}{2\pi} \frac{\sigma\sqrt{N}}{\lambda^{2N}} \int_{-\infty}^{\infty} \mathcal{S}_N(u) e^{izu} \widehat{\chi}(\epsilon_N u) du
\end{aligned}$$

We can substitute $t = u\sigma\sqrt{N}$ and then this becomes:

$$\frac{\sigma\sqrt{N}}{\epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)}) = \frac{\lambda^{-2N}}{2\pi} \int_{-\infty}^{\infty} \left(\mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right) e^{izt/\sigma\sqrt{N}} \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right) \right) dt.$$

We can write

$$\begin{aligned}
2\pi A(N,z) = \Bigg| \int_{-\infty}^{\infty} e^{izt/\sigma\sqrt{N}} \Bigg\{ &\lambda^{-2N} \mathcal{S}_N\left(\frac{t}{\sigma\sqrt{N}}\right) \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right) \\
&- \frac{\left(\int \chi(x)dx\right)\lambda^2}{\sqrt{2\pi}(\lambda-1)^2} e^{-t^2/2} \Bigg\} dt \Bigg|
\end{aligned}$$

In particular, we can bound

$$2\pi A(N,z) \le A_1(N,z) + A_2(N,z) + A_3(N,z)$$

and thus we can complete the proof of Proposition 7.1 with the bounds in Lemma 6.3 and Lemma 6.5.

In order to prove Theorem 2.2, we first need to replace $\psi_N(\chi_N^{(z)})$ with a sum over (prime) closed geodesics. More precisely, we define

$$\rho_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N} \sum_{\mathfrak{n}(\gamma)=n} \sum_{\mathfrak{n}(\gamma')=m} \chi_N^{(z)}(l(\gamma) - l(\gamma')).$$

We have the estimate

$$\rho_N(\chi_N^{(z)}) = \Xi_N(\chi_N^{(z)}) + O\left(\|\chi\|_\infty (\log N)^2 \lambda^{3N/2}\right),$$

where

$$\Xi_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{(\tau^n x, \tau^m y)=(x,y)} \chi_N^{(z)}(r^n(x) - r^m(y))$$

and the implied constant in the big-O term is independent of z. Clearly we have that

$$\frac{N^{5/2}}{\epsilon_N \lambda^{2N}} \Xi_N(\chi_N^{(z)}) \geq \frac{N^{1/2}}{\epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)}).$$

On the other hand, for any $0 < \alpha < 1$,

$$\begin{aligned}\frac{N^{5/2}}{\epsilon_N \lambda^{2N}} \Xi_N(\chi_N^{(z)}) &= \frac{N^{5/2}}{\epsilon_N \lambda^{2N}} \sum_{n,m=[\alpha N]+1}^{N} \frac{1}{nm} \sum_{(\tau^n x, \tau^m y)=(x,y)} \chi_N^{(z)}(r^n(x) - r^m(y)) \\ &\qquad + O\left(\|\chi\|_\infty (\log N)^2 N^{5/2} \epsilon_N^{-1} \lambda^{(\alpha-1)N}\right) \\ &\leq \frac{N^{1/2}}{\alpha \epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)}) + O\left(\|\chi\|_\infty (\log N)^2 N^{5/2} \epsilon_N^{-1} \lambda^{(\alpha-1)N}\right).\end{aligned}$$

Also, by Proposition 7.1 we have that

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \frac{N^{1/2}}{\epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)}) = \frac{\lambda^2 \int \chi(x)dx}{\sqrt{2\pi}\sigma(\lambda-1)^2}.$$

Thus, we see that

$$\begin{aligned} 0 &\leq \limsup_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{N^{5/2}}{\epsilon_N \lambda^{2N}} \Xi_N(\chi_N^{(z)}) - \frac{N^{1/2}}{\epsilon_N \lambda^{2N}} \psi_N(\chi_N^{(z)})\right) \\ &\leq \left(\frac{1}{\alpha} - 1\right) \frac{\lambda^2 \int \chi(x)dx}{\sqrt{2\pi}\sigma(\lambda-1)^2}.\end{aligned}$$

Since we may take α arbitrarily close to 1, the above limit exists and is equal to zero. We have shown the following.

Proposition 7.3.

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{N^{5/2}}{\epsilon_N \lambda^{2N}} \rho_N(\chi_N^{(z)}) - \frac{\lambda^2 \int \chi(x)dx}{\sqrt{2\pi}\sigma(\lambda-1)^2} e^{-z^2/2\sigma^2 N} \right| = 0.$$

The final step in the proof of Theorem 2.2 is to replace the smooth function χ by the indicator function $\chi_{[-1/2,1/2]}$ of the interval $[-1/2, 1/2]$. Given $\epsilon > 0$ we can choose compactly supported smooth functions $\chi_- \leq \chi_{[-1/2,1/2]} \leq \chi_+$ such that

$$\int \chi_{[-1/2,1/2]}(x)\,dx - \epsilon \leq \int \chi_-(x)\,dx \leq \int \chi_+(x)\,dx \leq \int \chi_{[-1/2,1/2]}(x)\,dx + \epsilon.$$

From this we can deduce that

$$\begin{aligned} -B\epsilon &\leq \liminf_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{\sigma N^{5/2}}{\epsilon_N \lambda^{2N}} \pi(N, I_N(z)) - \frac{\lambda^2}{\sqrt{2\pi}(\lambda-1)^2} e^{-\sigma^2 z^2/2N}\right) \\ &\leq \limsup_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{\sigma N^{5/2}}{\epsilon_N \lambda^{2N}} \pi(N, I_N(z)) - \frac{\lambda^2}{\sqrt{2\pi}(\lambda-1)^2} e^{-\sigma^2 z^2/2N}\right) \leq B\epsilon,\end{aligned}$$

where $B = \lambda^2/(\sigma(\lambda-1)^2)$. Since $\epsilon > 0$ can be chosen arbitrarily small, Theorem 2.2 follows.

8. Proof of Theorem 2.7

In this section we explain how to derive Theorem 2.7 from the preceding analysis. Our aim is to construct a Markov partition of the boundary associated to generators of the group.

Let $\mathbb{H}^N = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_1^2 + \cdots + x_N^2 < 1\}$ be the open unit ball in $\mathbb{R}^N$ equipped with the Poincaré metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_N^2}{(1 - (x_1^2 + \cdots + x_N^2))^2}.$$

The boundary of this space can be naturally identified with the unit sphere $\mathbb{S}^{N-1}$. A kleinian group is a discrete group of isometries. Let Γ be a co-compact Kleinian group acting on $\mathbb{H}^N$ which admits a fundamental polyhedron R satisfying the even corners condition defined in the introduction. Label the faces of R by $\{R_1, \ldots, R_m\}$ and let $g_i \in \Gamma$ denote the unique element for which $g_i R \cap R = R_i$. Then $S = \{g_1, \ldots, g_m\}$ is a symmetric set of generators for Γ. For each $i = 1, \ldots, m$, R_i extends to a co-dimension one hyperbolic hyperplane, which divides $\mathbb{H}^N \cup \mathbb{S}^{N-1}$ into two half-spaces. Let H_i denote the half-space which does not contain R and let $W_i = H_i \cap \mathbb{S}^{N-1}$. In general, the W_i will overlap; to obtain a partition we let $\mathcal{U} = \{U_1, \ldots, U_k\}$ denote the sets formed by taking the closure of all possible intersections of the W_i. Then $\mathbb{S}^{N-1} = \bigcup_{i=1}^k U_i$ and $\mathrm{int}(U_i) \cap \mathrm{int}(U_j) = \varnothing$ whenever $i \neq j$.

Choose an arbitrary ordering $\prec$ on S. Let $g \in \Gamma \backslash \{id\}$. If $g = g_{i_0} \cdots g_{i_{n-1}}$ then we say that $g_{i_0} \cdots g_{i_{n-1}}$ is lexically shortest if $|g|_S = n$ and if, whenever $g = h_{i_0} \cdots h_{i_{n-1}}$ with $h_{i_0}, \ldots, h_{i_{n-1}} \in S$, then $g_{i_j} \prec h_{i_j}$, where j is the smallest index at which the terms disagree. Clearly every group element is represented by a unique lexically shortest word.

Define a map $\tau : \coprod_{i=1}^k U_i \to \coprod_{i=1}^k U_i$ by $\tau|_{U_i}(x) = a_i^{-1} x$, where $\mathrm{int}(U_i) = \mathrm{int}(W_{j_1}) \cap \cdots \cap \mathrm{int}(W_{j_l})$ and where a_i is the $\prec$-smallest element of $\{g_{j_1}, \ldots, g_{j_l}\}$. If necessary, refining a finite number of times by considering intersections of sets in $\mathcal{U}, \tau^{-1}(\mathcal{U}), \ldots, \tau^{-n}(\mathcal{U})$, for some $n \geq 1$, τ will satisfy the Markov property: if $\tau(\mathrm{int}(U_i)) \cap \mathrm{int}(U_j) \neq \varnothing$ then $\tau(U_i) \supset U_j$[2]. We shall now define a $k \times k$ matrix A by

$$A(i,j) = \begin{cases} 1 & \text{if } \tau(U_i) \supset U_j \\ 0 & \text{otherwise.} \end{cases}$$

We define a map $\pi : \Sigma_A^+ \to \mathbb{S}^{N-1}$ by $\cap_{n=0}^{\infty} \tau^{-n} U_{x_n}$.

We may use the Markov map τ and the associated subshift of finite type Σ_A^+ to repeat the analysis of the earlier sections and obtain Theorem 2.7. More precisely, the sets in $\mathcal{U}$ correspond to (the stable projections of) Poincaré sections for the geodesic flow. In particular, it then follows from the work of Dolgopyat [9] and Stoyanov [31] that the transfer operators will satisfy the estimates in Lemmas 3.5 and 3.6.

9. Proof of Theorem 1.1

In this final section we consider the modications necessary for Schottky groups. To define these, consider the Poincaré disk model of the hyperbolic space $\mathbb{H}^N$. Let $C_1, \ldots, C_{2p}$ be $2p$ disjoint $(N-1)$-dimensional spheres in $\mathbb{R}^N$, each meeting the unit sphere $\mathbb{S}^{N-1}$ perpendicularly. For $i = 1, \ldots, p$, let a_i be the isometry which maps the exterior of C_i onto the interior of $C_{p+i \pmod{2p}}$. Then the group Γ generated by $S = \{a_1^{\pm 1}, \dots, a_p^{\pm 1}\}$ is called a Schottky group. Viewed as an abstract group, it is the free group on p generators and it is easy to see that

$$\lim_{n \to +\infty} \left(\#\{\gamma \in \mathcal{P} : |\gamma| \le n\}\right)^{1/n} = 2p - 1.$$

In the particular case of Fuchsian Schottky groups (i.e., when $N = 2$) the transfer operator estimates required for the proof are due to Naud [15] and Petkov and Stoyanov [19]. The basic principle is to show that there are bounds on the norm of iterates of the transfer operator by first showing that there are bounds on the integrals with respect to an equilibrium state. This in turn requires using an estimate on the non-uniform integrability of the stable and unstable laminations for the recurrent part of the geodesic flow.

We briefly recall the argument from Lemma 3 in [19] which establishes this estimate.

The lifts to $\mathbb{H}^n = \mathbb{R}^+ \oplus \mathbb{R}^{n-1}$ of stable and unstable manifolds (for recurrent tangent v_0) for the geodesic flow on $\mathbb{H}^n/\Gamma$ are orthogonal vectors to either:

(1) $(n-1)$-dimensional spheres tangent to boundary $\partial \mathbb{H}^n = \{0\} \times \mathbb{R}^{n-1}$; or
(2) $(n-1)$-dimensional hyperplanes $\{t\} \times \mathbb{R}^{n-1}$ for $t > 0$.

Let $v_0, v_1 \in S\mathbb{H}^n$ be unit tangent vectors which are lifts of recurrent vectors for the geodesic flow on the unit tangent bundle of $\mathbb{H}^n/\Gamma$.

Assume without loss of generality that v_0 converges to the point $0 \in \partial \mathbb{H}^n$ in the boundary under the geodesic flow and that the stable horocycle $W^s(v_0)$ corresponds to a sphere of radius 1. Assume without loss of generality that v_1 is orthogonal to the horocycle $\{t\} \times \mathbb{R}^{n-1}$ and that this flows vertical down to the point $x \in \partial \mathbb{H}^n$ in the boundary under the geodesic flow. The stable horocycle $W^s(v_1)$ corresponds to a sphere of radius 1.

The unstable horocycle $W^u(v_1)$ is $(n-1)$-dimensional hyperplanes $\{1\} \times \mathbb{R}^{n-1}$. The unstable horocycle $W^u(v_0)$ for v_0 is a $(n-1)$-dimensional sphere, and we denote the point where it touches the boundary at $y \in \mathbb{R}^{n-1}$ and write $R > 0$ for the radius.

We denote by $\Delta(v_0, v_1)$ the distance between the horospheres $W^s(v_0)$ and $W^s(v_1)$. An explicit calculation gives

$$\Delta(v_0, v_1) = \log\left(\frac{\|y - x\|}{2R}\right).$$

The necessary estimate describes how $\Delta(v_0, v_1)$ changes as v_0 changes in $W^u(v_0)$. More precisely, given a direction $b \in \mathbb{R}^{n-1}$ in the horocycle $\{1\} \times \mathbb{R}^{n-1}$ corresponding to a density point we can consider another direction $a \in \mathbb{R}^{n-1}$ which is not orthogonal. There exists $\epsilon > 0$ and $\delta > 0$ such that for $|t| < \epsilon$

$$|\Delta(v_0, v_1 + th) - \Delta(v_0, v_1)| \ge \delta |t|,$$

where $v_1 + th$ denotes the translation of the vector v_1 in $W^u(v_0)$.

References

[1] R. Adler and L. Flatto, *Geodesic flows, interval maps, and symbolic dynamics*, Bull. Amer. Math. Soc. 25 (1991), 229-334.
[2] M. Bourdon, *Actions quasi-convexes d'un groupe hyperbolique, flot géodésique*, Thesis, Orsay (1993).
[3] B. Bowditch and G. Mess, *A 4-dimensional Kleinian group*, Trans. Amer. Math. Soc. 344 (1994), 391-405.
[4] R. Bowen, *Equilibrium states and ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics 470, Springer, Berlin, 1975.
[5] R. Bowen, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. 95 (1973), 429-460.
[6] R. Bowen and C. Series, *Markov maps associated with Fuchsian groups*, Inst. Hautes Études Sci. Publ. Math 50 (1979), 153-170.
[7] A. Broise, *Transformations dilatantes de l'intervalle et theorèmes limites*, Asterisque 238, 5-109, Société Mathématique de France, 1996.

[8] F. Dal'bo, *Remarques sur le spectre des longueurs d'une surface et comptages*, Bol. Soc. Bras. Mat. 30 (1999), 199-221.
[9] D. Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. 147 (1998), 357-390.
[10] H. Huber, *Zur analytischen Theorie hyperbolischer Raum formen und Bewegungsgruppen, II*, Math. Ann. 142 (1961), 385-398.
[11] T. Kato, *Perturbation theory for linear operators*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
[12] Y. Kifer, *Large deviations, averaging and periodic orbits in dynamical systems*, Comm. Math. Phys. 162 (1994), 33-46.
[13] S. Lalley, *Distribution of periodic orbits of symbolic and Axiom A flows*, Adv. Appl. Math. 8 (1987), 154-193.
[14] S. Lalley, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits*, Acta Math. 163 (1989), 1-55.
[15] F. Naud, *Expanding maps on Cantor sets and analytic continuation of zeta functions*, Ann. Sci. École Norm. Sup. 38 (2005), 116-153.
[16] V. V. Nikulin, *On the classification of arithmetic groups generated by reflections in Lobachevskii spaces*, Math. USSR-Izv. 18 (1982), 99-123.
[17] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque 187-188 (1990), 1-268.
[18] V. Petkov and L. Stoyanov, *Correlations for pairs of periodic trajectories for open billiards*, Nonlinearity 22 (2009), 2657-2679.
[19] V. Petkov and L. Stoyanov, *Distribution of periods of closed trajectories in exponentially shrinking intervals*, Comm. Math. Phys. 310 (2012), 675-704.
[20] M. Pollicott and R. Sharp, *Rates of recurrence for $\mathbb{Z}^q$ and $\mathbb{R}^q$ extensions of subshifts of finite type*, J. London Math. Soc. 49 (1994), 401-416.
[21] M. Pollicott and R. Sharp, *The circle problem on surfaces of variable negative curvature*, Monat. Math. 123 (1997), 61-70.
[22] M. Pollicott and R. Sharp, *Comparison theorems and orbit counting in hyperbolic geometry*, Trans. Amer. Math. Soc. 350 (1998), 473-499.
[23] M. Pollicott and R. Sharp, *Exponential error terms for growth functions on negatively curved surfaces*, Amer. J. Math. 120 (1998), 1019-1042.
[24] M. Pollicott and R. Sharp, *Correlations for pairs of closed geodesics*, Invent. math. 163 (2006), 1-24.
[25] M. Pollicott and R. Sharp, *Distribution of ergodic sums for hyperbolic maps*, in Representation theory, dynamical systems, and asymptotic combinatorics, 167-183, Amer. Math. Soc. Transl. Ser. 2, 217, Amer. Math. Soc., Providence, RI, 2006.
[26] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, New York, 1978.
[27] D. Ruelle, *An extension of the theory of Fredholm determinants*, Publ. Math. Inst. Hautes Études Sci. 72 (1990), 175-193.
[28] J. Rousseau-Egele, *Un théorème de la limite locale pour une classe de transformations dilatantes et monotones par morceaux*, Ann. Probab. 11 (1983), 772-788.
[29] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) 20 (1956), 47-87.
[30] C. Series, *Symbolic dynamics for geodesic flows*, Acta Math. 146 (1981), no. 1-2, 103-128.
[31] L. Stoyanov, *Spectra of Ruelle transfer operators for Axiom A flows*, Nonlinearity 24 (2011), 1089-1120.
[32] L. Stoyanov, *Regular decay of ball diameters and spectra of Ruelle operators for contact Anosov flows*, Proc. Amer. Math. Soc. 140 (2012), 3463-3478.
[33] A. Vesnin, *Three-dimensional hyperbolic manifolds with a common fundamental polyhedron*, Math. Notes 49 (1991), 575-577.

Department of Mathematics, University of Warwick, Coventry CV4 7AL, U.K.
E-mail address: mpollic@maths.warwick.ac.uk

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, U.K.
Current address: Department of Mathematics, University of Warwick, Coventry CV4 7AL, U.K.
E-mail address: R.J.Sharp@warwick.ac.uk