ERROR TERMS FOR CLOSED ORBITS OF HYPERBOLIC FLOWS

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0. Introduction

In this paper we shall consider error terms in estimates for the number of closed orbits for a large class of C^1 flows $\phi_t : M \to M$, restricted to a hyperbolic set Λ. Let $\pi(T)$ be the number of closed orbits of least period at most $T > 0$. It is well known that

$$h = \lim_{T \to +\infty} \frac{1}{T} \log \pi(T),$$

where $h > 0$ denotes the topological entropy of the flow. It was shown by Parry and the first author in [7] that if ϕ is a weak-mixing Axiom A flow (restricted to a non-trivial basic set) then

$$\pi(T) \sim \frac{e^{hT}}{hT}, \text{ as } T \to +\infty$$

i.e. $\lim_{T \to +\infty} \frac{\pi(T)}{e^{hT}/hT} = 1$. This generalized a result of Margulis for geodesic flows over manifolds of negative sectional curvature [6].

It is an interesting problem to estimate the error terms in such asymptotic formulae. In the particular case of geodesic flows over compact negatively curved manifolds we showed that there was an exponential error term (with a suitable principal term) [10].

Our first result gives an error term in the case of weak-mixing transitive Anosov flows, in which case $\Lambda = M$.

Theorem 1. *Let $\phi_t : M \to M$ be a weak-mixing transitive Anosov flow. Then there exists $\delta > 0$ such that*

$$\pi(T) = \frac{e^{hT}}{hT} \left(1 + O\left(\frac{1}{T^\delta} \right) \right).$$

There are examples of Axiom A flows for which the error term may be arbitrarily bad [9]. In particular, it need not be the case that an error term as in the statement of Theorem 1 is satisfied. In order to obtain a positive result, we shall consider flows satisfying the following condition.

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Approximability condition. The flow ϕ has three closed orbits γ_1, γ_2 and γ_3 with distinct least periods $l(\gamma_1)$, $l(\gamma_2)$ and $l(\gamma_3)$, respectively, such that

$$\beta = \frac{l(\gamma_1) - l(\gamma_2)}{l(\gamma_2) - l(\gamma_3)}$$

is badly approximable, i.e., there exists $\alpha > 2$ and $C > 0$ such that we have $|\beta - \frac{p}{q}| \geq \frac{C}{q^\alpha}$, for all $p, q \in \mathbb{Z}$ $(q > 0)$.

The set of β satisfying this condition is a large set. For example, it is a set of full measure in the real line. Moreover, its complement has Hausdorff dimension zero.

Our second result is the following.

Theorem 2. *For a weak-mixing hyperbolic flow $\phi_t : \Lambda \to \Lambda$ satisfying the approximability condition there exists $\delta > 0$ such that*

$$\pi(T) = \frac{e^{hT}}{hT}\left(1 + O\left(\frac{1}{T^\delta}\right)\right).$$

The method of proof of both theorems follows the lines of the error term result for the prime number theorem in analytic number theory. In [7] a dynamical zeta function $\zeta(s)$ was used to obtain the principal term in the asymptotic formula for $\pi(T)$. In this paper we develop an approach introduced by Dolgopyat [2] to get more precise estimates on the location of the poles of $\zeta(s)$ and modulus of $(\zeta'/\zeta)(s)$. We employ these estimates to prove the theorems using techniques from analytic number theory.

We briefly outline the contents of this paper. In the first section we explain how, through the use of symbolic dynamics, the counting problem for hyperbolic flows can be reduced to one for periodic points for a subshift of finite type with respect to a weighting by a positive Hölder continuous function. In the second section we introduce a dynamical zeta function and derive some important properties of its analytic extension. This makes essential use of bounds on the norm of the associated Ruelle transfer operators (presented in Proposition 2). In section 3, we deduce Theorems 1 and 2 from these properties of the dynamical zeta functions. The method of proof here is inspired by approachs to proving error terms for counting prime numbers via the Riemman zeta function [3], [5]. Finally, in section 4 we explain the proof of Proposition 2 on the necessary spectral properties of the transfer operators. Although this section follows closely ideas of Dolgopyat, our exposition is somewhat different and is principally included for completeness.

1. Hyperbolic flows and Symbolic dynamics

Let M be a compact C^∞ manifold and let $\phi_t : M \to M$ be a C^1 flow. We call a closed ϕ-invariant subset $\Lambda \subset M$ *hyperbolic* if:

(1) there exist $D\phi$-invariant sub-bundles E^0, E^u and E^s such that $T_\Lambda M = E^0 \oplus E^u \oplus E^s$, where E^0 is the one dimensional sub-bundle tangent to the flow direction and there are constants $C, \lambda > 0$ such that $||D\phi_t|E^s|| \leq Ce^{-\lambda t}$ and $||D\phi_{-t}|E^u|| \leq Ce^{-\lambda t}$ for $t \geq 0$;
(2) $\phi_t : \Lambda \to \Lambda$ is transitive;
(3) closed orbits are dense in Λ; and
(4) there is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \phi_t U$.

The restriction $\phi_t : \Lambda \to \Lambda$ is called a *hyperbolic flow*. If $\Lambda = M$ then $\phi_t : M \to M$ is called a transitive *Anosov flow*. Given any sufficiently small $\epsilon > 0$, we define the (local) strong stable manifold by

$$W_\epsilon^{ss}(x) := \{y \in \Lambda : d_\Lambda(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0\}$$

and the (local) strong unstable manifold by

$$W_\epsilon^{su}(x) := \{y \in \Lambda : d_\Lambda(\phi_{-t}(x), \phi_{-t}(y)) \leq \epsilon \text{ for all } t \geq 0\}.$$

Given $x, y \in \Lambda$ sufficiently close, there exists a unique $|t| \leq \epsilon$ such that $W_\epsilon^{ss}(x) \cap W_\epsilon^{su}(\phi_t y) \neq \varnothing$. Furthermore, this non-empty set consists of a single point denoted by $[x, y]$.

A particularly desirable feature of such flows is that they can be studied through an associated symbolic model. In 1969, Ratner carried out such a construction for three dimensional Anosov flows, building on earlier work of Sinai for Anosov diffeomorphisms. Subsequently, she extended her results to arbitrary dimensions [11]. The general case of hyperbolic flows was treated by Bowen [1].

Given an aperiodic $k \times k$ matrix A with entries zero or one, we denote

$$X_A^+ = \left\{ x = (x_n)_{n=0}^\infty \in \prod_{n=0}^\infty \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1, \quad n \geq 0 \right\}.$$

For any $0 < \theta < 1$ we can define a metric $d(x, y) = \theta^N$ where, for $x \neq y$, we choose $N = N(x, y)$ to be the largest value for which $x_i = y_i$, for $0 \leq i \leq N$. We define the (one-sided) subshift of finite type $\sigma : X_A^+ \to X_A^+$ by $(\sigma x)_n = x_{n+1}$.

Given $1 \leq i \leq k$ we associate a *one-cylinder* by $[i] := \{x \in X_A^+ : x_0 = i\}$. Given a finite string $\underline{i} = (i_0, \ldots, i_{n-1})$ we denote the *n-cylinder*

$$[\underline{i}] = \{x \in X_A^+ : x_0 = i_0, \ldots, x_{n-1} = i_{n-1}\}.$$

(Equivalently, $[\underline{i}] := [i_0] \cap \sigma^{-1}[i_1] \cap \ldots \cap \sigma^{-(n-1)}[i_{n-1}]$.)

Let $r : X_A \to \mathbb{R}^+$ be a strictly positive Hölder continuous function. We can define

$$\pi^*(T) = \sum_{r^n(x) \leq T} 1,$$

where the sum is over period points $\sigma^n x = x$ with $r^n(x) \leq T$ where we denote $r^n(x) = r(x) + r(\sigma x) + \ldots + r(\sigma^{n-1} x)$.

The next result describes the connection between $\pi^*(T)$ and $\pi(T)$.

Proposition 1 (Bowen [1], Ratner [11]). *Given a hyperbolic flow $\phi_t : \Lambda \to \Lambda$ with topological entropy $h > 0$ there exists a subshift of finite type $\sigma : X_A^+ \to X_A^+$, a Hölder continuous function $r : X_A^+ \to \mathbb{R}^+$ and $\epsilon > 0$ such that $|\pi(T) - \pi^*(T)| = O\left(e^{(h-\epsilon)T}\right)$.*

Let us briefly recall how this result is obtained. The essential idea is to choose a finite family of local cross sections $\{T_1, \ldots, T_k\}$ to the flow. We choose $T_i = [U_i, S_i]$, $i = 1, \ldots, k$, where U_i and S_i are closed subsets of strong unstable and strong stable manifolds, respectively.

Let $\mathcal{P} : \coprod_{i=1}^k T_i \to \coprod_{i=1}^k T_i$ denote the Poincaré map. Define a $k \times k$ matrix A by

$$A(i,j) = \begin{cases} 1 & \text{if } \mathrm{int}T_i \cap \mathcal{P}^{-1}\mathrm{int}T_j \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

To this we can associate the space

$$X_A = \left\{ x = (x_n)_{n=-\infty}^{\infty} \in \prod_{n=-\infty}^{\infty} \{1, \dots, k\} : A(x_n, x_{n+1}) = 1 \quad \forall n \in \mathbb{Z} \right\}$$

with the metric $d(x,y) = \theta^N$ where for $x \neq y$ we choose $N = N(x,y)$ to be the largest value for which $x_i = y_i$ for $-N \leq i \leq N$. We define a (two sided) subshift of finite type $\sigma : X_A \to X_A$ by $(\sigma x) = x_{n+1}$.

Given $\bar{x} \in \coprod_{i=1}^k \mathrm{int}T_i$ such that $\mathcal{P}^n(\bar{x}) \in \coprod_{i=1}^k \mathrm{int}T_i$, for all $n \in \mathbb{Z}$ we can associate a sequence $x \in X_A$ by $\mathcal{P}^n(\bar{x}) \in \mathrm{int}T_{x_n}$. This naturally extends to a Hölder continuous map $p : X_A \to \coprod_{i=1}^k T_i$. By construction, we see that p is a semi-conjugacy, i.e., $p \circ \sigma = \mathcal{P} \circ p$.

We can associate a a strictly positive Hölder continuous function $r : X_A \to \mathbb{R}^+$ by $\phi_{r(x)}p(x) = p(\sigma x)$. We can introduce the suspension space

$$X_A^r = \{(x,t) \in X_A \times \mathbb{R} : 0 \leq t \leq r(x)\}$$

where $(x, r(x))$ and $(\sigma x, 0)$ are identified. We let $\sigma_t^r : X^r \to X^r$ be the flow defined by $\sigma_t^r(x,u) = (x, u+t)$, subject to the identifications.

We can extend $p : X_A \to \coprod_{i=1}^k T_i$ to a map $p : X_A^r \to \Lambda$ by $p(x,t) = \phi_t p(x)$. The map p is onto, and one-one on a residual set. Furthermore, p is a semi-conjugacy, i.e., $p \circ \phi_t = \sigma_t^r \circ p$.

Periodic orbits for the suspended flow $\sigma_t^r : X_A^r \to X_A^r$ correspond to periodic orbits for $\sigma : X_A \to X_A$ of the form $\{x, \sigma x, \dots, \sigma^{n-1}x\}$ where $\sigma^n x = x$ while $\sigma^m x \neq x$ for $1 \leq m < n$. The least period of the σ^r-orbit is precisely $r^n(x)$.

Since $T_i = [U_i, S_i]$ it follows that the function $r : X_A \to \mathbb{R}$ satisfies $r(x) = r(y)$ when $x_i = y_i$, for $i < 0$. In particular, $r : X_A \to \mathbb{R}$ can be identified with a function $r : X_A^+ \to \mathbb{R}$. Thus periodic orbits for the suspended flow can be identified with periodic orbits $\{x, \sigma x, \dots, \sigma^{n-1}x\}$ for $\sigma : X_A^+ \to X_A^+$ where their length is $r^n(x)$.

In general the map $p : X_A^r \to \Lambda$ will not be a bijection. In particular, there may not be a one-to-one correspondence between the periodic orbits for the two flows. However, this discrepancy can be accounted for either by the detailed analysis of Manning and Bowen [1] or by the simpler observation that the number of closed orbits passing through the boundaries grows at a slower exponential rate. The estimate in Proposition 1 follows from these considerations.

2. Transfer operators and zeta functions

When studying $\pi^*(T)$ a central role is played by a dynamical zeta function. This is a function of a complex variable determined by the lengths of periodic orbits for the suspended flow. An important technical tool in studying the zeta function is a family of linear operators. We first define the Banach space on which they act.

Given $0 < \theta < 1$ we define, for $f \in C(X_A^+, \mathbb{C})$, a value $|f|_\theta \in \mathbb{R}^+ \cup \{\infty\}$ by

$$|f|_\theta = \sup \left\{ \frac{|f(x) - f(y)|}{\theta^n} : x_i = y_i, 0 \leq i \leq n \text{ and } n \geq 0 \right\}.$$

If we define $\mathcal{F}_\theta = \{f \in C(X_A^+, \mathbb{C}) : |f|_\theta < +\infty\}$ then this is a Banach space with respect to the norm $||f||_\theta = |f|_\theta + |f|_\infty$. Since r is Hölder continuous we can choose $0 < \theta < 1$ such that $r \in \mathcal{F}_\theta$. This enables us to define the following family of operators.

Definition. For $s \in \mathbb{C}$, let $L_{-sr} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ be the *transfer operator* defined by

$$L_{-sr} f(x) = \sum_{\sigma y = x} e^{-sr(y)} f(y).$$

Given a continuous function $f : X_A^+ \to \mathbb{R}$ we define its *pressure* $P(f)$ by

$$P(f) = \sup \left\{ h(\nu) + \int f d\nu : \nu \text{ is a } \sigma\text{-invariant probability} \right\},$$

where $h(\nu)$ is the entropy of σ with respect to ν. When f is Hölder continuous then there is a unique measure μ realizing this supremum, which is called the measure of maximal entropy.

Lemma 1. *For $s \in \mathbb{C}$ the operator $L_{-Re(s)r} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ has a simple positive eigenvalue $e^{P(-Re(s)r)}$. Furthermore, the rest of the spectrum is contained in a strictly smaller disk.*

For each $s \in \mathbb{C}$ it is technically convenient to modify the function r so that $L_{-Re(s)r} 1 = e^{P(-Re(s)r)} 1$, where 1 denotes the constant function taking value 1. This can achieved by replacing r by $r + u \circ \sigma - u$, for some appropriate Hölder continuous function $u : X_A^+ \to \mathbb{R}$ [15].

The next lemma gives an estimate on the norm of iterates of the operator $L_{-sr} : \mathcal{F}_\theta \to \mathcal{F}_\theta$.

Lemma 2. *There exists $C_0 > 0$ such that*

$$\frac{||L_{-sr}^l f||}{e^{lP(-Re(s)r)}} \le C_0 |Im(s)| \, |f|_\infty + \theta^l |f|_\theta$$

for all $l \ge 0$ and any $f \in \mathcal{F}_\theta$.

Proof. This is essentially the Basic Inequality in [8]. (The only additional feature is the factor $|Im(s)|$ which follows from an inspection of the proof).

The next result gives estimates on L_{-sr} and the operator norm $|| \cdot ||$ under the hypothesis of either Theorem 1 or Theorem 2.

Proposition 2. *Assume that ϕ is either a transitive Anosov flow or an Axiom A flow satisfying the approximability condition. There exist constants $t_0 \ge 1$, $\tau > 0$, $C_1 > 0$, and $C > 0$, such that $\forall |Im(s)| \ge t_0$, $\forall m \ge 1$,*

$$||L_{-sr}^{2Nm}|| \le C_1 |Im(s)| e^{2NmP(-Re(s)r)} \left(1 - \frac{1}{|Im(s)|^\tau}\right)^{m-1}$$

where $|Im(s)| \geq t_0$ *and* $N = [C \log |Im(s)|]$.

We shall present a proof of this result in the final section. This will be based on techniques from [2].

Observe that if $n = 2Nm + l$, with $p = \left[\frac{n}{2N}\right] \geq 0$ and $0 \leq l \leq 2N - 1$ then, using Lemma 2 and Proposition 2, we can write, for all $|Im(s)| \geq t_0$,

$$\begin{aligned} ||L_{-sr}^n|| &\leq ||L_{-sr}^{2Nm}|| \; ||L_{-sr}^l|| \\ &\leq C_2 \; |Im(s)|^2 \; e^{nP(-Re(s)r)} \left(1 - \frac{1}{|Im(s)|^\tau}\right)^{m-1} \end{aligned} \tag{2.1}$$

where $C_2 = C_1(C_0 + 1) > 0$.

We will now define the zeta function and show how it is related to the transfer operators. Given $s \in \mathbb{C}$ we set

$$\zeta(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} Z_n(-sr),$$

where $Z_n(-sr) = \sum_{\sigma^n x = x} e^{-sr^n(x)}$. This converges to an analytic function for $Re(s) > h$. Moreover, $\zeta(s)$ has a simple pole at $s = h$ and, apart from this, has a non-zero analytic extension to a neighbourhood of $Re(s) \geq h$ [7], [8].

Lemma 3. *For each* $1 \leq i \leq k$, *fix a point* $x_i \in [i]$. *There exist constants* $C_3 > 0$ *and* $t_1 \geq t_0$ *such that, for all* $n \geq 1$ *and for* $|Im(s)| \geq t_1$,

$$\left| Z_n(-sr) - \sum_{i=1}^{k} L_{-sr}^n \chi_{[i]}(x_i) \right| \leq C_3 |Im(s)|^3 e^{nP(-Re(s)r)} \left(1 - \frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n}{2N}\right]}, \tag{2.2}$$

where $\chi_{[i]}$ *is the characteristic function for* $[i]$.

Proof. The proof follows the general lines of an argument from [13]. Consider all strings $\underline{i} = (i_0, \ldots, i_{n-1})$ with $[\underline{i}] \neq \varnothing$ and write $|\underline{i}| = n$. If $\underline{i} = [i]$ set $x_{\underline{i}} = x_i$. If $|\underline{i}| \geq 2$ we choose $x_{\underline{i}} \in [\underline{i}]$ such that $\sigma^n x_{\underline{i}} = x_{\underline{i}}$, if possible, and arbitrarily otherwise. With these choices we see that

$$Z_n(-sr) = \sum_{|\underline{i}|=n} (L_{-sr}^n \chi_{[\underline{i}]})(x_{\underline{i}}). \tag{2.3}$$

If $\underline{i} = (i_0, \ldots, i_{m-1})$ then we can adopt the notation $\underline{j}(\underline{i}) = (i_0, \ldots, i_{m-2})$, i.e., we look at the shorter string after deleting the last term.

We can write

$$\begin{aligned} &\sum_{|\underline{i}|=n} L_{-sr}^n \chi_{[\underline{i}]}(x_{\underline{i}}) - \sum_{i=1}^{k} L_{-sr}^n \chi_{[i]}(x_i) \\ &= \sum_{m=2}^{n} \left(\sum_{|\underline{i}|=m} L_{-sr}^n \chi_{[\underline{i}]}(x_{\underline{i}}) - \sum_{|\underline{j}|=m-1} L_{-sr}^n \chi_{[\underline{j}]}(x_{\underline{j}}) \right) \\ &= \sum_{m=2}^{n} \sum_{|\underline{i}|=m} \left(L_{-sr}^n \chi_{[\underline{i}]}(x_{\underline{i}}) - L_{-sr}^n \chi_{[\underline{i}]}(x_{\underline{j}(\underline{i})}) \right). \end{aligned}$$

We can therefore bound

$$\begin{aligned}&\left|\sum_{|\underline{i}|=n} L^n_{-sr}\chi_{[\underline{i}]}(x_{\underline{i}}) - \sum_{i=1}^{k} L^n_{-sr}\chi_{[i]}(x_i)\right| \\ &\leq \sum_{m=2}^{n} ||L^{n-m}_{-sr}|| \sum_{|\underline{i}|=m} |L^m_{-sr}\chi_{[\underline{i}]}|_\theta \ \theta^{m-1}.\end{aligned} \tag{2.4}$$

To get an upper bound on the Right Hand Side of (2.4) we can use the following estimates

(1) $||L^{n-m}_{-sr}|| \leq C_2|Im(s)|^2 e^{(n-m)P(-Re(s)r)}\left(1-\frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n-m}{N}\right]-1}$, by (2.1),

(2) since $L^m_{-sr}\chi_{[\underline{i}]}(x) = e^{-sr^m(\sigma_{\underline{i}}x)}$, where $\sigma_{\underline{i}}$ is the local inverse to $\sigma^m : [\underline{i}] \to X^+_A$,

$$\begin{aligned}&|L^m_{-sr}\chi_{[\underline{i}]}|_\theta \\ &\leq \sup_{x\in X^+_A}\left\{e^{-Re(s)r^m(\sigma_{\underline{i}}x)}\right\}\exp\left(\frac{|Re(s)||r|_\theta}{1-\theta}\right)\frac{(|Re(s)|+|Im(s)|)|r|_\theta}{1-\theta} \\ &\leq C'|Im(s)|\end{aligned}$$

if $|Im(s)| \geq t_0$, for some $C' > 0$,

(3) there exists $C'' > 0$ such that

$$\sum_{|\underline{i}|=m}\sup_{x\in X^+_A}\left\{e^{-Re(s)r^m(\sigma_{\underline{i}})}\right\} \leq C'' e^{mP(-Re(s)r)}.$$

We can compare the estimates above with the inequality (2.4) to deduce

$$\begin{aligned}&\sum_{m=2}^{n} ||L^{n-m}_{-sr}|| \sum_{|\underline{i}|=m} |L^m_{-sr}\chi_{[\underline{i}]}|_\theta d(x_{\underline{i}}, x_{\underline{j}(\underline{i})}) \\ &\leq C_2C'C''|Im(s)|^3 e^{nP(-Re(s)r)}\sum_{m=2}^{n}\left(1-\frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n-m}{2N}\right]-1}\theta^{m-1} \\ &\leq C_3|Im(s)|^3 e^{nP(-Re(s)r)}\left(1-\frac{1}{|Im(s)|^\tau}\right)^{[n/2N]},\end{aligned} \tag{2.5}$$

where the last inequality is valid for $|Im(s)| \geq t_1$ for some constant $t_1 \geq t_0$ and $C_3 > 0$. Comparing (2.3), (2.4) and (2.5) we see that we have proved Lemma 3.

We now apply Lemma 3 to get the following result on zeta functions.

Proposition 3. *There exist $\rho > 0$ and $t_2 \geq t_1$ such that the zeta function $\zeta(s)$ is analytic and non-zero in the region*

$$\mathcal{R}(\rho) = \left\{s : Re(s) > h - \frac{1}{|Im(s)|^\rho}, |Im(s)| \geq t_2\right\}.$$

Moreover, we can bound $\log|\zeta(s)| \leq C_4|Im(s)|^{3+\tau}\log|Im(s)|$, *for some* $C_4 > 0$.

Proof. Using (2.4) Proposition 2 we can bound

$$\begin{aligned}
|Z_n(-sr)| &\leq \left|Z_n(-sr) - \sum_{i=1}^{k} L_{-sr}^n \chi_{I_i}(x_i)\right| + \left|\sum_{i=1}^{k} L_{-sr}^n \chi_{I_i}(x_i)\right| \\
&\leq C_3|Im(s)|^3 e^{nP(-Re(s)r)}\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n}{2N}\right]} \\
&\qquad + kC_2|Im(s)|^2 e^{nP(-Re(s)r)}\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n}{2N}\right]-1} \\
&\leq (C_3 + kC_2)|Im(s)|^3 e^{nP(-Re(s)r)}\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n}{2N}\right]-1}.
\end{aligned} \tag{2.6}$$

Define the curve $A(t) := h - \frac{1}{|t|^\rho}$ and consider those s for which $Re(s) > A(t)$ and $|Im(s)| \geq t_1$. The map $t \mapsto P(-tr)$ is analytic with nowhere vanishing first derivative and, in particular, we can write

$$e^{P(-Re(s)r)} = 1 + P'(-hr)(Re(s) - h) + O((Re(s) - h)^2),$$

where $P'(-hr) < 0$ [12], [8]. Thus

$$\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{\frac{1}{2N}} < e^{-P(-Re(s)r)},$$

if $\rho > \tau$ and $|Im(s)| \geq t_2$, for some $t_2 \geq t_1$. This causes no loss of generality since increasing ρ simply corresponds to a narrowing of the region $\mathcal{R}(\rho)$. Hence in this region we have

$$\begin{aligned}
\log|\zeta(s)| &\leq \sum_{n=1}^{\infty} \frac{1}{n}|Z_n(-sr)| \\
&\leq (C_3 + kC_2)|Im(s)|^3 \sum_{n=1}^{\infty} \frac{e^{nP(-Re(s)r)}}{n}\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{\left[\frac{n}{2N}\right]-1} \\
&\leq \frac{(C_3 + kC_2)|Im(s)|^3 e^{P(-Re(s)r)}}{\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{2-1/2N}\left(1 - e^{P(-Re(s)r)}\left(1 - \frac{1}{|Im(s)|^\tau}\right)^{1/2N}\right)} \\
&\leq C_4|Im(s)|^{3+\tau}\log|Im(s)|.
\end{aligned} \tag{2.7}$$

This complete the proof of Proposition 3.

To proceed, we need the following standard result from complex analysis which allows us to convert the bound (2.7) into a bound for $\zeta'(s)/\zeta(s)$ in a smaller region.

FIGURE 1: PROOF OF PROPOSITION 4

Lemma 4. [14] *Suppose that $\zeta(s)$ is non-zero and analytic on a disk $\mathcal{D}(R) = \{s : |s - z| \leq R\}$. Suppose $\log|\zeta(s)|$ is bounded by $U > 0$ on $\mathcal{D}(R)$. Then for $0 < r < R$ and $s \in \mathcal{D}(r)$ we have the bound*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \frac{8R}{(R-r)^2}\left(U + |\log\zeta(z)|\right).$$

We can apply Lemma 4 in the following way. Given $s \in \mathcal{R}(\rho)$ we set $z = (h+1) + iIm(s)$. Choose $R = 1 + |Im(s)|^{-\rho}$ and $r = 1 + |Im(s)|^{-\rho}/2$. Then, by Lemma 4, we have

$$\begin{aligned}\left|\frac{\zeta'(s)}{\zeta(s)}\right| &\leq 32|Im(s)|^{2\rho}\left(|Im(s)|^{3+\tau} + |\log\zeta(h+1+iIm(s))|\right)\\ &\leq C_5|Im(s)|^{3(\rho+1)},\end{aligned}$$

for some $C_5 > 0$. Hence we have shown the following.

Proposition 4. *The logarithic derivative $\zeta'(s)/\zeta(s)$ is analytic in*

$$\mathcal{R}_1(\rho) = \left\{s : Re(s) > h - \frac{1}{2|Im(s)|^{\rho}}, |Im(s)| \geq t_2\right\},$$

and, in this domain, satisfies the bound

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq C_5|Im(s)|^{3(\rho+1)}.$$

In the following section it will be slightly more convenient for us to work with the complex function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} hr^n(x) e^{-shr^n(x)}.$$

By observing that $\eta(s) = h(\zeta'/\zeta)(sh)$ we conclude the following.

Corollary 4.1.

(1) *$\eta(s)$ is analytic for $Re(s) > 1$;*
(2) *$\eta(s)$ has an analytic extension to a neighbourhood of $Re(s) = 1$, except for a simple pole at $s = 1$;*
(3) *setting $t_3 = t_2/h > 0$, $\eta(s)$ has a non-zero analytic extension to*

$$\mathcal{R}_2(\rho) = \left\{ s \,:\, Re(s) > 1 - \frac{1}{2h^{(\rho+1)}|Im(s)|^\rho}, |Im(s)| \geq t_3 \right\};$$

(4) *setting $C_6 = C_5 h^{(3\rho+4)}$ and $\alpha = 3(\rho+1)$, we have $|\eta(s)| \leq C_6 |Im(s)|^\alpha$, for $s \in \mathcal{R}_2(\rho)$.*

3. Proof of Theorems 1 and 2

In this section we shall employ ideas from classical analytic number theory to prove our two main results. By Proposition 1, to prove the asymptotic formula for $\pi(T)$ in Theorems 1 and 2 it suffices to show the corresponding asymptotic formula for $\pi^*(T)$.

It will prove convenient to introduce an auxiliary function $\psi_0(T)$ defined by

$$\psi_0(T) = \sum_{e^{hr^n(x)} \leq T} hr^n(x),$$

where the summation is over all prime orbits $\{x, \sigma x, \dots, \sigma^{n-1}x\}$ satisfying the condition $e^{hr^n(x)} \leq T$. We then define inductively

$$\psi_k(T) = \int_1^T \psi_{k-1}(u)du = \frac{1}{k!} \sum_{e^{hr^n(x)} \leq T} hr^n(x) \left(T - e^{hr^n(x)}\right)^k,$$

for $k \geq 1$. We shall relate $\psi_k(T)$, $k \geq 1$, to $\eta(s)$ by applying the formula

$$\frac{1}{2\pi i} \int_{d-i\infty}^{d-i\infty} \frac{y^s}{s(s+1)\cdots(s+k)} ds = \begin{cases} 0 & \text{if } 0 < y < 1 \\ \frac{1}{k!}\left(1 - \frac{1}{y}\right)^k & \text{if } y \geq 1 \end{cases}$$

term by term to the series defining $\eta(s)$, where $d > 1$ [5, p.31]. The functions $\psi_k(T)$ and $\eta(s)$ are related by the following lemma.

Lemma 5. *For $d > 1$ we may write*

$$\psi_k(T) = \frac{1}{2\pi i} \int_{d+i\infty}^{d-i\infty} \eta(s) \frac{T^{s+k}}{s(s+1)\cdots(s+k)} ds. \tag{3.1}$$

To evaluate this integral, we shall change the contour of integration. The first step is to replace this integral by a finite integral. Write $R = R(T) = (\log T)^\epsilon$, for some fixed $0 < \epsilon < 1/\rho$, and truncate the integral in (3.1) to obtain the estimate

$$\left| \psi_k(T) - \frac{1}{2\pi i} \int_{d-iR}^{d+iR} \eta(s) \frac{T^{s+k}}{s(s+1)\cdots(s+k)} ds \right| \leq 2|\eta(d)| \frac{T^{d+k}}{R^{k+1}(\log T)}. \tag{3.2}$$

If we choose $d = 1 + \frac{1}{\log T}$, then since $\eta(d) = O((d-1)^{-1})$ we can bound the Right Hand Side of (3.2) by a term which is $O\left(\frac{\log T \cdot e \cdot T^{k+1}}{R^k \cdot \log T}\right) = O\left(\frac{T^{k+1}}{(\log T)^{k\epsilon}}\right)$.

By the residue theorem we can write

$$\begin{aligned} &\frac{1}{2\pi i} \int_{d+iR}^{d-iR} \eta(s) \frac{T^{s+k}}{s(s+1)\dots(s+k)} ds \\ &= \frac{T^{k+1}}{(k+1)!} + \frac{1}{2\pi i} \int_\Gamma \eta(s) \frac{T^{s+1}}{s(s+1)\dots(s+k)} ds, \end{aligned}$$

where Γ is the union of the line segments $[c(R)+iR, d+iR]$, $[d-iR, c(R)-iR]$ and $[c(R)-iR, c(R)+iR]$ and where $c(R) = 1 - \frac{1}{2h^{\rho+1}R^\rho}$.

Consider first the integral along the line segment $[c(R)-iR, c(R)+iR]$. We have the estimate

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{[c(R)-iR, c(R)+iR]} \eta(s) \frac{T^{s+1}}{s(s+1)\dots(s+k)} ds \right| &= O\left(T^{c(R)+k} \int_1^R t^{\alpha-(k+1)} dt \right) \\ &= O\left(T^{c(R)+k} R^{\alpha-k} \right). \end{aligned}$$

Since $R(T) = (\log T)^\epsilon$ we can bound

$$\begin{aligned} T^{c(R)+k} R^{\alpha-k} &= T^{\left[1 - \frac{1}{2h^{\rho+1}(\log T)^{\epsilon\rho}}\right] + k} (\log T)^{\epsilon(\alpha-k)} \\ &= T^{k+1} \left(e^{-\frac{\log T}{2h^{\rho+1}(\log T)^{\epsilon\rho}}} (\log T)^{\epsilon(\alpha-k)} \right). \end{aligned}$$

We see that the contribution to $\psi_k(T)$ is at most $O(T^{k+1}/(\log T)^\gamma)$, for any $\gamma > 0$, since $e^{-\frac{(\log T)^{1-K\rho}}{2h^{\rho+1}}}$ tends to zero faster than $(\log T)^{-\gamma}$.

If we consider the contours $[c(R)+iR, d+iR]$ and $[d-iR, c(R)-iR]$ then we have the bound

$$\begin{aligned} \left| \int_{[c(R)+iR, d+iR] \cup [d-iR, c(R)-iR]} \eta(s) \frac{T^{s+k}}{s(s+1)\cdots(s+k)} ds \right| &= O\left(R^{(\alpha-(k+1))} T^{d+k} \right) \\ &= O\left(\frac{T^{k+1}}{(\log T)^{(k+1-\alpha)\epsilon}} \right). \end{aligned}$$

Comparing these estimates we see that

$$\psi_k(T) = T^{k+1} + O\left(\frac{T^{k+1}}{(\log T)^{(k+1-\alpha)\epsilon}}\right).$$

Using repeatedly the inequality

$$\psi_{j-1}(T - \Delta T)\Delta T \le \psi_j(T) - \psi_j(T - \Delta T) \le \psi_{j-1}(T)\Delta T$$

where $\Delta T = T(\log T)^{-(k+1-\alpha)\epsilon/2^{k-j+1}}$ we see that

$$\psi_0(T) = T + O\left(\frac{T}{(\log T)^{\delta}}\right),$$

where $\delta = (k+1-\alpha)\epsilon/2^k$. Setting $\pi_0(T) = \sum_{e^{hr^n(x)} \le T} 1$, we see that

$$\begin{aligned} \pi_0(T) &= \int_2^T \frac{1}{\log u} d\psi_0(u) + O(1) \\ &= \frac{\psi_0(T)}{\log T} + \int_2^T \frac{\psi_0(u)}{u(\log u)^2} du. \end{aligned} \tag{3.3}$$

Recall the identity

$$\begin{aligned} \mathrm{li}(T) :&= \int_2^T \frac{1}{\log u} du \\ &= \frac{T}{\log T} + O(1) + \int_2^T \frac{du}{(\log u)^2} \end{aligned} \tag{3.4}$$

(so that, in particular, $\mathrm{li}(T) = T/\log T + O(T/(\log T)^2)$). Comparing (3.3) and (3.4) we see that

$$\begin{aligned} \pi_0(T) - \mathrm{li}(T) &= \frac{\psi_0(T) - T}{\log T} + \int_2^T \frac{\psi_0(u) - u}{(\log u)^2} du + O(1) \\ &= O\left(\frac{T}{(\log T)^{1+\delta}}\right) + O\left(\int_2^T \frac{du}{(\log u)^{2+\delta}}\right). \end{aligned}$$

Moreover, we can estimate

$$\begin{aligned} \int_2^T \frac{du}{(\log u)^{2+\delta}} &= \int_2^{T^{1/2}} \frac{du}{(\log u)^{2+\delta}} + \int_{T^{1/2}}^T \frac{du}{(\log u)^{2+\delta}} \\ &= O(T^{1/2}) + O\left(\frac{T}{(\log T)^{2+\delta}}\right) \end{aligned}$$

Thus $\pi_0(T) = \mathrm{li}(T) + O\left(\frac{T}{(\log T)^{1+\delta}}\right)$ from which the conclusion of Theorems 1 and 2 follows.

4. Proof of Proposition 2

In this section we give the proof of Proposition 2, which is used to obtain bounds on the zeta function. It is based on techniques introduced by Dolgopyat [2].

We recall the definition of the transfer operator $L_{-sr} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ given by the formula

$$L_{-sr}w(x) = \sum_{\sigma y = x} e^{-sr(y)} w(y).$$

We shall obtain estimates on the iterates L^N_{-sr}, where $N = [C \log |Im(s)|]$.

The next three technical lemmas give estimates which will be useful to us later.

Lemma 6. *(cf. [2]) Let μ be the unique equilibrium measure for $-Re(s)r$. Assume $w \in \mathcal{F}_\theta$ satisfies $|w|_\infty \le 1$ and $|w|_\theta \le |Im(s)|$ then for $0 \le n \le 2N$:*

$$|L^{2N}_{-sr}w|_\infty \le \int |L^n_{-sr}w| d\mu + O\left(||L^n_{-sr}|| \delta^{2N-n}\right) \tag{4.1}$$

for any δ chosen larger than the modulus of the second eigenvalue of $L_{-Re(s)r}$.

Proof. For $x \in X^+_A$, we can bound, for $0 \le n \le N$,

$$\begin{aligned} |L^{2N}_{-sr}w(x)| &\le L^{2N-n}_{-Re(s)r}(|L^n_{-sr}w|)(x) \\ &\le \int |L^n_{-sr}w| d\mu + O\left(||L^n_{-sr}|| \delta^{2N-n}\right). \end{aligned}$$

Recall that we are assuming that $L_{-Re(s)r}1 = e^{P(-Re(s)r)}1$. By replacing $-Re(s)r$ by $-Re(s)r - P(-Re(s)r)$ we may assume that $L_{-Re(s)r}1 = 1$. Using this new normalization, Lemma 2 becomes the following.

Lemma 7. *There exists $C_0 > 0$ such that*

$$||L^n_{-sr}w|| \le C_0 |Im(s)| |w|_\infty + \theta^n |w|_\theta, \quad \forall n \ge 0. \tag{4.2}$$

Lemma 8. *Given $\tau > 0$ there exists $\tau_0 > 0$ such that provided*

(1) *$|w|_\infty = 1$ and $|w|_\theta \le |Im(s)|$,*
(2) *there exists $x \in X^+_A$ and $0 \le n \le N$ with $|L^{2N}_{-sr}w(x)| \le 1 - \frac{1}{|Im(s)|^\tau}$,*

then

$$||L^{2N}_{-sr}w||_\infty \le 1 - \frac{1}{|Im(s)|^{\tau_0}} \tag{4.3}$$

for $|Im(s)|$ sufficiently large.

Proof. As is easily observed from (4.2), $||L^n_{-sr}|| \le (C_0+1)|Im(s)|$, for all $n \ge 0$. In particular, we see that whenever $y \in B_n(x) = \{y : d(x,y) \le \theta^n\}$, where n is chosen such that $\theta^n \le \left(2(C_0+1)|Im(s)|^{\tau+1}\right)^{-1} < \theta^{n-1}$, we have

$$\begin{aligned} |L^{2N}_{-sr}w(y)| &\le |L^{2N}_{-sr}w(x)| + (C_0+1)|Im(s)|\theta^n \\ &\le \left(1 - \frac{1}{|Im(s)|^\tau}\right) + (C_0+1)|Im(s)|\theta^n, \end{aligned} \tag{4.4}$$

for $n \geq 0$ and $w \in \mathcal{F}_\theta$ with $|w|_\infty = 1$ and $|w|_\theta \leq |Im(s)|$. Furthermore, from a standard characterization of equilibrium measures [12], there exists $D > 0$ (independent of n)

$$\mu(B_n(x)) \geq \theta^{nD} = \left(\frac{1}{2(C_0+1)|Im(s)|^{\tau+1}} \right)^D . \tag{4.5}$$

Thus we have from (4.4) and (4.5) that

$$\begin{aligned} \int |L_{-sr}^{2N} w| d\mu &\leq \int_{B_n(x)^c} |L_{-sr}^{2N} w| d\mu + \int_{B_n(x)} |L_{-sr}^{2N} w| d\mu \\ &\leq (1 - \mu(B_n(x))) + \mu(B_n(x)) \left(1 - \frac{1}{2|Im(s)|^\tau} \right) \\ &\leq 1 - \frac{1}{2^{1+D}(C_0+1)^D |Im(s)|^{\tau(1+D)+D}} \end{aligned} \tag{4.7}$$

for $|Im(s)|$ sufficiently large. Thus comparing (4.1) and (4.7) we see that

$$\begin{aligned} |L_{-sr}^{2N} w|_\infty &\leq \left(1 - \frac{1}{2^{1+D}(C_0+1)^D |Im(s)|^{\tau(1+D)+D}} \right) + O\left(|Im(s)|\delta^N\right) \\ &\leq \left(1 - \frac{1}{2^{1+D}(C_0+1)^D |Im(s)|^{\tau(1+D)+D}} \right) + O\left(|Im(s)|^{1+C\log\delta}\right) \\ &\leq 1 - \frac{1}{|Im(s)|^{\tau_0}} \end{aligned} \tag{4.8}$$

for $|Im(s)|$ sufficiently large, where we choose $\tau_0 > \min\{\tau(1+D)+D, C|\log\delta|-1\}$. Since we can assume that $C > 0$ has been chosen sufficiently large that $C|\log\delta| > 1$, this completes the proof of Lemma 8.

After these preliminary considerations, we can now move onto the proof of Proposition 2. With the earlier reductions it suffices to show that (under the hypotheses of Theorem 1 or Theorem 2) there exist $\tau > 0$, and $C > 0$ such that for $m \geq 1$,

$$||L_{-sr}^{2mN}|| \leq C_0 |Im(s)| \left(1 - \frac{1}{|Im(s)|^\tau} \right)^{m-1}, \tag{4.9}$$

where $|Im(s)|$ is sufficiently large. Observe that the negation of (4.9) is that for all $\tau > |r|_\infty / |\log\delta|$, and $C > 0$ there exist sequences s_k with $|Im(s_k)| \to +\infty$ and $m_k \to +\infty$ such that

$$||L_{-s_k r}^{m_k N}|| > 2C_0 |Im(s_k)| \left(1 - \frac{1}{|Im(s_k)|^\tau} \right)^{m_k - 1}. \tag{4.10}$$

The proof of Proposition 2 is by contradiction. The first step is to show that if (4.10) holds then there exist

(1) $\tau > \max\{2, |r|_\infty / |\log\delta|\}$,
(2) $1/|\log\delta| < C < \frac{\tau}{|r|_\infty}$,
(3) $s_k \in \mathbb{C}$ with $|Im(s_k)| \to +\infty$,
(4) $w_k \in \mathcal{F}_\theta$, with $|w_k|_\infty = 1$ and $|w_k|_\theta \leq Im(s_k)$

such that for all $0 \le n \le N$,

$$\inf_{x \in X_A^+} |L_{-s_k r}^n w_k(x)| \ge 1 - \frac{1}{|Im(s_k)|^\tau}. \tag{4.11}$$

The second step will be to show that (4.11) is incompatible with the hypothesis that $\phi_t : \Lambda \to \Lambda$ is either a transitive Anosov flow or a hyperbolic flow satisfying the approximability condition.

We start with the first step. Let us assume for a contradiction that (4.11) is false. In particular, we then have for all $\tau > |r|_\infty / |\log \delta|$ and $C \in (1/|\log \delta|, \tau/|r|_\infty)$, for all sufficiently large $|Im(s)|$ and functions $w \in \mathcal{F}_\theta$ with $|w|_\infty = 1$ and $|w|_\theta \le |Im(s)|$ there exists $0 \le n \le N$ and $x \in X_A^+$ such that

$$|L_{-sr}^n w(x)| \le 1 - \frac{1}{|Im(s)|^\tau}.$$

By applying Lemma 7 and Lemma 8 we obtain the following norm estimate. For any $m \ge 1$,

$$\begin{aligned} ||L_{-sr}^{2Nm} w|| &\le C_0 |Im(s)| |L_{-sr}^{2N(m-1)} w|_\infty + \theta^{2N(m-1)} |L_{-sr}^{2N} w|_\theta \\ &\le C_0 |Im(s)| \left(\left(1 - \frac{1}{|Im(s)|^\tau} \right)^{(m-1)} + \theta^{2N(m-1)} \right) \\ &\le C_0 |Im(s)| \left(1 - \frac{1}{|Im(s)|^{\tau_1}} \right)^{(m-1)}, \end{aligned}$$

for any $\tau_1 > \max\{C \log 4, \tau\}$, and we assume that $|Im(s)|$ is sufficiently large. However, with the choices $s = s_k$ and $m = m_k$ this contradicts (4.10) and this completes the proof of the first step.

For the second step in the proof of Proposition 2 we take as our hypothesis that (4.11) holds. Set $n_k = [\log |Im(s_k)|]$ and write

$$\begin{aligned} &w_k(x) = R_0(x) e^{i\theta_0(x)}, \\ &L_{-s_k r}^{n_k} w_k(x) = R_1(x) e^{i\theta_1(x)}, \text{ and} \\ &L_{-s_k r}^{2n_k} w_k(x) = R_2(x) e^{i\theta_2(x)}, \end{aligned}$$

where R_0, R_1, R_2 are the moduli of these functions, and $\theta_0, \theta_1, \theta_2$ are the arguments.

Set $\tau' = \tau - C|r|_\infty > 0$. We claim that whenever $\sigma^{n_k} y = x$ then

$$\exp(i\Theta_1(y,x)) = 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}} \right) \text{ and } \exp(i\Theta_2(y,x)) = 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}} \right), \tag{4.12}$$

where we denote

$$\Theta_1(y,x) = Im(s_k) r^{n_k}(y) - \theta_1(x) + \theta_0(y),$$

$$\Theta_2(y,x) = Im(s_k) r^{n_k}(y) - \theta_2(x) + \theta_1(y).$$

Since we are assuming $L^{n_k}_{-Re(s_k)r}1 = 1$ we can write

$$\begin{aligned}
&\sum_{\sigma^{n_k} y_1 = x} e^{-Re(s_k) r^{n_k}(y_1)} \left(1 - \exp\left(-i\Theta_1(y_1, x)\right) R_0(y_1)\right) \\
&= 1 - e^{-i\theta_1(x)} L^{n_k}_{-s_k r} w_k(x) \\
&= 1 - R_1(x).
\end{aligned} \tag{4.13}$$

Since by estimate (4.12) we can bound $1 - R_1(x) = O\left(1/|Im(s_k)|^\tau\right)$, we can estimate from (4.13) that for each $\sigma^{n_k} y_1 = x$ we have that

$$\begin{aligned}
1 - \exp\left(-i\Theta_1(y_1, x)\right) R_0(y_1) &= O\left(\frac{e^{n_k |r|_\infty}}{|Im(s_k)|^\tau}\right) \\
&= O\left(|Im(s_k)|^{C|r|_\infty} \frac{1}{|Im(s_k)|^\tau}\right) = O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right).
\end{aligned}$$

This proves the first part of (4.12). The second part follows similarly.

We want to show that (4.12) is inconsistent with either of the hypotheses on ϕ_t contained in Theorem 1 or Theorem 2.

The case of Anosov flows. Consider first the case of a transitive Anosov flow. We begin by making choices such that $\sigma^{n_k} y_0 = \sigma^{n_k} y_1 = x$ and $\sigma^{n_k} y_2 = \sigma^{n_k} y_3 = z$ where $d(y_0, y_2) = \theta^{n_k}$ and $d(y_1, y_3) = \theta^{n_k}$. We shall use Ξ to denote the set of values

$$\Delta(y_0, y_1, y_2, y_3) = r^{n_k}(y_0) + r^{n_k}(y_3) - r^{n_k}(y_1) - r^{n_k}(y_2),$$

where y_0, y_1, y_2, y_3 range over the above choices.

This corresponds to the following geometric situation. The points y_0, y_1, y_2 and y_3 correspond to points ξ_0, ξ_1, ξ_2 and ξ_3 in M which lie on the Markov sections $\coprod_{i=1}^k T_i$. Furthermore, each of the pairs ξ_0 ,ξ_2 and ξ_0, ξ_2 lie on the same section. Let $t = t(x, y) \in [-\epsilon, \epsilon]$ denote the unique value such that $W^{ss}_\epsilon(x) \cap W^{su}_\epsilon(\phi_t y) = [x, y]$. Then Δ takes the form $\Delta(y_0, y_1, y_2, y_3) = t(\xi_0, \xi_1) + t(\xi_2, \xi_3)$. It is not difficult to see, by continuity of the stable and unstable manifolds, that Ξ contains an interval. However, we shall show that if (4.11) holds then we obtain a contradiction.

From (4.11) we see that:

$$\begin{aligned}
\exp\left(i\Theta_1(y_0, x)\right) &= 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right), \\
\exp\left(i\Theta_2(y_2, z))\right) &= 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right), \\
\exp\left(i\Theta_1(y_1, x)\right) &= 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right) \text{ and} \\
\exp\left(i\Theta_2(y_3, z)\right) &= 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right).
\end{aligned}$$

Taking ratios of the first pair of expressions we see that

$$\exp\left(i(\Theta_1(y_0, x) - \Theta_2(y_2, z))\right) = 1 + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right)$$

and taking ratios of the second pair of expressions we see that

$$\exp\left(i(\Theta_1(y_1,x)-\Theta_2(y_3,z))\right)=1+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right).$$

Taking further ratios we get

$$\begin{aligned}&\exp\left(i(\Theta_1(y_0,x)-\Theta_2(y_2,z))-i(\Theta_1(y_1,x)-\Theta_2(y_3,z))\right)\\&=\exp\left(iIm(s_k)\Delta(y_0,y_1,y_2,y_3)\right)\\&=1+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right)\end{aligned}$$

In particular, we can assume that for $|Im(s_k)|$ sufficiently large,

$$\Xi\subset\bigcup_{n\in\mathbb{Z}}\left[\frac{2\pi n}{Im(s_k)}-\frac{1}{|Im(s_k)|^{\tau'+1}},\frac{2\pi n}{Im(s_k)}+\frac{1}{|Im(s_k)|^{\tau'+1}}\right]$$

Since $\tau'>0$ this is an obvious contradiction to Ξ containing an interval (or even being a set of non-zero measure). This completes the proof of Proposition 2 when the flow is Anosov.

Remark. Observe that in (4.11) τ can be chosen to be any value strictly larger than $|r|_\infty/|\log\delta|$. However, the smaller the value of τ we choose, the larger the analytic extension we have and consequently the smaller the error term we obtain for $\pi(T)$.

The case of hyperbolic flows with the approximability condition. Assume now that the flow is merely hyperbolic, but in addition satisfies the approximability condition.

Fix a value $x_0\in X_A^+$ and choose $y\in X_A^+$ sufficiently close that

$$|\theta_1(x_0)-\theta_1(y)|=O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right)\text{ and }|\theta_0(x_0)-\theta_0(y)|=O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right).$$

In particular, we have that

$$\theta_1(y)-\theta_0(y)=\Upsilon+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right),\tag{4.14}$$

where $\Upsilon:=\theta_1(x_0)-\theta_0(x_0)$. Providing n_k is sufficiently large, for any $x\in X_A^+$ we may choose a point $y\in\sigma^{-n_k}x$ satisfying (4.14). By taking the ratio of the two expressions in (4.12) we see that

$$\begin{aligned}1&=\exp\left(i\left((\theta_2(x)-\theta_1(x))-(\theta_1(y)-\theta_0(y))\right)\right)+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right)\\&=\exp\left(i\left((\theta_2(x)-\theta_1(x))-\Upsilon\right)\right)+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right),\end{aligned}$$

where for the last line we have used (4.14). Thus for any $x\in X_A^+$,

$$\begin{aligned}&\exp\left(i\left(Im(s_k)r^{n_k}(x)+\theta_1(x)-\theta_1(\sigma^{n_k}x)\right)\right)\\&=\exp\left(i\left(Im(s_k)r^{n_k}(x)+\theta_1(x)-\theta_2(\sigma^{n_k}x)+\Upsilon\right)\right)+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right)\\&=e^{i\Upsilon}+O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right),\end{aligned}\tag{4.15}$$

where for the last line we have used the second expression in (4.12) (with x substituted for y and $\sigma^{n_k}x$ substituted for x).

Let γ_1, γ_2 and γ_3 be the three distinct periodic orbits given by the approximablity condition. Using symbolic dynamics we can associate periodic orbits $\{x_1, \sigma x_1, \dots, \sigma^{n-1}x_1\}$ and $\{x_2, \sigma x_2, \dots, \sigma^{m-1}x_2\}$ for the subshift of finite type such that $l(\gamma_1) = r^n(x_1)$, $l(\gamma_2) = r^m(x_2)$ and $l(\gamma_3) = r^p(x_3)$. Let us assume for convenience that $\sigma x_1 = x_1$, $\sigma x_2 = x_2$ and $\sigma x_3 = x_3$(the general case being similar). Substituting these values of x into (4.15) we see that

$$Im(s_k)n_k l(\gamma_1) = Im(s_k)r^{n_k}(x_1) = \Upsilon + 2\pi p_k + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right),$$
$$Im(s_k)n_k l(\gamma_2) = Im(s_k)r^{n_k}(x_1) = \Upsilon + 2\pi q_k + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right), \text{ and}$$
$$Im(s_k)n_k l(\gamma_3) = Im(s_k)r^{n_k}(x_3) = \Upsilon + 2\pi r_k + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right),$$

for some p_k, q_k and $r_k \in \mathbb{Z}$, and taking differences gives the estimates

$$Im(s_k)n_k(l(\gamma_1) - l(\gamma_2)) = 2\pi(p_k - q_k) + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right), \text{ and}$$
$$Im(s_k)n_k(l(\gamma_2) - l(\gamma_3)) = 2\pi(q_k - r_k) + O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right).$$

In particular, we can conclude that $p_k' = p_k - q_k, q_k' = q_k - r_k \in \mathbb{Z}$ are solutions to

$$\frac{l(\gamma_1) - l(\gamma_2)}{l(\gamma_2) - l(\gamma_3)} - \frac{p_{k'}}{q_{k'}} = O\left(\frac{1}{|Im(s_k)|^{\tau'}}\right) = O\left(\frac{1}{q_{k'}^{\tau'}}\right).$$

Recall that we could choose τ' arbitrarily large. In particular, if we chose $\tau' > \alpha$ then this contradicts the approximability condition and so completes the proof.

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