GUREVIČ PRESSURE AND EQUIDISTRIBUTION FOR AMENABLE EXTENSIONS OF COUNTABLE STATE MARKOV SHIFTS

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Abstract. We obtain a weighted equidistribution theorem for amenable skew product extensions of countable state Markov shifts satisfying the BIP property. We also show, without requiring the BIP property, that Gurevič pressure for an amenable skew product agrees with the Gurevič pressure for the abelianized system. This had been proved by Dougall and Sharp in the case where the base is a subshift of finite type. The equality of Gurevič pressures is a key part of the proof of the equidistribution result.

1. Introduction

It is well-known that, for mixing subshifts of finite type $\sigma : \Sigma \to \Sigma$ (and hyperbolic systems more generally), weighted averages of periodic points become equidistributed with respect to the Gibbs measure for the weighting potential, and this continues to hold in the countable state case provided the so-called *big images and pre-images* (BIP) property is satisfied. Precisely, if δ_x is the Dirac measure at x and $\tau_{x,n} = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\sigma^j x}$ then, for suitably regular $\varphi : \Sigma \to \mathbb{R}$, and writing

$$\varphi^n := \varphi + \varphi \circ \sigma + \cdots \varphi \circ \sigma^{n-1},$$

the sequence of probability measures

$$\left(\sum_{\sigma^n x = x} e^{\varphi^n(x)}\right)^{-1} \sum_{\sigma^n x = x} e^{\varphi^n(x)} \tau_{x,n}$$

converges weakly to μ_φ, the Gibbs measure for φ, as $n \to \infty$. (For subshifts of finite type, see Bowen [2] when $\varphi = 0$ and Parry [25], Parry and Pollicott [26] for general Hölder continuous φ. The result in [25] and [26] is given for flows but the shift case follows directly by considering a constant suspension. For the countable state case with BIP, the equidistribution result can be obtained from the stronger large deviation results in Takahasi [35].) However, more interesting phenomena occur if we consider extensions of Σ by a (finitely generated, infinite) countable group G and only consider those periodic points for $\sigma : \Sigma \to \Sigma$ which correspond to periodic points in the extension. More precisely, given $\psi : \Sigma \to G$, define the skew product extension

$T_\psi : \Sigma \times G \to \Sigma \times G$ by

$$T_\psi(x,g) = (\sigma x, g\psi(x)).$$

Then we consider points $x \in \Sigma$ such that $T_\psi^n(x,e) = (x,e)$, i.e. $\sigma^n x = x$ and $\psi_n(x) = e$, where e is the identity in G and

$$\psi_n(x) := \psi(x)\psi(\sigma x)\cdots\psi(\sigma^{n-1}x).$$

(Note that $T_\psi^n(x,e) = (x,e)$ if and only if $T_\psi^n(x,g) = (x,g)$ for all $g \in G$.) Given a (sufficiently nice) function $\varphi : \Sigma \to \mathbb{R}$, we might seek to understand the limiting behaviour of the measures

$$\mathfrak{M}_n := \left(\sum_{\substack{\sigma^n x = x \\ \psi_n(x)=e}} e^{\varphi^n(x)} \right)^{-1} \sum_{\substack{\sigma^n x = x \\ \psi_n(x)=e}} e^{\varphi^n(x)} \tau_{x,n}$$

This problem was studied when Σ is a subshift of finite type and $G = \mathbb{Z}^d$, in which case, subject to a mixing condition on the extension, the weak limit exists and is equal to the Gibbs measure for $\varphi + \langle \xi, \psi \rangle$, for some (unique) $\xi = \xi(\varphi) \in \mathbb{R}^d$ (where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^d$). This value ξ is characterised by minimizing the pressure function $P(\varphi + \langle w, \psi \rangle, \sigma)$ over $w \in \mathbb{R}^d$, or by the variational principle

$$h(\mu_{\varphi+\langle \xi,\psi\rangle}) = \sup \left\{ h(\nu) + \int \varphi \, d\nu \, : \, \nu \in \mathcal{M}_\sigma(\Sigma), \ \int \psi \, d\nu = 0 \right\},$$

where $\mathcal{M}_\sigma(\Sigma)$ is the set of σ-invariant Borel probability measures on Σ and $h(\nu)$ is the measure-theoretic entropy of ν with respect to σ. (In the new work discussed below, the existence of such a ξ will become an assumption: see Assumption (III) below.) In the technically more involved situation of Anosov flows and homology, the analogous convergence statement for $\varphi = 0$ was proved by Sharp [32] (see also Lalley [20] and Babillot and Ledrappier [1]) and for general Hölder continuous φ by Coles and Sharp [7], and it is easy to recover the shift case from the results in these papers. It is now apparent that a version of this result should hold whenever G is amenable: again a result for Anosov flows was proved by Dougall and Sharp [13] and one can adapt this to subshifts of finite type (indeed, the analysis in [13] proceeds via subshifts of finite type and symbolic dynamics to obtain the flow result). In the amenable case, the limit measure is exactly the same as one would obtain if one abelianized the group G. More precisely, if $\overline{G} \cong \mathbb{Z}^d$ is the torsion-free part of the abelianization of G and $\pi : G \to \overline{G}$ is the projection homomorphism, then the limiting measure is $\mu_{\varphi+\langle \xi,\bar{\psi}\rangle}$, where $\bar{\psi} : \Sigma \to \overline{G}$ is defined by $\bar{\psi} = \pi \circ \psi$ and $\xi = \xi(\varphi)$ is as above. (If $d = 0$ then $\xi = 0$ and one simply obtains μ_φ in the limit.)

Our first main result is to extend this equidistribution for amenable skew products to mixing countable state Markov shifts satisfying BIP. We say that $\sigma : \Sigma \to \Sigma$ satisfies the *big images and pre-images* (BIP) property if there exist $b_1, \dots, b_N \in S$ such that, for all $a \in S$, there exist $i, j \in \{1, \dots, N\}$ with $A(b_i, a) = 1$ and $A(a, b_j) = 1$. Note that topologically transitive subshifts of finite type trivially satisfy this condition and, indeed, BIP shifts share many thermodynamic properties with subshifts of finite type, in particular the existence of Gibbs measures. We assume that:

- $\sigma : \Sigma \to \Sigma$ satisfies BIP;
- G is finitely generated amenable group;
- $\psi : \Sigma \to G$ satisfies $\psi(x) = \psi(x_1)$; and
- $\varphi : \Sigma \to \mathbb{R}$ is locally Hölder continuous

Furthermore, we make additional assumptions on φ and ψ.

Assumption (I): $T_\psi : \Sigma \times G \to \Sigma \times G$ is topologically mixing.

Let
$$\delta(\varphi, \bar{\psi}) := \sup\{r \geq 0 : \varphi + r|\bar{\psi}| \text{ is summable}\},$$
where $|\cdot|$ denotes the 2-norm on $\mathbb{R}^d$.

Assumption (II): $\delta(\varphi, \bar{\psi}) > 0$.

We define $\mathfrak{p} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ by
$$\mathfrak{p}(w) := P_{\mathrm{G}}(\varphi + \langle w, \bar{\psi} \rangle, \sigma),$$
where $P_{\mathrm{G}}(\cdot, \sigma)$ is Gurevič pressure with respect to σ (defined in section 2). In particular, $\mathfrak{p}$ is finite for $|w| < \delta(\varphi, \bar{\psi})$. Write $\delta = \delta(\varphi, \bar{\psi})$ and
$$B(\delta) = \{w \in \mathbb{R}^d : |w| < \delta\}$$

Assumption (III). The function $w \mapsto \mathfrak{p}(w)$ has a unique minimum at some (necessarily unique) $\xi \in B(\delta)$.

For $x \in \Sigma$, let δ_x denote the Dirac measure at x and write
$$\tau_{x,n} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma^j x}.$$

For a sequence of countable sets $\Lambda_n \subset \Sigma$, we write
$$\Pi_\varphi(\Lambda_n) = \sum_{x \in \Lambda_n} e^{\varphi^n(x)}$$
and introduce a sequence of probability measures on Σ,
$$\mathfrak{M}_\varphi(\Lambda_n) = \frac{1}{\Pi_\varphi(\Lambda_n)} \sum_{x \in \Lambda_n} e^{\varphi^n(x)} \tau_{x,n}.$$

We consider the following choices for Λ_n:

(1)
$$\Lambda^\psi(n) = \{x \in \Sigma : \sigma^n x = x,\ \psi_n(x) = 0\};$$

(2) for $a \in S$,
$$\Lambda^\psi_a(n) = \{x \in [a] : \sigma^n x = x,\ \psi_n(x) = 0\};$$

(3) for $o \in \Sigma$,
$$\Lambda^\psi(n,o) = \{x \in \Sigma : \sigma^n x = o,\ \psi_n(x) = e\};$$
(4) for $a \in S$ and $o \in [a]$,
$$\Lambda_a^\psi(n,o) = \{x \in [a] : \sigma^n x = o,\ \psi_n(x) = e\}.$$

Theorem 1.1. *Let $\sigma : \Sigma \to \Sigma$ be a countable state Markov shift which satisfies BIP and let G be a finitely generated group. Let $\varphi : \Sigma \to \mathbb{R}$ be locally Hölder continuous and let $\psi : \Sigma \to G$ satisfy $\psi(x) = \psi(x_1)$, such that φ and ψ satisfy Assumptions (I), (II) and (III). If G is amenable then $\mathfrak{M}_\varphi(\Lambda^\psi(n))$, $\mathfrak{M}_\varphi(\Lambda_a^\psi(n))$, $\mathfrak{M}_\varphi(\Lambda^\psi(n,o))$ and $\mathfrak{M}_\varphi(\Lambda_a^\psi(n,o))$ all converge to $\mu_{\varphi+\langle\xi,\bar\psi\rangle}$, as $n \to \infty$, with respect to the weak* topology on $\mathcal{M}(\Sigma)$.*

Remark 1.2. Assumption (II) is the assertion that
$$\sup_{x\in\Sigma} \sum_{\sigma y = x} e^{\varphi(y)+r|\bar\psi(y)|} \tag{1.1}$$
is finite for some $r > 0$. This is somewhat analogous to Cramér's condition for random walks (existence of an exponential moment) or to the entropy gap assumption that appears in [3] and [17], for example. Of course, for a subshift of finite type, (1.1) is automatically finite for all $r > 0$.

The equidistribution result is proved via large deviations, and to prove the latter, we need to show that, when G is amenable, the Gurevič pressure of φ with respect to the skew product T_ψ is equal to the Gurevič pressure of φ with respect to the abelianized skew product $T_{\bar\psi}$ (which, in turn, is equal to the Gurevič pressure of $\varphi + \langle\xi,\bar\psi\rangle$ with respect to σ). The largest part of the paper is devoted to proving the first equality, formally stated as Theorem 1.3 below, To motivate this, we briefly outline some older results for random walks, spectral geometry and group theory, before returning to dynamical questions.

A classical result of Kesten for symmetric random walks on countable groups is that the exponential rate of return to the identity is equal to 1 if the group is amenable and is less than 1 otherwise [19]. Variants of this amenability dichotomy were subsequently obtained in the settings of the spectrum of the Laplacian for coverings of Riemannian manifolds [4], [5], group theory and graph theory [6], [16], [23], [24] and critical exponents for discrete groups [8], [9], [11], [12], [28], [33]. The latter results also have a dynamical interpretation in terms of geodesic flows (over manifolds with negative sectional curvatures) and, more generally, Anosov flows with a time-reversing involution. In this setting, with time-reversal symmetry, one compares the growth rate of the number of periodic orbits of the flow (which is equal to the topological entropy) with the corresponding quantity for the lift of the flow to a regular cover: the growth rates agree if and only if the covering group is amenable. For general Anosov flows, without this additional symmetry, the amenability dichotomy fails (even for abelian covers). However, in [13], Dougall and the author showed that a modified result holds, where the comparison is between growth rates for the given cover and the maximal abelian subcover: these two quantities agree if and only if the covering group is amenable. The argument in [13] (with some errors rectified in the associated correction) used symbolic dynamics, where the Anosov flow is coded by a suspension flow over a subshift of finite type. In particular, it boils down to showing equality of Gurevič pressure for a skew product extension associated to the covering group with the corresponding quantity for its abelianization. Here we give a new proof, modelled on the argument given for random walks in [14], which works in the countable state case and which gives a cleaner proof. (Both proofs have their roots in ideas of Roblin [28] and Stadlbauer [34].) We stress that, for the direction "amenability implies equality" stated in Theorem 1.3 below, the shift $\sigma : \Sigma \to \Sigma$ does not need to satisfy BIP. (The recent paper [15] obtains the conclusion of Theorem 1.3 when the base is a full branch Gibbs–Markov map satisfying BIP.)

We now state this result in detail (although some definitions will be deferred until section 2). Given a locally Hölder continuous function $\varphi : \Sigma \to \mathbb{R}$, we have induced functions $\tilde{\varphi} : \Sigma \times G \to \mathbb{R}$ and $\bar{\varphi} : \Sigma \times \overline{G} \to \mathbb{R}$, defined by $\tilde{\varphi}(x, \cdot) = \varphi(x)$ and $\bar{\varphi}(x, \cdot) = \varphi(x)$, respectively. To avoid over-cluttering formulae, we will slightly abuse notation by denoting all three functions φ, $\bar{\varphi}$ and $\tilde{\varphi}$ by φ. We have the following inequalities for the (Gurevič) pressure of these functions:

$$P_{\mathrm{G}}(\varphi, T_\psi) \leq P_{\mathrm{G}}(\varphi, T_{\bar{\psi}}) \leq P_{\mathrm{G}}(\varphi + \langle w, \bar{\psi} \rangle, \sigma), \tag{1.2}$$

where the second inequality holds for all $w \in \mathbb{R}^d$. (See Lemma 3.1 for the proof of these inequalities.)

Theorem 1.3. *Suppose that*

- *$\sigma : \Sigma \to \Sigma$ is a (one-sided) countable state Markov shift,*
- *G is a countable group,*
- *$T_\psi : \Sigma \times G \to \Sigma \times G$ is a topologically transitive skew-product extension, where $\psi((x_i)_{i=1}^\infty) = \psi(x_1)$, and*
- *$\varphi : \Sigma \to \mathbb{R}$ is a locally Hölder continuous function.*

If G is amenable then $P_{\mathrm{G}}(\varphi, T_\psi) = P_{\mathrm{G}}(\varphi, T_{\bar{\psi}})$.

Remark 1.4. Since $T_\psi : \Sigma \times G \to \Sigma \times G$ is topologically transitive, $\sigma : \Sigma \to \Sigma$ is also topologically transitive, so we do not need to include this as a separate assumption on the base.

If Σ satisfies BIP and φ and ψ satisfy the same conditions needed for Theorem 1.1, then the converse of statement also holds.

Proposition 1.5. *Let $\sigma : \Sigma \to \Sigma$ be a countable state Markov shift which satisfies BIP and let G be a finitely generated group. Let $\varphi : \Sigma \to \mathbb{R}$ be locally Hölder continuous and let $\psi : \Sigma \to G$ satisfy $\psi(x) = \psi(x_1)$, such that*

φ and ψ satisfy Assumptions (I), (II) and (III). If $P_{\mathrm{G}}(\varphi, T_\psi) = P_{\mathrm{G}}(\varphi, T_{\bar{\psi}})$ then G is amenable.

We remark that includes the case of subshifts of finite type (where the moment condition is automatic). The proof, which relies heavily on results of Stadlbauer [33] is given in section 7.

We will now outline the contents of the rest of the paper. Section 2 and section 3 provide background on countable state Markov shifts, Gurevič pressure and skew product extensions. Section 4 contains the proof of Theorem 1.3. Section 5 considers the BIP property, Gibbs measures and makes the assumptions we impose for our equidistribution results, and section 6 contains the statement and proof of these results. Finally, section 7 establishes Proposition 1.5.

2. Countable state Markov shifts

Let S be a countable set and let A be a matrix indexed by $S \times S$ with entries in $\{0, 1\}$. We call S the set of states and A the transition matrix. We write $A(s, s')$ for the (s, s') entry of A. We then let

$$\Sigma = \{x = (x_n)_{n=0}^{\infty} : x_n \in S \text{ and } A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}^+\}.$$

Let $\sigma : \Sigma \to \Sigma$ be the shift map defined by $(\sigma x)_n = x_{n+1}$. Then we call (Σ, σ) a (one-sided) countable state Markov shift. If S is finite, then (Σ, σ) is called a (one-sided) subshift of finite type.

A word $w = (w_1, \dots, w_n) \in S^n$ is called *allowed* if $A(w_i, w_{i+1}) = 1$ for all $i = 1, \dots, n-1$ and we call $[w] = \{x \in \Sigma : x_i = w_i, \ \forall i = 1, \dots, n\}$ the cylinder defined by w. Let $\mathcal{W}_n \subset S^n$ denote the set of all allowed words of length n and let $\mathcal{W} := \bigcup_{n=1}^{\infty} \mathcal{W}_n$. For $a \in S$, we write

$$\mathcal{W}_n(a) := \{w = (w_1, \dots, w_n) \in \mathcal{W}_n : w_1 = a \text{ and } A(w_n, a) = 1\}.$$

If w is a finite word and x is either a finite word or element of Σ then we let wx denote the concatenation.

The topology on Σ is the topology generated by cylinder sets. The shift map is continuous and we have that Σ is compact if and only if S is finite. We say that $\sigma : \Sigma \to \Sigma$ is *topologically transitive* if there is a point with a dense orbit. We say that $\sigma : \Sigma \to \Sigma$ is *topologically mixing* if for all $a, b \in S$ there exists $N(a, b)$ such that for all $n \geq N(a, b)$ there exists $w \in \mathcal{W}_n$ with $awb \in \mathcal{W}$. (This is equivalent to the usual definition of topological mixing in topological dynamical systems.)

We say that $\varphi : \Sigma \to \mathbb{R}$ is *locally Hölder continuous* if

$$\sup_{w \in \mathcal{W}_n} \sup_{x,y \in [w]} |\varphi(x) - \varphi(y)| \leq C\theta^n,$$

for some $C > 0$ and $0 < \theta < 1$, for all $n \geq 1$. If we wish to specify θ then we say that φ is θ-locally Hölder continuous. We write

$$\varphi^n(x) := \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1} x).$$

A locally Hölder function satisfies the following estimates.

Lemma 2.1 (Lemma 1 of [29])**.** *Let $\varphi : \Sigma \to \mathbb{R}$ be locally Hölder continuous. Then, for all $a \in S$ and all $o, o' \in [a]$ there exists $B > 0$ such that*

$$|\varphi^n(wo) - \varphi^n(wo')| \leq B$$

for all $w \in \mathcal{W}_n$ such that $wa \in \mathcal{W}_{n+1}$.

Assume that σ is topologically transitive and that $\varphi : \Sigma \to \mathbb{R}$ is locally Hölder continuous. Then we can define the *Gurevič pressure* $P_{\mathrm{G}}(\varphi, \sigma) \in \mathbb{R} \cup \{\infty\}$ by

$$P_{\mathrm{G}}(\varphi, \sigma) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\sigma^n x = x} e^{\varphi^n(x)} \mathbb{1}_{[a]}(x),$$

for any choice of $a \in S$ [29]. If σ is topologically mixing, then the limsup may be replaced by a limit.

The transfer operator L_φ acts on real-valued function on Σ by the formula

$$(L_\varphi v)(x) = \sum_{\sigma y = x} e^{\varphi(y)} v(y)$$

(which may be infinite). We say that φ is *summable* if $\|L_\varphi 1\|_\infty < \infty$. For later use, we note that, using the local Hölder continuity of φ, this is equivalent to the condition

$$\sum_{a \in S} \exp\left(\sup\{\varphi(y) \,:\, y \in [a]\}\right) < \infty$$

used in [18], [21] and [22]. If φ is summable then $P_{\mathrm{G}}(\varphi, \sigma) < \infty$.

We may also characterize the Gurevič pressure directly in terms of the transfer operator.

Lemma 2.2 (Theorem 4.4 of [31])**.** *If $P_{\mathrm{G}}(\varphi, \sigma) < \infty$ and $v : \Sigma \to \mathbb{R}$ is continuous, non-negative, not identically zero and with support contained inside a finite union of cylinders then*

$$\limsup_{n \to \infty} \frac{1}{n} \log(L_\varphi^n v)(x) = P_{\mathrm{G}}(\varphi, \sigma)$$

for all $x \in \Sigma$. If the shift is topologically mixing then the limsup may be replaced by a limit.

3. Skew product extension

We now consider skew product extensions. Let G be a finitely generated countable group (with the discrete topology), with identity e. Suppose we are given a function $\psi : \Sigma \to G$ with $\psi((x_i)_{i=1}^\infty) = \psi(x_1)$. (Any continuous function from Σ to G is locally constant and can be reduced to this form by recoding.) Then we can define the skew product extension

$$T_\psi : \Sigma \times G \to \Sigma \times G$$

by

$$T_\psi(x, g) = (\sigma x, g\psi(x)).$$

Note that
$$T_\psi^n(x,g) = (\sigma^n x, g\psi_n(x)),$$
where
$$\psi_n(x) = \psi(x)\psi(\sigma x)\cdots\psi(\sigma^{n-1}x).$$
We will always assume that T_ψ is topologically transitive.

The system $T_\psi : \Sigma \times G \to \Sigma \times G$ is also a countable state Markov shift with set of states $S \times G$ and a transition matrix $\mathbb{A}$ defined by
$$\mathbb{A}((i,g),(j,h)) = 1 \quad \Longleftrightarrow \quad A(i,j) = 1 \text{ and } g\psi(i) = h.$$
Thus, Gurevič pressure is also defined for the extension.

Given an extension with group G, we will also need to consider the abelianized system as follows. Let G^{ab} denote the abelianization of G and let $\overline{G}$ denote the torsion-free part of G^{ab}, i.e. $\overline{G}$ is the quotient of G^{ab} by its torsion subgroup. Then $\overline{G}$ is isomorphic to $\mathbb{Z}^d$, for some $d \geq 0$. We use additive notation and let 0 to denote the identity. Let $\pi : G \to \overline{G}$ be the projection homomorphism. Define $\bar\psi : \Sigma \to \overline{G}$ by $\bar\psi = \pi \circ \psi$; then we have the skew product $T_{\bar\psi} : \Sigma \times \overline{G} \to \Sigma \times \overline{G}$. (If $d = 0$ then we can regard $T_{\bar\psi}$ as σ.)

Given $\varphi : \Sigma \to \mathbb{R}$, define $\tilde\varphi : \Sigma \times G \to \mathbb{R}$ by $\tilde\varphi(x,g) = \varphi(x)$ and $\bar\varphi : \Sigma \times \overline{G} \to \mathbb{R}$ by $\bar\varphi(x,k) = \varphi(x)$. Following the convention in the introduction, we write all of these functions as φ. The next result proves the inequalities (1.2) in the introduction.

Lemma 3.1. *For all $w \in \mathbb{R}^d$, w have*
$$P_{\mathrm{G}}(\varphi, T_\psi) \leq P_{\mathrm{G}}(\varphi, T_{\bar\psi}) \leq P_{\mathrm{G}}(\varphi + \langle w, \bar\psi\rangle, \sigma).$$

Proof. Let $a \in S$, then $[a] \times \{e\}$ is a cylinder set for $\Sigma \times G$ and $[a] \times \{0\}$ is a cylinder set for $\Sigma \times \overline{G}$. It is clear that $T_\psi^n(x,e) = (x,e)$ implies $T_{\bar\psi}^n(x,0) = (x,0)$ and therefore
$$\sum_{T_\psi^n(x,e)=(x,e)} e^{\varphi^n(x,e)} \mathbb{1}_{[a]\times\{e\}}(x,e) \leq \sum_{T_\psi^n(x,0)=(x,0)} e^{\varphi^n(x,0)} \mathbb{1}_{[a]\times\{0\}}(x,0).$$
Hence, $P_{\mathrm{G}}(\varphi, T_\psi) \leq P_{\mathrm{G}}(\varphi, T_{\bar\psi})$. By a similar argument, $P_{\mathrm{G}}(\varphi, T_{\bar\psi}) \leq P_{\mathrm{G}}(\varphi, \sigma)$. Furthermore, if $T_{\bar\psi}^n(x,0) = (x,0)$ then $\sigma^n x = x$ and $\bar\psi^n(x) = 0$, so we see that
$$P_{\mathrm{G}}(\varphi, T_{\bar\psi}) = P_{\mathrm{G}}(\varphi + \langle w, \bar\psi\rangle, T_{\bar\psi}),$$
for all $w \in \mathbb{R}^d$. $\square$

Each of the skew products has its own transfer operator (associated to φ), which we distinguish by using different fonts, as follows:
$$(\mathcal{L}_\varphi v)(x,g) := \sum_{T_\psi(y,h)=(x,g)} e^{\varphi(y)} v(y,h) = \sum_{\sigma y = x} e^{\varphi(y)} v(y, g\psi(y)^{-1})$$

and

$$(\mathfrak{L}_\varphi v)(x,m) := \sum_{T_{\bar\psi}(y,k)=(x,m)} e^{\varphi(y)} v(y,k) = \sum_{\sigma y = x} e^{\varphi(y)} v(y, m - \bar\psi(y))$$

The formulae for iterates are

$$(\mathcal{L}_\varphi^n v)(x,g) = \sum_{T_\psi^n(y,h)=(x,g)} e^{\varphi^n(y)} v(y,h) = \sum_{\sigma^n y = x} e^{\varphi^n(y)} v(y, g\psi_n(y)^{-1})$$

and

$$(\mathfrak{L}_\varphi^n v)(x,m) = \sum_{T_{\bar\psi}^n(z,k)=(x,m)} e^{\varphi(y)} v(z,k) = \sum_{\sigma y = x} e^{\varphi(y)} v(y, m - \bar\psi_n(y)).$$

4. Amenability implies equality

In the section we prove Theorem 1.3. Since we know from (1.2) that $P_{\mathrm{G}}(\varphi, T_\psi) \le P_{\mathrm{G}}(\varphi, T_{\bar\psi})$, we only need to prove that $P_{\mathrm{G}}(\varphi, T_{\bar\psi}) \le P_{\mathrm{G}}(\varphi, T_\psi)$. To simplify the notation, write shall $\rho = e^{P_{\mathrm{G}}(\varphi, T_\psi)}$.

We begin by outlining the strategy employed in the proof. We shall work with the operator $\mathcal{L}_\varphi$ but we only need to consider functions defined on a countable set $\mathcal{Z} \times G$, where $\mathcal{Z}$ is the set of all pre-images (under σ) of a prescribed "base point". The first task is to construct a bounded positive function $\Phi : \mathcal{Z} \times G \to \mathbb{R}^{>0}$ from limits of ratios of what one might think of as "Green's functions" $\zeta_{z,g}(t)$, with a parameter decreasing to ρ. This is carried out in Lemma 4.2 and Lemma 4.3. Then, in Lemma 4.4, we show that $\mathcal{L}_\varphi \Phi = \rho\Phi$. The next step, in Lemma 4.6, is to replace Φ with a function $\phi : \mathcal{Z} \times G \to \mathbb{R}^{>0}$ such that $\mathcal{L}_\varphi^n \phi \le \rho^n \phi$ and $\phi(z,\cdot)/\phi(z,e)$ is always homomorphism from G to $\mathbb{R}^{>0}$. This is the only point in the proof where the amenability of G is used: we consider an invariant mean on $\ell^\infty(G,\mathbb{R})$ applied to a bounded sequence derived from Φ, along with Jensen's inequality. The proof of Theorem 1.3 is then nearly complete. In Lemma 4.7, using the homomorphism property, ϕ descends to a function $\bar\phi : \mathcal{Z} \times \overline{G} \to \mathbb{R}^{>0}$ satisfying $\mathfrak{L}_\varphi^n \bar\phi \le \rho^n \bar\phi$. Finally, Lemma 2.2 relates $P_{\mathrm{G}}(\varphi, T_{\bar\psi})$ to the growth rate of the iterates of $\mathfrak{L}_\varphi$ and hence we obtain $P_{\mathrm{G}}(\varphi, T_{\bar\psi}) \le \log \rho$.

Fix $a \in S$; for $o \in [a]$ and $g \in G$, we will work with the quantity

$$\sum_{\substack{\sigma^n x = o \\ x_1 = a \\ \psi_n(x) = g}} e^{\varphi^n(x)} = \sum_{\sigma^n y = o} e^{\varphi^n(y)} \mathbb{1}_{[a]\times\{e\}}(x, g\psi_n(x)^{-1}) = (\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o,g).$$

Since T_ψ is topologically transitive, it follows from Lemma 2.2 that

$$\limsup_{n\to\infty} \frac{1}{n} \log(\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o,g) = P_{\mathrm{G}}(\varphi, T_\psi). \tag{4.1}$$

Define

$$\eta_{o,g}(t) = \sum_{n=1}^\infty t^{-n} (\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o,g).$$

This converges for $t > \rho$ but may either diverge or converge at $t = \rho$. In the latter case, we need to modify the series so that it diverges at $t = \rho$. We use the following standard lemma.

Lemma 4.1 (Lemma 3.2 of [10]). *Let a_n be a sequence of positive real numbers such that $\limsup_{n\to\infty} a_n^{1/n} = \rho$. Then there is a sequence b_n of positive real numbers such that $\lim_{n\to\infty} b_{n+1}/b_n = 1$, $\sum_{n=1}^{\infty} a_n b_n t^{-n}$ converges for all $t > \rho$, but $\sum_{n=1}^{\infty} a_n b_n \rho^{-n}$ diverges.*

Let b_n be the sequence given by the lemma corresponding to

$$a_n = (\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o, e)$$

and write $c_n = 1/b_n$. Let

$$\zeta_{o,g}(t) = \sum_{n=1}^{\infty} t^{-n} \frac{1}{c_n} (\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o, g).$$

If $o' \in [a]$ then there is a natural bijection between the pre-images of o and o', so we can compare $\zeta_{o,g}(t)$ and $\zeta_{o',g}(t)$ term-by-term. Using Lemma 2.1, for all $g \in G$ and $t > \rho$,

$$e^{-B}\zeta_{o,g}(t) \le \zeta_{o',g}(t) \le e^{B}\zeta_{o,g}(t) \tag{4.2}$$

and so the same sequence c_n works for all $o \in [a]$.

Lemma 4.2. *Let $\rho' > \rho$. For each $a \in S$ and each $g \in G$,*

$$0 < \inf_{o\in[a]} \inf_{\rho<t\le\rho'} \frac{\zeta_{o,g}(t)}{\zeta_{o,e}(t)} \le \sup_{o\in[a]} \sup_{\rho<t\le\rho'} \frac{\zeta_{o,g}(t)}{\zeta_{o,e}(t)} < \infty.$$

Proof. We begin by observing that, for every $g, h \in G$, we have

$$\begin{aligned}(\mathcal{L}_\varphi^{n+k} \mathbb{1}_{[a]\times\{e\}})(o, g) &\ge \sum_{\substack{\sigma^k z = o \\ z_1 = a \\ \psi_k(z) = h^{-1}g}} e^{\varphi^k(z)} \sum_{\substack{\sigma^n x = z \\ x_1 = a \\ \psi_n(x) = h}} e^{\varphi^n(x)} \\ &= \sum_{\substack{\sigma^k z = o \\ z_1 = a \\ \psi_k(z) = h^{-1}g}} e^{\varphi^k(z)} [(\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(z, h)]\end{aligned}$$

and (due to the transitivity of T_ψ) we can find a $k \geq 1$ so that the right hand side is positive. Thus we obtain

$$\begin{aligned}\zeta_{o,g}(t) &= \sum_{m=1}^{k} \frac{t^{-n}}{c_n} (\mathcal{L}_\varphi^m \mathbb{1}_{[a]\times\{e\}})(o,g) + \sum_{n=1}^{\infty} \frac{t^{-(n+k)}}{c_{n+k}} (\mathcal{L}_\varphi^{n+k} \mathbb{1}_{[a]\times\{e\}})(o,g) \\ &\geq \sum_{m=1}^{k} \frac{t^{-n}}{c_n} (\mathcal{L}_\varphi^m \mathbb{1}_{[a]\times\{e\}})(o,g) \\ &+ t^{-k} \sum_{\substack{\sigma^k z = o \\ z_1 = a \\ \psi_k(z) = h^{-1}g}} e^{\varphi^k(z)} \sum_{n=1}^{\infty} \frac{t^{-n}}{c_n} \frac{c_n}{c_{n+k}} [(\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(z,h)].\end{aligned}$$

Since the numbers c_n are positive and, for each fixed k, $\lim_{n\to\infty} c_n/c_{n+k} = 1$, we have $\inf_{n\geq 1} c_n/c_{n+k} > 0$. Furthermore,

$$\inf_{o\in[a]} \sum_{\substack{\sigma^k z = o \\ z_1 = a \\ \psi_k(z) = h^{-1}g}} e^{\varphi^k(z)} > 0.$$

Hence, for $t \in (\rho, \rho']$, we have

$$\zeta_{o,g}(t) \geq C_1(g,k,a) + \frac{C_2(g,h,k,a)}{C_3(k)} \zeta_{o,h}(t),$$

for positive C_1, C_2 and C_3. We conclude that

$$\inf_{\rho < t \leq \rho'} \frac{\zeta_{o,g}(t)}{\zeta_{o,h}(t)} > 0.$$

Since g and h are arbitrary, we have

$$0 < \inf_{\rho < t \leq \rho'} \frac{\zeta_{o,g}(t)}{\zeta_{o,e}(t)} \leq \sup_{\rho < t \leq \rho'} \frac{\zeta_{o,g}(t)}{\zeta_{o,e}(t)} < \infty.$$

Using (4.2), the bounds can be made uniform over $o \in [a]$. □

Define

$$\mathcal{Z} = \bigcup_{n=0}^{\infty} \{z \in \Sigma : \sigma^n z = o\}$$

the set of pre-images of o; this is a countable subset of Σ.

Lemma 4.3. *We can find a sequence $t_n \downarrow \rho$ such that the following limit exists and lies in $(0,\infty)$ for all $(z,g) \in \mathcal{Z} \times G$:*

$$\Phi(z,g) := \lim_{n\to\infty} \frac{\zeta_{z,g}(t_n)}{\zeta_{z,e}(t_n)}.$$

Furthermore, for all $g \in G$,

$$0 < \inf_{z\in\mathcal{Z}\cap[a]} \Phi(z,g) \leq \sup_{z\in\mathcal{Z}\cap[a]} \Phi(z,g) < \infty. \tag{4.3}$$

Proof. Let $\{(z_m,g_m)\}_{m=1}^\infty$ be an enumeration of $\mathcal{Z}\times G$ and let $t_n^{(0)}$ be an arbitrary sequence in $(\rho,\rho']$ that is strictly decreasing to ρ. Then the numbers $\zeta_{z_1,g_1}(t_n^{(0)})/\zeta_{z_1,e}(t_n^{(0)})$ are a sequence in the compact interval

$$\left[\inf_{o\in[a]}\inf_{\rho<t\le\rho'}\frac{\zeta_{o,g}(t)}{\zeta_{o,e}(t)},\sup_{o\in[a]}\sup_{\rho<t\le\rho'}\frac{\zeta_{o,g}(t)}{\zeta_{o,e}(t)}\right].$$

Hence, $t_n^{(0)}$ has a subsequence $t_n^{(1)}$ such that $\zeta_{z_1,g_1}(t_n^{(1)})/\zeta_{z_1,e}(t_n^{(1)})$ converges. Continue inductively to construct successive subsequences $t_n^{(m)}$ of $t_n^{(m-1)}$ such that $\zeta_{z_m,g_m}(t_n^{(m)})/\zeta_{z_m,e}(t_n^{(m)})$ converges. Now take t_n to be the diagonal sequence $t_n^{(n)}$ and convegence will hold for all $(z,g)\in\mathcal{Z}\times G$. The inequalities (4.3) are immediate. $\square$

Lemma 4.4. *For all $z\in\mathcal{Z}\cap[a]$ and all $n\ge 1$,*

$$(\mathcal{L}_\varphi^n\Phi)(z,g)=\rho^n\Phi(z,g).$$

Proof. To simplify notation, we shall write the proof for the case $z=o$ and note that, using (4.2), the same argument holds for general $z\in\mathcal{Z}\cap[a]$. Fix $k\ge 1$ and let $\epsilon>0$. Since $\lim_{n\to\infty}c_{n-k}/c_n=1$, we can choose n_0 such that $1-\epsilon\le c_{n-k}/c_n\le 1+\epsilon$, for all $n\ge n_0$. Without loss of generality, we may take $n_0\ge k$, We will also use that

$$(\mathcal{L}_\varphi^n\mathbb{1}_{[a]\times\{e\}})(o,g)=\sum_{\sigma^kz=o}e^{\varphi^k(z)}\,(\mathcal{L}_\varphi^{n-k}\mathbb{1}_{[a]\times\{e\}})(z,g\psi_k(z)^{-1}).$$

Then, for $n\ge n_0$,

$$\begin{aligned}&\frac{1-\epsilon}{c_{n-k}}\sum_{\sigma^kz=o}e^{\varphi^k(z)}\,(\mathcal{L}_\varphi^{n-k}\mathbb{1}_{[a]\times\{e\}})(z,g\psi_k(z)^{-1})\le\frac{1}{c_n}(\mathcal{L}_\varphi^n\mathbb{1}_{[a]\times\{e\}})(o,g)\\ &\le\frac{1+\epsilon}{c_{n-k}}\sum_{\sigma^kz=o}e^{\varphi^k(z)}\,(\mathcal{L}_\varphi^{n-k}\mathbb{1}_{[a]\times\{e\}})(z,g\psi_k(z)^{-1}).\end{aligned}\tag{4.4}$$

Setting

$$C_1(g,t,n_0)=\sum_{n=1}^{n_0}\frac{t^{-n}}{c_n}(\mathcal{L}_\varphi^n\mathbb{1}_{[a]\times\{e\}})(o,g),$$

by (4.4), we have

$$\begin{aligned}&t^k\sum_{n=1}^\infty\frac{t^{-n}}{c_n}(\mathcal{L}_\varphi^n\mathbb{1}_{[a]\times\{e\}})(o,g)\le t^kC_1(g,t,n_0)\\ &+t^k(1+\epsilon)\sum_{\sigma^kz=o}e^{\varphi^k(z)}\sum_{n=n_0+1}^\infty\frac{t^{-n}}{c_{n-k}}(\mathcal{L}_\varphi^{n-k}\mathbb{1}_{[a]\times\{e\}})(z,g\psi_k(z)^{-1}).\end{aligned}$$

Replacing t by the sequence t_m from Lemma 4.3, dividing by $\zeta_{z,e}(t_m)$, we obtain

$$t_m^k \Phi(o,g) \le \frac{t_m^k C_1(g,t_m,n_0)}{\zeta_{z,e}(t_m)} + \frac{(1+\epsilon)}{\zeta_{z,e}(t_m)} \sum_{\sigma^k z = o} e^{\varphi^k(z)} \sum_{n=n_0+1}^{\infty} \frac{t_m^{-(n-k)}}{c_{n-k}} (\mathcal{L}_\varphi^{n-k} \mathbb{1}_{[a]\times\{e\}})(z, g\psi_k(z)^{-1}).$$

Then, letting $m \to \infty$, so that $t_m \to \rho$, this gives us

$$\rho^k \Phi(o,g) \le (1+\epsilon) \sum_{\sigma^k z = o} e^{\varphi^k(z)} \Phi(z, g\psi_k(z)^{-1})$$

and, since ϵ is arbitrary,

$$\rho^k \Phi(o,g) \le \sum_{\sigma^k z = o} e^{\varphi^k(z)} \Phi(z, g\psi_k(z)^{-1}).$$

For the lower bound, we have

$$\begin{aligned} & t^k \sum_{n=1}^{\infty} \frac{t^{-n}}{c_n} (\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o,g) \\ & \ge (1-\epsilon) \sum_{\sigma^k z = o} e^{\varphi^k(z)} \sum_{n=n_0+1}^{\infty} \frac{t^{-(n-k)}}{c_{n-k}} (\mathcal{L}_\varphi^{n-k} \mathbb{1}_{[a]\times\{e\}})(z, g\psi_k(z)^{-1}) \\ & = (1-\epsilon) \sum_{\sigma^k z = o} e^{\varphi^k(z)} \sum_{n=n_0-k+1}^{\infty} \frac{t^{-n}}{c_n} (\mathcal{L}_\varphi^{n} \mathbb{1}_{[a]\times\{e\}})(z, g\psi_k(z)^{-1}), \end{aligned}$$

which gives

$$\rho^k \Phi(o,g) \ge (\mathcal{L}_\varphi^k \Phi)(o,g).$$

□

Corollary 4.5. *For all $(z,\gamma) \in \mathcal{Z} \times G$, we have*

$$0 < \inf_{g\in G} \frac{\Phi(z, g\gamma^{-1})}{\Phi(o,g)} \le \sup_{g\in G} \frac{\Phi(z, g\gamma^{-1})}{\Phi(o,g)} < \infty.$$

Proof. Here we again use the topological transitivity of T_ψ. Given $(z,\gamma) \in \mathcal{Z} \times G$, we can find $k \ge 1$ such that $\sigma^k z = o' \in [a]$ and $\psi_k(z) = \gamma$. This gives us

$$\begin{aligned} e^{\varphi^k(z)} \Phi(z, g\gamma^{-1}) &\le \sum_{\sigma^k y = o'} e^{\varphi^k(y)} \Phi(y, g\psi_k(y)^{-1}) \\ &= \rho^k \Phi(o', g) \le e^{2B} \rho^k \Phi(o,g). \end{aligned}$$

Hence, $\sup_{g\in G} \Phi(z, g\gamma^{-1})/\Phi(o,g)$ is finite.

Next we find $x \in [a]$ and $k \geq 1$ such that $\sigma^k x = z$ and $\psi_k(x) = \gamma^{-1}$. Then

$$\begin{aligned} e^{-2B} \epsilon^{\varphi^k(x)} \Phi(o,g) &\leq e^{\varphi^k(x)} \Phi(x,g) \\ &\leq \sum_{\sigma^k y = o'} e^{\varphi^k(y)} \Phi(y, g\gamma^{-1}\psi_k(y)^{-1}) = \rho^k \Phi(z, g\gamma^{-1}). \end{aligned}$$

Hence, $\inf_{g \in G} \Phi(z, g\gamma^{-1})/\Phi(o,g)$ is positive. □

We are now able to use the amenability of G to replace Φ with a function $\phi : \mathcal{Z} \times G \to \mathbb{R}^{>0}$ which, after normalization, is a homomorphism in the second factor. This will allow us to descend to $\overline{G}$.

Lemma 4.6. *There is a function $\phi : \mathcal{Z} \times G \to \mathbb{R}^{>0}$ such that*

(i) for all $g \in G$ and $n \geq 1$,

$$(\mathcal{L}_\varphi^n \phi)(o,g) \leq \rho^n \phi(o,g),$$

(ii) for all $z \in \mathcal{Z}$, the function

$$\gamma \mapsto \frac{\phi(z,\gamma)}{\phi(z,e)}$$

is a homomorphism from G to $\mathbb{R}^{>0}$.

Furthermore, for all $a \in S$ and all $\gamma \in G$, we have

$$\inf_{z \in \mathcal{Z} \cap [a]} \phi(z,\gamma) > 0.$$

Proof. The construction of the function ϕ uses the amenability of G. Let M be a right-invariant Banach mean on $\ell^\infty(G)$. By Jensen's inequality, if α is convex and $f \in \ell^\infty(G)$ then

$$M(\alpha(f)) \geq \alpha(M(f)).$$

We apply this to the functions

$$f_{z,\gamma}(g) = \log \frac{\Phi(z, g\gamma^{-1})}{\Phi(o,g)},$$

where $(z,\gamma) \in \mathcal{Z} \times G$. It follows from Corollary 4.5 that $f_{z,\gamma} \in \ell^\infty(G)$, for all $(z,\gamma) \in \mathcal{Z} \times G$. Thus, we have

$$M\left(g \mapsto \frac{\Phi(z, g\gamma^{-1})}{\Phi(o,g)}\right) = M\left(g \mapsto \exp\log \frac{\Phi(z, g\gamma^{-1})}{\Phi(o,g)}\right) \geq \exp M(f_{z,\gamma}).$$

Now define $\phi : \mathcal{Z} \times G \to \mathbb{R}^{>0}$ by

$$\phi(z,\gamma) = \exp M(f_{z,\gamma}).$$

We will first prove that the inequality in part (i) holds. We need a little care as M is only finitely additive. Thus, we work with finite subsets of G

that exhaust G. Let $\{g_k\}_{k=1}^{\infty}$ be an enumeration of G and, for any $N \geq 1$, let $G_N = \{g_1, \ldots, g_N\}$. Then Lemma 4.4 gives us

$$\begin{aligned}
\rho^n &= M\left(g \mapsto \frac{(\mathcal{L}_{\varphi}^n \Phi)(o,g)}{\Phi(o,g)}\right) = M\left(g \mapsto \frac{1}{\Phi(o,g)} \sum_{\sigma^n z = o} e^{\varphi^n(z)} \Phi(z, g\psi_n(z)^{-1})\right) \\
&\geq M\left(g \mapsto \frac{1}{\Phi(o,g)} \sum_{\substack{\sigma^n z = o \\ \psi_n(z) \in G_N}} e^{\varphi^n(z)} \Phi(z, g\psi_n(z)^{-1})\right) \\
&= \sum_{\substack{\sigma^n z = o \\ \psi_n(z) \in G_N}} e^{\varphi^n(z)} M\left(g \mapsto \frac{\Phi(z, g\psi_n(z)^{-1})}{\Phi(o,g)}\right) \\
&\geq \sum_{\substack{\sigma^n z = o \\ \psi_n(z) \in G_N}} e^{\varphi^n(z)} \exp M\left(g \mapsto \log \frac{\Phi(z, g\psi_n(z)^{-1})}{\Phi(o,g)}\right) \\
&= \sum_{\substack{\sigma^n z = o \\ \psi_n(z) \in G_N}} e^{\varphi^n(z)} \phi(z, g\psi_n(z)^{-1}).
\end{aligned}$$

Taking the supremum over N gives the inequality in (i).

Next we show that part (ii) holds. First note that we have

$$\begin{aligned}
\log \phi(o, \gamma\delta) &= M\left(g \mapsto \log \frac{\Phi(o, g\delta^{-1}\gamma^{-1})}{\Phi(o,g)}\right) \\
&= M\left(g \mapsto \log \frac{\Phi(o, g\delta^{-1}\gamma^{-1}}{\Phi(o, g\delta^{-1})}\right) + M\left(g \mapsto \log \frac{\Phi(o, g\delta^{-1})}{\Phi(o,g)}\right) \\
&= \phi(o,\gamma) + \phi(o,\delta)
\end{aligned}$$

and

$$\begin{aligned}
\log \phi(o, \gamma^{-1}) &= M\left(g \mapsto \log \frac{\Phi(o, g\gamma)}{\Phi(o,g)}\right) = M\left(g \mapsto \log \frac{\Phi(o,g)}{\Phi(o, g\gamma^{-1})}\right) \\
&= M\left(g \mapsto -\log \frac{\Phi(o, g\gamma^{-1})}{\Phi(o,g)}\right) = -\log \phi(o,\gamma).
\end{aligned}$$

Hence $\gamma \mapsto \phi(o,\gamma)$ is a homomorphism. Next, we observe that

$$\begin{aligned}
\log \phi(z,\gamma) &= M\left(g \mapsto \log \frac{\Phi(z, g\gamma)}{\Phi(o,g)}\right) \\
&= M\left(g \mapsto \log \frac{\Phi(o, g\gamma)}{\Phi(o,g)}\right) + M\left(g \mapsto \log \frac{\Phi(z, g\gamma)}{\Phi(o, g\gamma)}\right) \\
&= M\left(g \mapsto \log \frac{\Phi(o, g\gamma)}{\Phi(o,g)}\right) + M\left(g \mapsto \log \frac{\Phi(z, g)}{\Phi(o,g)}\right) \\
&= \log \phi(o,\gamma) + \log \phi(z, e).
\end{aligned}$$

Hence

$$\gamma \mapsto \frac{\phi(z,\gamma)}{\phi(z,e)} = \phi(o,\gamma)$$

is a homomorphism.

To finish, we note that, by Corollary 4.5, $\phi(z,\gamma) > 0$ and one easily sees that this positive lower bound can be taken uniformly in $z \in \mathcal{Z} \cap [a]$. $\square$

Lemma 4.7. *There is a function $\bar{\phi} : \mathcal{Z} \times \overline{G} \to \mathbb{R}^{>0}$ satisfying*

$$(\mathfrak{L}_\varphi^n \bar{\phi})(o,g) \leq \rho^n \bar{\phi}(o,g),$$

for all $n \geq 1$, and, for all $a \in S$ and all $m \in \overline{G}$, we have

$$\inf_{z \in \mathcal{Z} \cap [a]} \bar{\phi}(z,m) > 0.$$

Proof. By part (ii) of Lemma 4.6, for each $z \in \mathcal{Z}$, there is a homomorphism $\Upsilon_z : \overline{G} \to \mathbb{R}^{>0}$ such that

$$(\Upsilon_z \circ \pi)(\gamma) = \frac{\phi(z,\gamma)}{\phi(z,e)}.$$

We deduce that if $\pi(\gamma) = \pi(\delta)$ then $\phi(z,\gamma) = \phi(z,\delta)$, and we define $\bar{\phi} : \mathcal{Z} \times \overline{G} \to \mathbb{R}^{>0}$ by

$$\bar{\phi}(z,m) = \phi(z,\gamma)$$

for any $\gamma \in G$ satisfying $\pi(\gamma) = m$.

We then have

$$\begin{aligned}
(\mathfrak{L}_\varphi^n \bar{\phi})(o,0) &= \sum_{\sigma^n z = o} e^{\varphi^n(z)} \bar{\phi}(z, \bar{\psi}_n(z)^{-1}) \\
&= \sum_{\sigma^n z = o} e^{\varphi^n(z)} \phi(z, \psi_n(z)^{-1}) \\
&= (\mathcal{L}_\varphi^n \phi)(o,e) \leq \rho^n \phi(o,e) = \rho^n \bar{\phi}(o,0).
\end{aligned}$$

The final statement follows from Lemma 4.6 and the definition of $\bar{\phi}$. $\square$

We now complete the proof of Theorem 1.3. Let

$$c = \inf_{z \in \mathcal{Z} \cap [a]} \bar{\phi}(z,0) > 0.$$

Using Lemma 2.2, we have that

$$\begin{aligned}
P_{\mathrm{G}}(\varphi, T_{\bar{\psi}}) &= \limsup_{n \to \infty} \frac{1}{n} \log(\mathfrak{L}_\varphi^n (c \mathbb{1}_{[a] \times \{0\}}))(o,0) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \log(\mathfrak{L}_\varphi^n \bar{\phi})(o,0) \leq \log \rho,
\end{aligned}$$

as required.

Remark 4.8. We could reformulate Theorem 1.3 and the above proof in terms of the skew product $T_{\psi^{\mathrm{ab}}} : \Sigma \times G^{\mathrm{ab}} \to \Sigma \times G^{\mathrm{ab}}$ without significantly changing the argument. We observe that $\langle \xi, \cdot \rangle$ represents a choice of homomorphism from $\overline{G}$ to $\mathbb{R}$ and there is a natural isomorphism between

$\mathrm{Hom}(\overline{G},\mathbb{R})$ and $\mathrm{Hom}(G^{\mathrm{ab}},\mathbb{R})$. Then $\langle \xi, \bar{\psi}\rangle$ is replaced by $\chi \circ \psi^{\mathrm{ab}}$, for an appropriate choice of $\chi \in \mathrm{Hom}(G^{\mathrm{ab}},\mathbb{R})$.

5. The BIP property and skew product extensions

We now introduce the stronger condition on Σ required for the equidistribution result we described in the introduction. Let $\sigma : \Sigma \to \Sigma$ be a countable state Markov shift satisfying the BIP property. (The terminology BIP appears in, for example, the work of Sarig. If $\sigma : \Sigma \to \Sigma$ is topologically mixing then BIP is equivalent to the finite primitivity condition used by Mauldin and Urbański [21], [22]. Thus, we can cite results from both sets of authors.)

Let $\varphi : \Sigma \to \mathbb{R}$ be a locally Hölder continuous function. We say that a σ-invariant probability measure μ on Σ is a *Gibbs measure* for φ if there are constants A, B and P (depending only on φ) such that

$$A \le \frac{\mu([w])}{e^{\varphi^n(x)-nP}} \le B$$

for all $w \in \mathcal{W}_n$, for all $n \ge 1$ and all $x \in [w]$. If φ has a Gibbs measure then $P = P_{\mathrm{G}}(\varphi,\sigma)$.

From now on, we assume that $\sigma : \Sigma \to \Sigma$ is topologically mixing and satisfies BIP. Let $\varphi : \Sigma \to \mathbb{R}$ be locally Hölder continuous and summable, so that $P_{\mathrm{G}}(\varphi,\sigma)$ is finite. Then φ has a unique Gibbs measure, which we denote by μ_φ, and $P = P_{\mathrm{G}}(\varphi)$ (Theorem 1 of [30]) . We also have a characterization of Gurevič pressure in terms of periodic points:

$$P_{\mathrm{G}}(\varphi,\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} e^{\varphi^n(x)} \tag{5.1}$$

(Corollary 1 of [30]). (Note that the equality (5.1) may fail in the absence of the BIP property.)

In the presence of BIP, we also have stronger results for our transfer operators, acting on an appropriate Banach space. Let $\mathcal{F}_\theta^b$ denote the space of bounded θ-locally Hölder continuous functions $v : \Sigma \to \mathbb{C}$ with the norm $\|\cdot\|_\theta = \|\cdot\|_\infty + |\cdot|_\theta$, where

$$|v|_\theta = \sup_{n\ge 1} \sup_{w\in\mathcal{W}_n} \sup_{x,y\in[w]} \frac{|v(x)-v(y)|}{\theta^n}.$$

If $\varphi : \Sigma \to \mathbb{R}$ is *summable* then $L_\varphi : \mathcal{F}_\theta^b \to \mathcal{F}_\theta^b$ is a well-defined bounded linear operator (Lemma 2.4.1 of [22]).

We have the following analyticity result.

Lemma 5.1 (Theorem 2.6.13 and Proposition 2.6.13 of [22])**.** *Let $\sigma : \Sigma \to \Sigma$ be topologically mixing and satisfy BIP and let $\varphi, \theta : \Sigma \to \mathbb{R}$ be weakly Hölder continuous. Suppose there exists $\epsilon > 0$ such that $\varphi + t\theta$ is summable for $|t| < \epsilon$. Then $t \mapsto P_{\mathrm{G}}(\varphi + t\theta,\sigma)$ is real analytic on $(-\epsilon,\epsilon)$ and*

$$\left.\frac{dP_{\mathrm{G}}(\varphi+t\theta,\sigma)}{dt}\right|_{t=0} = \int \theta \, d\mu_\varphi.$$

We need an analogue of equation (5.1) for T_ψ. If $T_\psi : \Sigma \times G \to \Sigma \times G$ is mixing then we have the following lemma.

Lemma 5.2. *Let $\sigma : \Sigma \to \Sigma$ be topologically mixing and satisfy BIP and, furthermore, let $T_\psi : \Sigma \times G \to \Sigma \times G$ be topologically mixing. Then we have*

$$P_{\mathrm{G}}(\varphi, T_\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} e^{\varphi^n(x)}.$$

Proof. It is immediate from the definition of Gurevič pressure that

$$P_{\mathrm{G}}(\varphi, T_\psi) \leq \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} e^{\varphi^n(x)}.$$

For the other direction, we adapt the proof of Theorem 1 in [30]. Given $w \in \mathcal{W}_n$, let $\varphi_+^n(w) = \sup\{\varphi^n(y) : y \in [w]\}$. Now fix $a \in S$. By the mixing of T_ψ, there exists $n_0 \geq 1$ and, for $i = 1, \ldots, N$, $w_{a,b_i}, w_{b_i,a} \in \mathcal{W}_{n_0}$ such that $aw_{a,b_i}b_i \in \mathcal{W}_{n_0+2}$, $b_i w_{b_i,a} a \in \mathcal{W}_{n_0+2}$, $\psi_{n_0}(w_{a,b_i}) = e$ and $\psi_{n_0}(w_{b_i,a}) = e$. By the BIP property, given $w \in \mathcal{W}_n$, there exists $i, j \in \{i, \ldots, N\}$ such that $b_i w b_j \in \mathcal{W}_{n+2}$. Hence

$$aw_{a,b_i} b_i w b_j w_{b_j,a} a \in \mathcal{W}_{n+M+1},$$

where $M = 2n_0 + 3$, and concatenating $aw_{a,b_i}b_i w b_j w_{b_j,a}$ gives a periodic point of period $n + M$. Furthermore, if $\psi_n(w) = e$ then

$$|\psi_{n+M}(aw_{a,b_i} b_i w b_j w_{b_j,a})|$$

is bounded independently of n. We conclude that there exists $n_1 \geq 1$ and $C > 0$ so that

$$\sum_{|g| \leq n_1} \sum_{\substack{\sigma^{n+M} x = x \\ \psi_{n+M}(x) = g}} e^{\varphi^{n+M}(x)} \geq C \sum_{\substack{w \in \mathcal{W}_n \\ \psi_n(w) = e}} e^{\varphi_+^n(w)} \geq C(\mathcal{L}_\varphi^n \mathbb{1}_{[a]\times\{e\}})(o, g), \tag{5.2}$$

for any choice of $o \in [a]$.

Now, the mixing of T_ψ implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = g}} e^{\varphi^n(x)}$$

is independent of $g \in G$. Combining this with (5.2), we see that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} e^{\varphi^n(x)} \geq P_{\mathrm{G}}(\varphi, T_\psi),$$

as required. $\square$

We now suppose that φ and ψ satisfying Assumption I, Assumption II and Assumption III, stated in section 1. To simplify notation, we identify $\bar\psi : \Sigma \to \overline{G}$ with a function $f : \Sigma \to \mathbb{Z}^d$ and we now denote the abelian skew product by T_f.

Assumption (I) immediately implies that $T_f : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ is also topologically mixing. Furthermore, Assumption (I) implies that f is *aperiodic*: if $e^{2\pi i \langle t, f(x)\rangle} = zu(\sigma x)/u(x)$ for all $x \in \Sigma$, where $t \in \mathbb{R}^d$, $z \in \mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$ and $u \in C(\Sigma, \mathrm{U}(1))$, then $t = 0 \in \mathbb{R}^d/\mathbb{Z}^d$ and $z = 1$. If Assumption (II) holds then we easily see that $\varphi + \langle w, f\rangle$ is summable, $L_{\varphi+\langle w,f\rangle} : \mathcal{F}_\theta^b \to \mathcal{F}_\theta^b$ is a well-defined bounded linear operator and

$$\mathfrak{p}(w) := P_{\mathrm{G}}(\varphi + \langle w, f\rangle, \sigma)$$

is finite for $|w| < \delta(\varphi, f)$.

We also need to consider the transfer operator with complex potentials. As in section 1, $\delta = \delta(\varphi, f)$ and we write

$$B_{\mathbb{C}}(\delta) = \{s \in \mathbb{C}^d : |\mathrm{Re}(s)| < \delta\}.$$

One can easily check that $s \mapsto L_{\varphi+\langle s,f\rangle}$ is analytic on $B_{\mathbb{C}}(\delta)$ (cf. Corollary 2.6.10 in [22]). Furthermore, for s in a neighbourhood of $B(\delta)$ in $B_{\mathbb{C}}(\delta)$, $L_{\varphi+\langle s,f\rangle}$ has a simple isolated maximal eigenvalue $\lambda(s)$, such that

$$\lambda(w) = e^{\mathfrak{p}(w)},$$

for $w \in B(\delta)$ (Proposition 2.8 in [18]). As a consequence, using spectral perturbation theory, we have that $s \mapsto e^{\mathfrak{p}(s)}$ is analytic for s is a neighbourhood of $B(\delta)$ in $B_{\mathbb{C}}(\delta)$.

We may also calculate the derivatives of $\mathfrak{p}$. From Lemma 5.1, we then have

$$\nabla \mathfrak{p}(w) = \int f \, d\mu^w,$$

where

$$\mu^w := \mu_{\varphi+\langle w.f\rangle}.$$

Furthermore, the function $\mathfrak{p}$ is strictly convex. (Strict convexity fails only if unless there is $w \neq 0$ such that $\langle w, f\rangle$ is cohomologous to a constant, which violates the transitivity of T_f.)

Under Assumption (III), we have

$$\nabla \mathfrak{p}(\xi) = \int f \, d\mu^\xi = 0.$$

The value ξ is important and allows us to relate the Gurevič pressure $P_{\mathrm{G}}(\varphi, T_f)$ to a Gurevič pressure with respect to σ. First, we give a technical lemma. We observe that since f is valued in $\mathbb{Z}^d$, $L_{\varphi+\langle \xi+2\pi i t, f\rangle}$ is well-defined for $t \in \mathbb{R}^d/\mathbb{Z}^d$.

Lemma 5.3. *For $t \in \mathbb{R}^d/\mathbb{Z}^d \setminus \{0\}$, $L_{\varphi+\langle \xi+2\pi i t, f\rangle}$ has spectral radius strictly less than $e^{\mathfrak{p}(\xi)}$.*

Proof. As noted above Assumption (I) implies that f is aperiodic. It then follows from Theorem 2.14 of [18] that $L_{\varphi+\langle \xi+2\pi i t, f\rangle}$ has spectral radius strictly less than $e^{\mathfrak{p}(\xi)}$. $\square$

Proposition 5.4. *We have*

$$P_{\mathrm{G}}(\varphi, T_f) = P_{\mathrm{G}}(\varphi + \langle \xi, f\rangle, \sigma).$$

Proof. By the definition of Gurevič pressure (and recalling our convention that $\varphi : \Sigma \times \mathbb{Z}^d \to \mathbb{R}$ is defined by $\varphi(x, \cdot) = \varphi(x)$), we have

$$P_{\mathrm{G}}(\varphi, T_f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ f^n(x)=0}} e^{\varphi^n(x)} \mathbb{1}_{[a]}(x).$$

On the other hand, we may make the following calculation. Choose $o \in [a]$ and set

$$\mathcal{Z}(n, a, o) = \{y \in [a] \,:\, \sigma^n y = o\}.$$

There is a natural bijection $q : \mathrm{Fix}_n(\sigma) \cap [a] \to \mathcal{Z}(n.a, o)$ given by $q(x) = x_1 \ldots x_n o$. We have

$$\begin{aligned}
\sum_{\substack{\sigma^n x = x \\ f^n(x)=0x}} e^{\varphi^n(x)} \mathbb{1}_{[a]}(x) &= \sum_{\substack{\sigma^n x = x \\ f^n(x)=0}} e^{\varphi^n(x)+\langle \xi, f^n(x)\rangle} \mathbb{1}_{[a]}(x) \\
&= \int_{\mathbb{R}^d/\mathbb{Z}^d} \sum_{x \in \mathrm{Fix}_n(\sigma)\cap[a]} e^{\varphi^n(x)+\langle \xi + 2\pi i t, f^n(x)\rangle}\, dt \\
&= \int_{\mathbb{R}^d/\mathbb{Z}^d} \sum_{y \in \mathcal{Z}(n,a,z)} e^{\varphi^n(y)+\langle \xi + 2\pi i t, f^n(y)\rangle}\, dt + O(\theta^n e^{n\mathfrak{p}(\xi)}) \\
&= \int_{\mathbb{R}^d/\mathbb{Z}^d} (L^n_{\varphi+\langle \xi+2\pi i t, f\rangle} \mathbb{1}_{[a]})(z) + O(\theta^n e^{n\mathfrak{p}(\xi)}),
\end{aligned}$$

for some $0 < \theta < 1$. Using Lemma 5.3 and the analysis in [27], we can show that the right hand side behaves like $e^{n\mathfrak{p}(\xi)} n^{-d/2}$, so the result follows. □

6. Equidistribution for skew product extensions

In this section, we give the proof of Theorem 1.1. We give the proof for the sequence $\mathfrak{M}_\varphi(\Lambda^\psi(n))$, the other cases being similar. The proof is based on the following lemma.

Lemma 6.1. *Let $g : \Sigma \to \mathbb{R}$ be a bounded locally Hölder continuous function. Given $\epsilon > 0$, there exists $C > 0$ and $\eta > 0$ such that*

$$\frac{1}{\Pi_\varphi(\Lambda^\psi(n))} \sum_{\substack{x \in \Lambda^\psi(n) \\ |\int g\, d\tau_{x,n} - \int g\, d\mu_\xi| > \epsilon}} e^{\varphi^n(x)} \le C e^{-\eta n}.$$

Proof. We consider $x \in \Lambda^\psi(n)$ such that $\int g\, d\tau_{x,n} > \int g\, d\mu_\xi + \epsilon$ and $\int g\, d\tau_{x,n} < \int g\, d\mu_\xi - \epsilon$ separately. For $s > 0$, we have

$$\sum_{\substack{x \in \Lambda^\psi(n) \\ \int g\, d\tau_{x,n} > \int g\, d\mu^\xi + \epsilon}} e^{\varphi^n(x)} \le \sum_{x \in \Lambda^\psi(n)} e^{\varphi^n(x) + s g^n(x) - ns\int g\, d\mu^\xi - ns\epsilon},$$

so that

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{\substack{x\in\Lambda^\psi(n)\\ \int g\,d\tau_{x,n}>\int g\,d\mu^x i+\epsilon}} e^{\varphi^n(x)} \le P_{\mathrm{G}}(\varphi+\langle\xi,f\rangle+sg,\sigma)-s\int g\,d\mu^\xi - s\epsilon.$$

Similarly, for $s<0$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{\substack{x\in\Lambda^\psi(n)\\ \int g\,d\tau_{x,n}<\int g\,d\mu^\xi-\epsilon}} e^{\varphi^n(x)} \le P_{\mathrm{G}}(\varphi+\langle\xi,f\rangle+sg,\sigma)-s\int g\,d\mu^\xi + s\epsilon.$$

If we write $\mathfrak{p}_g(s) := P_{\mathrm{G}}(\varphi+\langle\xi,f\rangle+sg)$ then, applying Lemma 5.1, we have a bound of the form

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{\substack{x\in\Lambda^\psi(n)\\ |\int g\,d\tau_{x,n}-\int g\,d\mu_\xi|>\epsilon}} e^{\varphi^n(x)} \le \mathfrak{p}_g(s)-s\mathfrak{p}_g'(0)-\epsilon|s|,$$

which is strictly smaller than $P_{\mathrm{G}}(\varphi+\langle\xi,f\rangle)$ for sufficiently small values of s. The proof is completed by noting that, by Theorem 1.3 and Proposition 5.4,

$$\lim_{n\to\infty}\frac{1}{n}\log\Pi_\varphi(\Lambda^\psi(n)) = P_{\mathrm{G}}(\varphi+\langle\xi,f\rangle,\sigma).$$

□

Proof of Theorem 1.1. Write $\mathfrak{M}_n=\mathfrak{M}_\varphi(\Lambda^\psi(n))$. We need to show that for every bounded continuous function $g:\Sigma\to\mathbb{R}$ we have

$$\lim_{n\to\infty}\int g\,d\mathfrak{M}_n = \int g\,d\mu^\xi.$$

It is enough to prove this when g is a bounded locally Hölder continuous function. We have

$$\begin{aligned}\int g\,d\mathfrak{M}_n-\int g\,d\mu^\xi &= \frac{1}{\Pi_\varphi(\Lambda^\psi(n))}\sum_{\substack{x\in\Lambda^\psi(n)\\ |\int \chi\,d\tau_{x,n}-\int \chi d\mu^\xi|>\epsilon}} e^{\varphi^n(x)}\int g\,d\tau_{x,n}\\ &\quad+\frac{1}{\Pi_\varphi(\Lambda^\psi(n))}\sum_{\substack{x\in\Lambda^\psi(n)\\ |\int \chi\,d\tau_{x,n}-\int g d\mu^\xi|\le\epsilon}} e^{\varphi^n(x)}\int g\,d\tau_{x,n}-\int \chi\,d\mu^\xi.\end{aligned}$$

By Lemma 6.1, the first term on the right hand side tends to zero exponentially fast. Also,

$$\begin{aligned}&\left|\frac{1}{\Pi_\varphi(\Lambda^\psi(n))}\sum_{\substack{x\in\Lambda^\psi(n)\\ \left|\int g\,d\tau_{x,n}-\int g d\mu^\xi\right|\le\epsilon}} e^{\varphi^n(x)}\int g\,d\tau_{x,n}-\int g\,d\mu^\xi\right|\\ &\le \epsilon+\frac{\left|\int g\,d\mu^\xi\right|}{\Pi_\varphi(\Lambda^\psi(n))}\sum_{\substack{x\in\Lambda^\psi(n)\\ \left|\int g\,d\tau_{x,n}-\int g d\mu^\xi\right|>\epsilon}} e^{\varphi^n(x)}\le \epsilon+Ce^{-\eta(\epsilon)n},\end{aligned}$$

which, since ϵ is arbitrary, gives the result.

□

7. Equality implies amenability

In this section we show that, under Assumptions (I), (II) and (III), the converse of Theorem 1.3 holds, i.e. if $P_{\mathrm{G}}(\varphi,T_\psi)=P_{\mathrm{G}}(\varphi,T_{\bar\psi})$ then G is amenable. As above, we write $f=\bar\psi$.

To do this, we need to specify a function space for $\mathcal{L}_\varphi$ to act on. Let $\mathcal{H}$ denote the set of continuous functions $v:\Sigma\times G\to\mathbb{C}$ such that $g\mapsto\|v(\cdot,g)\|_\infty$ is in $\ell^2(G)$, with norm

$$\|v\|_{\mathcal{H}}=\left(\sum_{g\in G}\|v(\cdot,g)\|_\infty^2\right)^{1/2},$$

and let $\mathrm{spr}_{\mathcal{H}}(\mathcal{L}_\varphi)$ denote the spectral radius of $\mathcal{L}_\varphi$ on $\mathcal{H}$. We always have

$$P_{\mathrm{G}}(\varphi,T_\psi)\le\log\mathrm{spr}_{\mathcal{H}}(\mathcal{L}_\varphi)\le P_{\mathrm{G}}(\varphi,\sigma).$$

We state a result due to Stadlbauer.

Proposition 7.1 (Stadlbauer [33], Theorem 5.4)**.** *Suppose that $\sigma:\Sigma\to\Sigma$ satisfies BIP and that $T_\psi:\Sigma\times G\to\Sigma\times G$ is topologically transitive. If G is not amenable then* $\log\mathrm{spr}_{\mathcal{H}}(\mathcal{L}_\varphi)<P_{\mathrm{G}}(\varphi,\sigma)$.

We can now prove the desired result.

Proof of Proposition 1.5. Let $\xi\in\mathbb{R}^d$ be as in the previous section and let $\varphi_\xi=\varphi+\langle\xi,f\rangle$. Suppose that G is not amenable. Then

$$P_{\mathrm{G}}(\varphi,T_\psi)=P_{\mathrm{G}}(\varphi_\xi,T_\psi)\le\log\mathrm{spr}_{\mathcal{H}}(\mathcal{L}_{\varphi_\xi})<P_{\mathrm{G}}(\varphi_\xi,\sigma)=P_{\mathrm{G}}(\varphi,T_{\bar\psi}),$$

where the strict inequality uses Proposition 7.1. □

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