# PSEUDO-ANOSOV FOLIATIONS ON PERIODIC SURFACES

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## 0. Introduction

In this note we shall study the lifts of stable foliations of pseudo-Anosov diffeomorphism to certain infinite abelian covers. This is motivated, at least in part, by recent progress in understanding the ergodic properties of the analogous horocycle flows on infinite surfaces [3],[13]. Our aim is to show that many of the results from that context hold in this natural and technically simpler setting.

It has long been appreciated that the stable foliation of an Anosov diffeomorphism of a compact manifold and the horocycle foliation of the unit tangent bundle of a compact negatively curved surface have many similar properties, although in the former case the technical analysis is often more straightforward. In particular, it is a classical fact that the horocycle foliation is uniquely ergodic [10] and, as shown by Bowen and Marcus, this property also holds for the stable foliation of an Anosov (or Axiom A) diffeomorphism [6]. Furthermore, this property extends to the broader class of pseudo-Anosov maps of surfaces. Let us recall the following result of Thurston.

**Theorem (Thurston)** [20]**.** *A pseudo-Anosov homeoeomorphism $f : M \to M$ satisfies the following:*

(i) *the stable foliation $\mathcal{W}^s$ for $f : M \to M$ is uniquely ergodic (i.e., there exists a unique transverse measure $\mu^s$, up to multiplication by a non-zero scalar);*
(ii) *the transverse measure satisfies $f_* \mu^s = e^{h(f)} \mu^s$, where $h(f) > 0$ denotes the topological entropy of $f$.*

Any non-periodic irreducible diffeomorphism of a compact surface $M$ (possibly with boundary) with negative Euler characteristic is isotopic to a pseudo-Anosov map. There is a particularly nice account, including a treatment of the proof of this theorem using symbolic dynamics, in [4]. It is natural to seek extensions of these results in the setting of infinite volume surfaces. We shall consider the particular case of $\mathbb{Z}^d$-covers. In this direction, it is a relatively recent result, due to Babillot and Ledrappier, that the horocycle foliation is ergodic but, remarkably, *not* uniquely ergodic on $\mathbb{Z}^d$-covers of compact surfaces [3]. In this paper we prove analogous results for $\mathbb{Z}^d$-covers of suitable pseudo-Anosov maps. However, in this context the proofs are technically easier.

We shall consider pseudo-Anosov homeomorphisms satsifying the three hypotheses given below. Suppose that $M$ is a compact surface with boundary components $\partial M_1, \cdots, \partial M_l$ and that $f : M \to M$ is a pseudo-Anosov homeomorphism. Then

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the boundary components are obviously permuted under $f$. Let $N$ be the compact surface that comes from $M$ by collapsing a subset of the boundary components to periodic points. Let $\overline{f} : N \to N$ denote the associated homeomorphism on $N$. We shall need to make the following hypothesis.

*Hypothesis I.* The map $\overline{f}$ is isotopic to the identity.

In the original context of horocycle flows no such hypothesis was required, since flows are isotopic to the identity. We shall restrict ourselves to $\mathbb{Z}^d$-covers $\widehat{M}$ which arise as follows. Let $p : M \to N$ be the natural map. We can identify $H_1(N, \mathbb{R})$ with $H_1(M, \mathbb{R})/\ker p_*$, where $p_* : H_1(M, \mathbb{R}) \to H_1(N, \mathbb{R})$ is the induced linear map. Let $\widehat{M}$ be a covering surface for $M$ with covering group $\Gamma$.

*Hypothesis II.* Let $\Gamma \cong \mathbb{Z}^d$ (with $1 \leq d \leq \dim H_1(N, \mathbb{R})$) be a subgroup of $H_1(M, \mathbb{R})$ such that $\Gamma \cap \ker(p_*) = \{0\}$.

This is equivalent to $p_*\Gamma$ being isomorphic to $\Gamma$. Hypothesis II guarantees that there exists a lift $\widehat{f} : \widehat{M} \to \widehat{M}$ (see Lemma 2.6 below). Let $\widehat{\mathcal{W}^s}$ be the lift of the foliation $\mathcal{W}^s$ from $M$ to $\widehat{M}$. In particular, the leaves $\widehat{W}^s(x)$ are related by the covering group $\Gamma \cong \mathbb{Z}^d$, i.e., $\gamma\widehat{W^s(x)} = \widehat{W^s(\gamma x)}$, where $\gamma \in \Gamma$.

To each periodic orbit $\{x, f(x), \cdots, f^{n-1}(x)\}$ for $f$ in the interior of $M$, we may associate an element $\gamma_x \in \Gamma$ by the formula $\widehat{f}^{n-1}(\widehat{x}) = \gamma_x\widehat{x}$, where $\widehat{x}$ is a lift of $x$ to $\widehat{M}$. The set $\{\gamma_x : f^n x = x\}$ spans $\Gamma$ [10].

*Hypothesis III.* There exists an $f$-periodic point $x$ such that $\gamma_x = 0$.

Let $\widehat{\mu}^s$ be the lift of the transverse measure $\mu^s$, i.e., we consider the measures on sufficiently small transverse sections for $\mathcal{W}^s$ associated to $\mu^s$ and lift the measure on a transverse section to a corresponding section $\widehat{\mathcal{W}^s}$.

**Theorem A.** *Assume that Hypotheses I, II and III hold. Then the transverse measure $\widehat{\mu}^s$ is ergodic and satisfies $\widehat{f}_*\widehat{\mu}^s = e^{h(f)}\widehat{\mu}^s$.*

By ergodicity of the foliation we understand that any Borel set on a transverse section which is the union of orbits under the holonomy either has zero measure or its complement has zero measure. There is a corresponding result for a transverse measure $\widehat{\mu}^u$ to the unstable foliation $\widehat{\mathcal{W}^u}$. One of the main ingredients in the proof is the use of the homology of periodic orbits in the mapping torus for $f : M \to M$. In section 1 we recall some basic properties of pseudo-Anosov diffeomorphisms and their relation with homology. In section 2 we describe the symbolic dynamics for $f$ and for $\widehat{f}$. In section 3 we show how this approach can be used to give a proof of Theorem A. In section 4 we show that there are other transverse measures. In section 5 we describe some ergodic theorems for $\widehat{\mu}^s$.

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## 1. Pseudo-Anosov Maps

We begin by recalling some basic definitions. We shall consider measurable foliations of a compact surface $M$ (possibly with boundary) which are genuine foliations except at a finite set of singularities. Given such a foliation, a holonomy map between two transverse sections is a map which takes points to images which lie in the same leaf. A transverse measure is a family of measures on transverse sections which is preserved by holonomy.

*Definition.* A homeomorphism $f : M \to M$ is called pseudo-Anosov if there exist two foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, which are transverse except at the singularities and on the boundary, equipped with transverse measures $\mu^s$, $\mu^u$ such that $f$ preserves the foliations and

(a) $f_*\mu^s = e^{h(f)}\mu^s$,
(b) $f_*\mu^u = e^{-h(f)}\mu^u$.

In fact, the measures $\mu^s$ and $\mu^u$ are unique, up to a scalar [4],[20].

The classification of the singularities is completely understood [20], [9].

Let us consider some simple examples to which Theorem A applies.

*Example 1.* The following construction is described in [14]. Take a diffeomorphism $\overline{f} : N \to N$ of a torus $N = \mathbb{T}^2$ with 3 periodic points whose rotation vectors $\rho_1, \rho_2, \rho_3$ span $H_1(N, \mathbb{R})$. Using a standard method we may blow up the periodic points to obtain a map $f_0 : M \to M$ on a surface $M$ with genus one and three boundary components. More precisely, we remove each periodic point $x$ and replace it with a (small) boundary circle $C_x$. Away from these circles, $f_0$ acts as $\overline{f}$, while on the circles it acts by the projective action of the derivative. It was shown in [14] that $f_0$ is isotopic to a pseudo-Anosov map $f : M \to M$. Corresponding to a $\mathbb{Z}$-cover of $N$ there will be a $\mathbb{Z}$-cover $\widehat{M}$ of $M$. In particular, $\widehat{M}$ is an infinite cylinder with periodic holes (Figure 1). There is an analogous construction for $\mathbb{Z}^2$-covers. There are similar constructions on higher genus surfaces [16], [12].

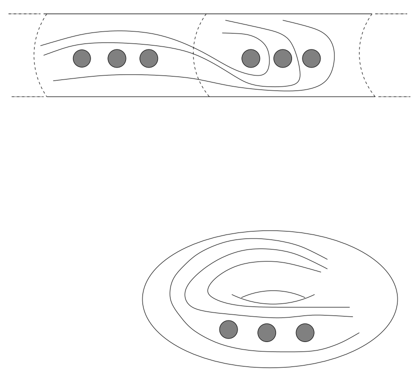

FIGURE 1

*Example 2.* (This example is not quite covered by Hypotheses I and II, since $\ker p_* = h_1(M, \mathbb{R})$. Nevertheless, it provides a good illustration of our results.) Let $M$ be a disk with three holes. We can associate to this a $\mathbb{Z}$-cover $\widehat{M}$, as in the Figure 2 (cf. [7]). Let $f : M \to M$ be a pseudo-Anosov homeomorphism. In particular, if $N$ is the disk that corresponds to collapsing the three holes then the resulting map $\overline{f} : N \to N$ must necessarily be isotopic to the identity on the disk.

## 2. Symbolic dynamics

A particularly useful feature of pseudo-Anosov maps is that they admit a description in terms of symbolic dynamics which, in turn, is based on the use of Markov partitions. This has the advantage of giving a symbolic characterization of transverse measures which is central to our proof. In order to define Markov partitions,

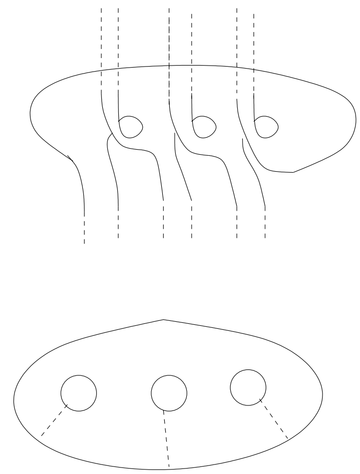

FIGURE 2

we first recall the definitions of the stable and unstable manifolds. Given $x \in M$ we denote

$$W^s(x) = \{y \in M : d(f^n x, f^n y) \to 0 \text{ as } n \to +\infty\}$$

and

$$W^u(x) = \{y \in M : d(f^{-n} x, f^{-n} y) \to 0 \text{ as } n \to +\infty\}.$$

These sets form the leaves of the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ introduced in the preceding section, i.e., $\mathcal{W}^s = \{W^s(x)\}$ and $\mathcal{W}^u = \{W^u(x)\}$. Since we are considering measurable foliations, we need not discuss the complication of the zero measure set of points that lie on the boundary and other singular points.

**Lemma 2.1.** *The foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ have one dimensional transverse leaves except on a set of zero measure.*

There are local versions of the stable and unstable manifolds. Let $\epsilon > 0$ be sufficiently small. Let

$$W^s_\epsilon(x) = \{y \in M : d(f^n x, f^n y) \le \epsilon \text{ for } n \ge 0\}$$

and

$$W^u_\epsilon(x) = \{y \in M : d(f^{-n} x, f^{-n} y) \le \epsilon \text{ for } n \ge 0\}.$$

These are embedded one dimensional disks except at the boundaries or at singularities. One can proceed with the construction of a Markov partition $\mathcal{T} = \{T_1, \dots, T_k\}$ (the interiors of each element not containing any singularities) by analogy with the usual approach for Anosov diffeomorphisms. Here, for $x, y \in T_i$, $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ is a single point, which we denote by $[x, y]$. Each element can be written in the form $T_i = \overline{[U_i, S_i]}$, for $i = 1, \dots, k$, where $U_i$ and $S_i$ are open intervals in local unstable and stable manifolds and $\overline{[U_i, S_i]}$ denotes the closure of $[U_i, S_i]$.

The family $\mathcal{T}$ consists of closed sets such that

(1) $M = \cup_{i=1}^k T_i$;
(2) $T_i = \overline{\text{int}(T_i)}$, for $1 \le i \le k$;
(3) if $x \in \text{int}(T_i)$ and $fx \in \text{int}(T_j)$ then $f([x, S_i]) \subset [Tx, S_j]$; and
(4) if $x \in \text{int}(T_i)$ and $f^{-1}x \in \text{int}(T_l)$ then $f^{-1}([U_l, x]) \subset [U_l, x]$.

We have assumed that $U$ and $S$ to be open to avoid complications with the singularities and boundary components of $M$, which lie in the boundaries of elements of the partition.

**Lemma 2.2.** *Every pseudo-Anosov homeomorphism has a Markov partition.*

*Proof.* This was essentially proved in [9] by modifying the original construction of Adler and Weiss for Anosov diffeomorphisms on surfaces. Only minor modifications are required to deal with boundary components. □

We can associate to $\mathcal{T}$ a $k \times k$ matrix $A$ with entries

$$A(i,j) = \begin{cases} 1 & \text{if } f^{-1}(\mathrm{int}(T_j)) \cap \mathrm{int}(T_i) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.3** [9, p.205]**.** *$A$ is aperiodic (i.e., there exists $m \geq 1$ such that $A^m(i,j) > 0$, for $i, j = 1, \ldots, k$).*

We use the matrix $A$ to define a shift space $\Sigma$ by

$$\Sigma = \left\{ \underline{x} = (x_n) \in \prod_{n\in\mathbb{Z}} \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1, \text{ for all } n \in \mathbb{Z} \right\}.$$

We equip this with the metric

$$d(\underline{x}, \underline{y}) = \sum_{n\in\mathbb{Z}} \frac{1}{2^{|n|}} (1 - \delta_{x_n, y_n}),$$

where $\delta_{i,j}$ is the usual Kronecker delta. The shift map $\sigma : \Sigma \to \Sigma$ defined by $(\sigma\underline{x})_n = x_{n+1}$ is a homeomorphism. There is a unique measure of maximal entropy $\nu$ for $\sigma$ and there is a unique measure of maximal entropy $\mu$ for $f$ (cf. [4]).

**Lemma 2.4.** *The map $\pi : \Sigma \to M$ given by $\pi(\underline{x}) = \cap_{n\in\mathbb{Z}} f^{-n}(T_{x_n})$ is well defined $\nu$ almost everywhere. Furthermore, $\pi : \Sigma \to M$ is an isomorphism, i.e., $\pi$ is almost everywhere one-to-one, $\pi \circ \sigma = f \circ \pi$ and $\pi_* \nu = \mu$ (cf. [4]).*

Due to the presence of boundary components for $M$, the map $\pi$ is not well defined on a set of zero measure. However, since we are only interested in ergodic properties, this does not present a serious problem. The local product structure on each $T_i$, $i = 1, \cdots, k$, (and the corresponding for $\nu$ on $\Sigma$) allows us to write $\mu = \mu^s \times \mu^u$ locally. To study the foliations, we need to consider a corresponding one-sided shift. Let

$$\Sigma^+ = \left\{ \underline{x} = (x_n) \in \prod_{n\in\mathbb{Z}^+} \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}^+ \right\}$$

and let $\sigma : \Sigma^+ \to \Sigma^+$ be the shift map defined by $(\sigma\underline{x})_n = x_{n+1}$. This is a local homeomorphism. Let $\nu$ also denote the measure of maximal entropy for $\sigma : \Sigma^+ \to \Sigma^+$. The natural map $\pi : \Sigma^+ \to \cup_{i=1}^k U_i$ is well defined $\nu$ almost everywhere and the measure $\nu$ corresponds to a measure $\mu^s$, say, on $\cup_{i=1}^k U_i$.

*Definition.* We define an equivalence relation on $\Sigma^+$ by $\underline{x} \sim \underline{y}$ when there exists $N \geq 0$ such that $x_i = y_i$ for $i \geq N$.

**Lemma 2.5.** *For almost all ($\nu$) $\underline{x} \in \Sigma$, if $\underline{y} \sim \underline{x}$ then $\pi(\underline{x})$ and $\pi(\underline{y})$ lie on the same leaf of $\mathcal{W}^s$. Conversely, for almost all ($\mu^u$) $x \in \cup_i U_i$, if $y$ lies on the same stable leaf as $x$ then there are sequences $\underline{x}, \underline{y} \in \Sigma^+$ with $\pi(\underline{x}) = x$, $\pi(\underline{y}) = y$ which satisfy $\underline{x} \sim \underline{y}$.*

*Proof.* This is a simple modification of the proof of Ruelle and Sullivan in the case of Axiom A diffeomorphisms [19]. If two points $x, y \in \cup_i U_i$ lie on the same stable manifold then either $f^n x, f^n y$ lie in the same elements of the Markov partition $T_{x_n}$, say, for $n$ sufficiently large, or the distance of both $f^n x$ and $f^n y$ from the boundaries $\cup_{i=1}^k \partial T_i$ tend to zero. However, the latter case only holds for sets of zero measure, from which the conclusion follows. □

Whenever $x_0, \ldots, x_{n-1}$ is an admissible word (i.e. a finite sequence such that $A(x_i, x_{i+1}) = 1$, $i = 0, \ldots, n-2$), we define an $n$-cylinder by

$$[x_0, \ldots, x_{n-1}] = \{\underline{y} \in \Sigma^+ : x_i = y_i, 0 \le i \le n-1\}.$$

We say that a probability measure $m$ on $\Sigma^+$ is preserved by $\sim$ if

$$m([x_0, \ldots, x_{n-1}]) = m([y_0, \ldots, y_{n-1}])$$

whenever $x_{n-1} = y_{n-1}$. It is easy to see that there is a unique probability measure $\nu$ on $\Sigma^+$ satisfying this property and that this corresponds to a unique transverse measure for $\mathcal{W}^s$ (up to multiplication by a scalar). Moreover, $\nu$ is precisely the measure of maximal entropy on $\Sigma^+$ [6].

**Lemma 2.6.** *We can lift the map $f : M \to M$ to a diffeomorphism $\widehat{f} : \widehat{M} \to \widehat{M}$.*

*Proof.* The map $p : M \to N$ corresponds to collapsing certain of the boundary components to points, and the map $\overline{f} : N \to N$ lifts to a map $\widehat{\overline{f}} : \widehat{N} \to \widehat{N}$, since $\overline{f}$ is isotopic to the identity. However, since $p$ is defined locally we can consider a locally defined inverse operation which reinserts copies of the boundary components back into $\overline{N}$. This allows us to define $\widehat{f} : \widehat{M} \to \widehat{M}$. □

We can lift the Markov partition $\{T_i\}_{i=1}^k$ from $M$ to $\widehat{M}$ to give rectangles

$$\{T_{i,a} := T_i \times \{a\} : 1 \le i \le k,\ a \in \mathbb{Z}^d\}.$$

This partition of $\widehat{M}$ also has the Markov property with respect to the lifted map $\widehat{f}$. Providing the original Markov partition was sufficiently small, this allows us to define a function $g : \Sigma \to \mathbb{Z}^d$ such that $g(x)$ depends only on $(x_0, x_1)$ and so that whenever $A(i,j) = 1$ then

$$\mathrm{int}(T_{i,a}) \cap \widehat{f}^{-1}(\mathrm{int}(T_{j,a+g(i,j)})) \ne \emptyset.$$

If $\sigma^n \underline{x} = \underline{x}$ then we refer to $g^n(\underline{x}) := g(\underline{x}) + g(\sigma \underline{x}) + \cdots + g(\sigma^{n-1}\underline{x}) \in \mathbb{Z}^d$ as the $g$-weighting of the periodic orbit. A particular consequence of the definition is that, if $f^n x = x$, $x = \pi(\underline{x})$, then $g^n(\underline{x})$ is the element of $\mathbb{Z}^d$ corresponding to $\gamma_x \in \Gamma$.

We can define a skew product $\widehat{\sigma} : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ by

$$\widehat{\sigma}(\underline{x}, a) = (\sigma \underline{x}, a + g(x_0, x_1)).$$

This is a homeomorphism. We can define $\widehat{\pi} : \Sigma^+ \times \mathbb{Z}^d \to \cup_i U_i \times \mathbb{Z}^d$ by $\widehat{\pi}(\underline{x}, a) = (\pi(\underline{x}), a)$, a.e. By analogy with the approach for the case of compact surfaces, we can reduce the classification of transverse measures for $\widehat{\mathcal{W}}^s$ to the following.

*Definition.* We define an equivalence relation on $\Sigma^+ \times \mathbb{Z}^d$ by $(\underline{x}, a) \sim (\underline{y}, b)$ when:

(1) there exists $N \geq 0$ such that $x_i = y_i$ for $i \geq N$; and
(2) $a + g^N(\underline{x}) = b + g^N(\underline{y})$.

Assume that $\underline{x} \sim \underline{y}$ and, in particular, $\sigma^n \underline{x} = \sigma^n \underline{y}$. Then

$$(\underline{x}, a) \sim \left( \underline{y}, a + \sum_{i=0}^{n-1} (g(x_i) - g(y_i)) \right)$$

We observe the following trivial result.

**Lemma 2.7.** *We can write that $(\underline{x}, a) \sim (\underline{y}, b)$ if and only if there exists $n \geq 0$ such that $\widehat{\sigma}^n(\underline{x}, a) = \widehat{\sigma}^n(\underline{y}, b)$.*

We say that a measure $m$ on $\Sigma^+ \times \mathbb{Z}^d$ is preserved by $\sim$ if

$$m([x_0, \ldots, x_{n-1}] \times \{a\}) = m\left( [y_0, \ldots, y_{n-1}] \times \left\{ a + \sum_{i=1}^{n-1} (g(x_i) - g(y_i)) \right\} \right)$$

for any cylinders $[x_0, \ldots, x_{n-1}]$ and $[y_0, \ldots, y_{n-1}]$, with $x_{n-1} = y_{n-1}$. Such measures will be infinite. The following is the analogue of Lemma 2.5

**Lemma 2.8.** *For almost all ($\nu$), $\underline{x} \in \Sigma$, if $(\underline{y}, a) \sim (\underline{x}, b)$ then $\widehat{\pi}(\underline{x}, a)$ and $\widehat{\pi}(\underline{y}, b)$ lie on the same leaf of $\widehat{\mathcal{W}}^s$. Conversely, for almost all ($\mu^u$) $x \in \cup_i U_i$, if $y$ lies on the same stable leaf as $x$ then there are points $(\underline{x}, a), (\underline{y}, b) \in \Sigma^+ \times \mathbb{Z}^d$ with $\widehat{\pi}(\underline{x}, a) = x$, $\widehat{\pi}(\underline{y}, b) = y$ which satisfy $(\underline{x}, a) \sim (\underline{y}, b)$.*

*Proof.* This is very similar to the proof of Lemma 2.5. In the present case we use instead that $(\underline{x}, a) \sim (\underline{y}, b)$ if and only if $\widehat{\sigma}^n(\underline{x}, a) = \widehat{\sigma}^n(\underline{y}, b)$, for some $n \geq 0$. However, the result again follows from properties of the Markov partition. More precisely, if two points $x, y \in \cup_i U_i \times \mathbb{Z}^d$ lie on the same stable manifold then either $f^n x, f^n y$ lie in the same elements of the Markov partition $T_{x_n} \times \{z_n\}$, say, for $n$ sufficiently large, or the distance of both $\widehat{f}^n x, \widehat{f}^n y$ from the boundaries $\cup_i \partial T_i \times \mathbb{Z}^d$ tends to zero. However, as before, the latter case only holds for sets of zero measure, from which the conclusion follows. □

Let us denote by $\widehat{\nu}$ the lift of $\nu$ to $\Sigma^+ \times \mathbb{Z}^d$ using the projection onto the first coordinate, i.e., $\widehat{\nu}([x_0, \ldots, x_{n-1}] \times \{a\}) = \nu([x_0, \ldots, x_{n-1}])$.

## 3. Proof of Theorem A

The results of the preceding section, particularly Lemma 2.8, allow us to reduce the proof of Theorem A to showing the following symbolic version.

**Proposition 3.1.** *The equivalence relation $\sim$ is ergodic with respect to $\widehat{\nu}$, i.e., if $B \subset \Sigma^+ \times \mathbb{Z}^d$ is a Borel set which is the union of equivalence classes then either $B$ or its complement has zero measure.*

The simplest approach to proving this result is the direct use of [2, Theorem 2.1], which implies ergodicity of the equivalence relation once we can establish that $g : \Sigma \to \mathbb{Z}^d$ is aperiodic i.e., that there is no non-trivial (Hölder) function $\phi : \Sigma \to \mathbb{C}^d$ such that

$$\phi(\sigma \underline{x}) = \phi(\underline{x}) e^{i\theta} e^{2\pi i \langle a, g(\underline{x})\rangle} \tag{3.1}$$

with $a \in \mathbb{R}^d$ and $\theta \in \mathbb{R}$. (Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$.) By ergodicity, we see that the modulus $|\phi|$ must be constant. This aperiodicity condition is reminiscent of the weak mixing condition for flows [17].

It is useful to replace the group $\Gamma \cong \mathbb{Z}^d$ by a finite index subgroup $\Delta \subset \Gamma$ such that *every* element $b \in \Delta$ is the difference of $g$-weightings of closed orbits for $\sigma$. Define

$$\Delta = \{g^n(\underline{x}) - g^n(\underline{y}) : \underline{x}, \underline{y} \in \mathrm{Fix}_n, \text{ for some } n \geq 1\},$$

where $\mathrm{Fix}_n = \{\underline{x} \in \Sigma : \sigma^n \underline{x} = \underline{x}\}$. (The well-known fact that $\Delta$ is a group follows from the aperiodicity of the matrix $A$.)

Choose periodic points $\sigma^n \underline{x} = \underline{x}$ and $\sigma^{n+1} \underline{y} = \underline{y}$ and define

$$c_g = g^{n+1}(\underline{y}) - g^n(\underline{x}).$$

A key component in the proof of Proposition 3.1 is the following result.

**Lemma 3.2 (Marcus and Tuncel** [15]**).**
*(i)* $c_g$ *is well-defined modulo* $\Delta$.
*(ii) For any* $i \geq 1$, $ic_g + \Delta = \bigcup_{n=1}^{\infty} \{g^{n+i}(\underline{x}) - g^n(\underline{y}) : \sigma^{n+i}\underline{x} = \underline{x},\ \sigma^n \underline{y} = \underline{y}\}$.
*(iii)* $\Gamma/\Delta$ *is a cyclic group generated by* $c_g + \Delta$.

*Proof.* This result was shown in [15] for real valued functions (and using multiplicative notation). Essentially the same proof works here but, to make this clear, we need to give the details. It is helpful to think of periodic orbits for $\sigma$ as cycles in the directed graph $G(A)$ associated to $A$. Note that then $g$ is a function defined on the edges of $G(A)$. If the cycle $\gamma$ corresponds to the orbit $\sigma^n \underline{x} = \underline{x}$ then we write $|\gamma| = n$ and $g(\gamma) = g^n(\underline{x})$.

The proof rests on the following:

*Claim.* If $\gamma, \gamma', \delta, \delta'$ are cycles in $G(A)$ such that $|\gamma| - |\gamma'| = |\delta| - |\delta'|$ then

$$g(\gamma) - g(\gamma') = g(\delta) - g(\delta') \pmod{\Delta}.$$

To see this, first note that, since $A$ is aperiodic, we can fix a vertex $v$ in $G(A)$ and, for some $N \geq 1$, find paths $\tau_{vw}$ of length $N$ from $v$ to each other vertex $w$ in $G(A)$ and paths $\tau_{wv}$ of length $N$ from each vertex $w$ to $v$. Thus we may modify $\gamma, \gamma', \delta, \delta'$ so that they each start and end at $v$. More precisely, choose vertices $w(\gamma), w(\gamma'), w(\delta), w(\delta')$ in $\gamma, \gamma', \delta, \delta'$ and, for example, replace $\gamma$ by the concatenation

$$\tau_{vw(\gamma)} * \gamma * \tau_{w(\gamma)v} * \tau_{vw(\gamma')} * \tau_{w(\gamma')v} * \tau_{vw(\delta)} * \tau_{w(\delta)v} * \tau_{vw(\delta')} * \tau_{w(\delta')v}.$$

The quantities $g(\gamma) - g(\gamma')$ and $g(\delta) - g(\delta')$ are unchanged by these modifications.

We can then consider the concatenations $\gamma * \delta'$ and $\gamma' * \delta$ of such strings and note that $|\gamma * \delta'| = |\gamma' * \delta|$. Then

$$(g(\gamma) - g(\gamma')) - (g(\delta) - g(\delta')) = g(\gamma * \delta') - g(\gamma' * \delta) \in \Delta,$$

so that

$$g(\gamma) - g(\gamma') = g(\delta) - g(\delta') \pmod{\Delta},$$

as required.

(i) This follows immediately from the claim by taking $|\gamma| = |\gamma'|+1$ and $|\delta| = |\delta'|+1$.

(ii) It is easy to see that, for any $i \geq 1$,

$$ic_g = g(\delta) - g(\delta'),$$

where $\delta, \delta'$ are cycles with $|\delta| - |\delta'| = i$. Using the claim, this gives that

$$ic_g + \Delta = \{g(\gamma) - g(\gamma') : \gamma, \gamma' \text{ cycles with } |\gamma| - |\gamma'| = i\},$$

which is (ii).

(iii) Taking $\gamma'$ to be the empty cycle in (ii), we obtain

$$\{g(\gamma) : \gamma \text{ a cycle}\} \subset \bigcup_{i \in \mathbb{Z}} (ic_g + \Delta) \subset \Gamma.$$

So $\bigcup_{i\in\mathbb{Z}}(ic_g + \Delta)$ is a subgroup of $\Gamma$ which contains the generators of $\Gamma$. Hence $\Gamma = \bigcup_{i\in\mathbb{Z}}(ic_g + \Delta)$ and (ii) follows. □

Therefore, we can apply Lemma 3.2 to conclude that $\Delta$ has finite index in $\mathbb{Z}^d$. By Hypothesis III, there is an $f$-periodic point $x$ such that $\gamma_x = 0$. Hence there exists $\sigma^n \underline{x} = \underline{x}$ such that $g^n(\underline{x}) = 0$. Then we also have $\sigma^{2n}\underline{x} = \underline{x}$ and $g^{2n}(\underline{x}) = 0$. By part (ii) of Lemma 3.2, there exists $a \in \Delta$ such that $nc_g + a = 0$, so that $-nc_g \in \Delta$. By part (iii) of the lemma, we conclude that $\Delta$ has finite index in $\Gamma$.

We are now ready to return to the proof of Proposition 3.1.

*Proof of Proposition 3.1.* Assume that we have a solution to the aperiodicity identity (3.1). Then we can write

$$\phi(\sigma^n \underline{x}) = \phi(\underline{x}) e^{in\theta} e^{2\pi i \langle a, g^n(\underline{x})\rangle}.$$

If $\sigma^n \underline{x} = \underline{x}$ and $\sigma^n \underline{y} = \underline{y}$ are periodic points (of the same period $n$) we can deduce that

$$e^{2\pi i \langle a, g^n(\underline{x}) - g^n(\underline{y})\rangle} = 1.$$

In particular, $e^{2\pi i\langle a,b\rangle} = 1$ and thus that $\langle a, b\rangle \in \mathbb{Z}$, for all $b \in \Delta$. Since $\Delta$ has finite index in $\Gamma \cong \mathbb{Z}^d$, we have $a \in \mathbb{Q}^d$. Returning to (3.1) we see that this describes the dynamics of a finite extension of the shift space, and thus for a finite extension of the original pseudo-Anosov map, for which ergodicity is easily seen. In particular, $a \in \mathbb{Z}^d$ and $\phi$ is trivial, as required. As explained before, the result follows using [2].

*Remark.* One can also give a more direct, if somewhat messier, proof of Proposition 3.1 by modifying the approach in [8]).

## 4. Other transverse measures

For pseudo-Anosov maps $f : M \to M$ on compact manifolds $M$ we know by Thurston's theorem that the foliation $\mathcal{W}^s$ is uniquely ergodic, i.e, $\mu^s$ is the only transverse measure for this foliation (up to a scalar). However, in the case of the $\mathbb{Z}^d$-covers $\widehat{f} : \widehat{M} \to \widehat{M}$ considered above, there are infinitely many transverse measures for $\widehat{\mathcal{W}^s}$. This is completely analogous to the results of Babillot and Ledrappier for horocycle flows [3].

**Theorem B.** *Assume that Hypotheses I, II and III hold. For each $\alpha \in \mathbb{R}^d$ there is a distinct ergodic transverse measure $\mu^s_\alpha$ for $\widehat{\mathcal{W}^s}$. Moreover, these satisfy $T_a \mu^s_\alpha = e^{\langle a, \alpha \rangle} \mu^s_\alpha$, for all $a \in \mathbb{Z}^d$, where $T_a$ denotes translation by $a$, and there are no other ergodic transverse Radon measures.*

*Proof.* We begin with a proof of existence. For each $\alpha \in \mathbb{R}^d$, let $m_\alpha$ denote the unique probability measure on $\Sigma^+$ satisfying $\sigma^* m_\alpha = e^{P(\langle \alpha, g \rangle) - \langle \alpha, g \rangle} m_\alpha$. (This $m_\alpha$ is the eigenmeasure for the asscociated transfer operator $L_{\langle \alpha, g \rangle}$.) In particular, if cylinders $[x_0, \dots, x_{n-1}]$ and $[y_0, \dots, y_{n-1}]$ satisfy $x_{n-1} = y_{n-1}$ then

$$\begin{aligned} m_\alpha([x_0, \dots, x_{n-1}]) &= e^{-P(\langle \alpha, g \rangle) - \sum_{i=0}^{n-2} \langle \alpha, g(x_i, x_{i+1}) \rangle} m_\alpha([x_{n-1}]) \\ &= e^{-P(\langle \alpha, g \rangle) - \sum_{i=0}^{n-2} \langle \alpha, g(x_i, x_{i+1}) \rangle} m_\alpha([y_{n-1}]) \\ &= m_\alpha([y_0, \dots, y_{n-1}]). \end{aligned} \tag{4.1}$$

We want to define transverse measures for $\widehat{\mathcal{W}^s}$ by defining measures on $\Sigma^+ \times \mathbb{Z}^d$ which are invariant under $\sim$ and then projecting these to $\widehat{M}$. For each $\alpha \in \mathbb{R}^d$ we define a measure $\nu_\alpha$ on $\Sigma^+ \times \mathbb{Z}^d$ by $d\nu_\alpha = dm_\alpha \times e^{\langle \alpha, a \rangle} da$, where $da$ denotes the usual counting measure on $\mathbb{Z}^d$. For any cylinder $[x_0, \cdots, x_{n-1}]$ and $a \in \mathbb{Z}^d$ we have

$$\nu_\alpha([x_0, \cdots, x_{n-1}] \times \{a\}) = m_\alpha([x_0, \cdots, x_{n-1}]) e^{\langle \alpha, a \rangle} \tag{4.2}$$

We need to show that these measures are invariant. Assume that

$$[x_0, \cdots, x_{n-1}] \times \{a\} \sim [y_0, \cdots, y_{n-1}] \times \{b\}$$

then (cf. Lemma 2.7)

$$\begin{aligned} [x_{n-1}] \times \{a\} &= \widehat{\sigma}^{n-1}([x_0, \cdots, x_{n-1}] \times \{a + \sum_{i=0}^{n-1} g(x_i, x_{i+1})\}) \\ &= \widehat{\sigma}^{n-1}([y_0, \cdots, y_{n-1}] \times \{a + \sum_{i=0}^{n-1} g(x_i, x_{i+1})\}) \\ &= [y_{n-1}] \times \left\{ a + \sum_{i=0}^{n-1} (g(x_i, x_{i+1}) - g(y_i, y_{i+1})) \right\} \end{aligned} \tag{4.3}$$

and so can deduce from the equivalence that

$$a + \sum_{i=0}^{n-1} g(x_i, x_{i+1}) = b + \sum_{i=0}^{n-1} g(y_i, y_{i+1}). \tag{4.4}$$

Therefore, comparing (4.1), (4.2), (4.3) and (4.4) we have that

$$\begin{aligned}
&\nu_\alpha([x_0, \cdots, x_{n-1}] \times \{a\}) \\
&= m_\alpha([x_0, \cdots, x_{n-1}])e^{\langle \alpha, a\rangle} \\
&= m_\alpha([x_{n-1}])e^{-P(\langle \alpha, g\rangle) + \langle \alpha, a + \sum_{i=0}^{n-2} g(x_i, x_{i+1})\rangle} \\
&= m_\alpha([y_0, \cdots, y_{n-1}])e^{\langle \alpha, a + \sum_{i=0}^{n-2}(g(x_i, x_{i+1}) - g(y_i, y_{i+1}))\rangle} \\
&= \nu_\alpha([y_0, \cdots, y_{n-1}] \times \{b\})
\end{aligned} \tag{4.5}$$

We can modify the proof of Theorem A to deduce that $\nu_\alpha$ is ergodic. More precisely, we apply the same argument to the measure $\overline{\nu}$ on $\Sigma^+ \times \mathbb{Z}^d$ defined by

$$\overline{\nu}([x_0, \cdots, x_{n-1}] \times \{a\}) := e^{-\langle \alpha, a\rangle} \nu_\alpha([x_0, \cdots, x_{n-1}]),$$

which by construction is the same for equivalent cylinders on $\Sigma^+ \times \mathbb{Z}^d$. We then observe that the analogue of Lemma 2.8 holds (with $\nu_\alpha$ replacing $\nu$).

To complete the proof of Theorem B we need to know that there are no other Radon measures. One approach is to apply directly the following result of Aaronson, Nakada, Sarig and Solomyak.

**Proposition 4.1** ([2, Theorem 2.2])**.** *If $\mu$ is a Radon measure which satsifies (4.5) then (up to multiplication by a scalar) it is of the form $\nu_\alpha$, for some $\alpha$.*

It then only remains to apply the analogue of Lemma 2.8.

*Remark.* For a more direct proof if suffices to show that any invariant measure $m$ on $\Sigma^+ \times \mathbb{Z}^d$ is absolutely continuous to its translation $\tau m$, whenever $\tau : \Sigma^+ \times \mathbb{Z}^d \to \Sigma^+ \times \mathbb{Z}^d$ is any translation in the second coordinate of the form $\tau(x, a) = (x, a + b)$, where $a, b \in \mathbb{Z}^d$. From this it is easy to deduce that the measure $m$ is of the form described by considering the Radon-Nikodym derivative.

## 5. Other applications

In this section we briefly describe how the symbolic dynamics in Section 2 can be used to study related problems for maps $\widehat{f} : \widehat{M} \to \widehat{M}$ satisfying Hypotheses I, II and III.

**5.1 Bounded Rational Ergodicity.** We recall the following definition.

*Definition.* The map $\widehat{f}$ is called *boundedly rationally ergodic* if there is a Borel set $A$ with $0 < \widehat{\mu}(A) < +\infty$ and $M > 0$ such that for all $n \geq 1$ we have

$$\left\| \sum_{k=0}^{n-1} \chi_A \circ \widehat{f}^k \right\|_{L^\infty} \leq M \int_A \left( \sum_{k=0}^{n-1} \chi_A \circ \widehat{f}^k \right) d\widehat{\mu}.$$

In this context we have the following result.

**Proposition 5.1.** *The measure $\widehat{\mu}$ satisfies bounded rational ergodicity.*

This follows by an application of Theorem 4.1 from [2]. We say that a *generalized law of large numbers* holds if there is a function $L : \{0, 1\}^{\mathbb{Z}} \to \mathbb{R}^+$ such that for every Borel set $A$ we have that $L\left(\chi_A(\widehat{f}^n x)\right) = \widehat{\mu}(A)$ for a.e. $x$.

By comparison with [2], we obtain the following corollary.

**Corollary.** *The measure $\widehat{\mu}$ satisfies a generalized law of large numbers.*

By contrast, the other invariant measures $\widehat{\mu}_\alpha$ ($\alpha \neq 0$) can be shown not to have a generalized law of large numbers. (This is related to the concept of "squashability".) We refer the reader to [1] for more details.

**5.2 Hyperfiniteness.** The notion of hyperfiniteness for equivalence relations is a particularly important one in the ergodic theory of foliations. Let $Y$ be a measurable space with a sigma finite measure $\mu$. An equivalence relation $\sim_Y$ on $Y$ is a limit of equivalence relations $\sim_{Y,m}$ ($m \geq 1$) if we can write almost every equivalence classes $[x]_Y$ for $\sim_Y$ as unions of equivalence classes $[x]_{Y,m}$ for $\sim_{Y,m}$, i.e.,

$$[x]_Y = \cup_{m=1}^{\infty} [x]_{Y,m}, \text{ a.e.} \tag{5.1}$$

We call an equivalence relation hyperfinite if the equivalence relation is a limit of equivalence relations whose equivalence classes contain only finitely many elements, almost everywhere. In the context of foliations, we can take $Y = \cup_i U_i$ to be a transverse section (or union of transverse sections) to the foliation and let $\mu$ be a transverse measure supported on $Y$. Bowen showed that the foliation $\mathcal{W}^s$ with the transverse measure $\mu^s$ for an Anosov diffeomorphism is hyperfinite, which measure that the corresponding equivalence on sections given by points being equivalent if they lie on the same leaf [5]. The proof uses symbolic dynamics and also applies to the foliation $\mathcal{W}^s$ for pseudo-Anosov maps. There is a corresponding result for the $\mathbb{Z}^d$-cover, and we summarize the two as follows.

**Proposition 5.2.**

(1) *The foliation $\mathcal{W}^s$ with the transverse measure $\mu^s$ is hyperfinite on $M$.*
(2) *The foliation $\widehat{\mathcal{W}}^s$ with the transverse measure $\widehat{\mu}^s$ is hyperfinite on $\widehat{M}$.*

*Proof.* The proof of the first part is simple modification of that in [5]. We can consider a family of equivalence relations $\sim_{\Sigma^+,m}$ (for $m \geq 0$) on $\Sigma^+$ where $\underline{x} = (x_n)$ and $\underline{y} = (y_n)$ satisfy $\underline{x} \sim_{\Sigma^+,m} \underline{y}$ if $x_i = y_i$ for $i \geq m$. It is immediately apparent from the definitions that the equivalence classes for $\sim$ on $\Sigma^+ \times \mathbb{Z}^d$ satisfy

$$[x] = \cup_{m=1}^{\infty} [x]_{\Sigma^+,m}, \text{ a.e.}$$

and the equivalence classes $[x]_{\Sigma^+,m}$ are finite, a.e. In particular, we see that (5.1) holds. The proof of the second part uses Lemma 2.7 with $Y = \cup_i U_i \times \mathbb{Z}^d$. We can consider a family of equivalence relations $\sim_{\Sigma^+ \times \mathbb{Z}^d,m}$ (for $m \geq 0$) on $\Sigma^+ \times \mathbb{Z}^d$ where

$$(\underline{x}, a) \sim_{\Sigma^+ \times \mathbb{Z}^d,m} (\underline{y}, a') \text{ if } x_i = y_i, \text{ for } i \geq m, \text{ and } g^m(\underline{x}) + a = g^m(\underline{y}) + a'.$$

Clearly, the equivalence classes for $\sim$ on $\Sigma^+ \times \mathbb{Z}^d$ satisfy $[x] = \cup_{m=1}^{\infty} [x]_{\Sigma^+ \times \mathbb{Z}^d,m}$, a.e. and the equivalence classes $[x]_{\Sigma^+ \times \mathbb{Z}^d,m}$ are finite, a.e. □

Similar results hold for the measures $\mu_\alpha^s$, described in Theorem B, and the corresponding measures for $\mathbb{Z}^d$-covers of horocycle flows in [3].

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