RELATIVE GROWTH IN HYPERBOLIC GROUPS

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Abstract. In this note we obtain estimates on the relative growth of normal subgroups of non-elementary hyperbolic groups, particularly those with free abelian quotient. As a corollary, we deduce that the associated relative growth series fail to be rational.

1. Introduction and Results

Let G be a non-elementary hyperbolic group equipped with a finite symmetric generating set. Write $W_n = \{g \in G : |g| = n\}$ for the collection of elements of word length n. By a result of Coornaert [6], the growth rate of its cardinality $\#W_n$ is purely exponential, i.e. there exist constants $\lambda > 1$ and $C_1, C_2 > 0$ such that

$$C_1\lambda^n \leq \#W_n \leq C_2\lambda^n$$

for all $n \geq 1$. Now suppose that N is a subgroup of G. An interesting question to ask is how $\#(W_n \cap N)$, which we call the relative growth of N, grows in comparison to $\#W_n$. A result of Gouëzel, Matheus and Maucourant [11] states that if N has infinite index in G then

$$\lim_{n\to\infty} \frac{\#(W_n \cap N)}{\#W_n} = 0. \tag{1.1}$$

This is a subtle result that relies strongly on the hyperbolicity of G. If we suppose further that N is normal and the quotient G/N is isomorphic to $\mathbb{Z}^\nu$ for some $\nu \geq 1$, then we have access to more structure. With this additional information it seems reasonable to expect that we can describe the relative growth of N more precisely.

Pollicott and Sharp [18] studied this problem when G is the fundamental groups of a compact orientable surface of genus at least two and N is the commutator subgroup. Sharp [19] extended this to cover hyperbolic groups G that may be realised as convex cocompact groups of isometries of real hyperbolic space whose fundamental domain can be chosen to be a finite sided polyhedron R such that $\bigcup_{g\in G} \partial R$ is a union of geodesic hyperplanes, with generators given by the side pairings. The fundamental groups of compact surfaces were shown to satisfy this condition by Bowen and Series [2]. In addition, this class includes free groups on at least two generators and certain higher dimensional examples (see Bourdon's thesis [1]). In these cases, it was shown that there exists an integer $D \geq 1$ such that, along the subsequence Dn, the relative growth $\#(W_{Dn} \cap N)$ grows asymptotically like $\lambda^{Dn}/(Dn)^{\nu/2}$, as $n \to \infty$. The aim of this note is to extend this result so that it applies all non-elementary hyperbolic groups.

Before we state our main result, we recall the following standard definitions. Given two real valued sequences a_n and b_n, we say that $a_n \sim b_n$ if $a_n/b_n \to 1$, as $n \to \infty$. Furthermore, if b_n is positive, we say that $a_n = O(b_n)$ if there exists a constant $C > 0$ such that $|a_n| \leq Cb_n$, for all $n \geq 1$.

Theorem 1.1. *Let G be a non-elementary hyperbolic group equipped with a finite symmetric generating set and let $N \lhd G$ be a normal subgroup with $G/N \cong \mathbb{Z}^\nu$ for some $\nu \geq 1$. Then*

$$\#(W_n \cap N) = O\left(\frac{\lambda^n}{n^{\nu/2}}\right)$$

as $n \to \infty$. Furthermore, there exists $D \in \mathbb{Z}_{\geq 0}$ and $C > 0$ such that

$$\#(W_{Dn} \cap N) \sim \frac{C\lambda^{Dn}}{(Dn)^{\nu/2}}$$

as $n \to \infty$.

This theorem has the following immediate corollary.

Corollary 1.2. *Let G be a non-elementary hyperbolic group equipped with a finite symmetric generating set and let $N \lhd G$ be a normal subgroup such that the abelianisation of G/N has rank $\nu \geq 1$. Then*

$$\#(W_n \cap N) = O\left(\frac{\lambda^n}{n^{\nu/2}}\right)$$

as $n \to \infty$.

Proof. Write the abelianisation of G/N as $\mathbb{Z}^\nu \times F$, where F is finite. There are then natural surjective homomorphisms $\phi : G \to G/N$ and $\psi : G/N \to \mathbb{Z}^\nu$. Set $\phi_0 = \psi \circ \phi$ and $N_0 = \ker \phi_0$. Then $N \subset N_0$. Furthermore, by Theorem 1.1, $\#(W_n \cap N_0) = O(\lambda^n n^{-\nu/2})$, giving the required estimate. □

Remark 1.3. The relative growth in Corollary 1.2 may occur at a slower exponential rate. Indeed, Coulon, Dal'Bo and Sambusetti recently showed that $\#(W_n \cap N) = O(\lambda_0^n)$, for some $0 < \lambda_0 < \lambda$ precisely when G/N is *not* amenable [7]. In fact, their result does not require normality of the subgroup, in which case amenability is replaced by co-amenability of N in G, i.e. that the G-action on the coset space G/N is amenable.

To prove Theorem 1.1, we would like to employ the strategy used by the second author in [19]. However, there are significant technical obstacles which we need to overcome in order to use this method. We summarise these below.
(i) Firstly, as mentioned above, in [19] there are strong restrictions on the hyperbolic groups and their generating sets. This makes it much easier to study the relative growth quantity $\#(W_n \cap N)$. In the current paper we need to find a new approach that works for general non-elementary hyperbolic groups, that will allow us to express $\#(W_n \cap N)$ in terms of quantities which we can analyse. To achieve this we appeal to ideas and techniques used in [5].
(ii) Secondly, we need a good understanding of how real valued group homomorphisms on hyperbolic groups grow as we increase the word length of the input. Again, recent work of the first author [5] allows us to deduce the required properties of these homomorphisms.

We end this section with a discussion of relative growth series. We define the relative growth series for N in G (with respect to the given generators) to be the power series

$$\sum_{n=0}^{\infty} \#(W_n \cap N) z^n.$$

When $N = G$, this is the standard growth series and, for hyperbolic groups, is well-known to be the series of a rational function [4], [10]. The requirement that a power series be rational imposes a strong constraint on the coefficients: if $\sum_{n=0}^{\infty} a_n z^n$ is rational then there are complex numbers $\xi_1, \ldots, \xi_m$ and polynomials $P_1, \ldots, P_m$ such that

$$a_n = \sum_{j=1}^{m} P_j(n) \xi_j^n$$

(Theorem IV.9 of [8]). Comparing with the asymptotic in Theorem 1.1, we see that $\#(W_n \cap N)$ does not satisfy this constraint. Thus we obtain the following.

Corollary 1.4. *Suppose G is a non-elementary hyperbolic group equipped with a finite symmetric generating set. Let $N \lhd G$ be a normal subgroup with $G/N \cong \mathbb{Z}^\nu$, for some $\nu \geq 1$. Then, the relative growth series*

$$\sum_{n=1}^{\infty} \#(W_n \cap N) z^n$$

is not the series of a rational function.

Remark 1.5. (i) The first result of this type is due to Grigorchuk, who showed that the relative growth series is not rational when G is the free group on two generators and N is the commutator subgroup (see [13]). A similar result was obtained for the fundamental groups of compact surfaces of genus ≥ 2 in [18] and this was extended to a wider class of hyperbolic groups in [19].
(ii) We note that, as Corollary 1.4 requires the asymptototic along a subsequence in Theorem 1.1, it does not apply to general infinite index subgroups of hyperbolic groups. In fact, Grigorchuk showed that if N is a finite index subgroup of a free group than its relative growth series is rational [12].

2. Preliminaries

We first recall the definition of a hyperbolic group. A metric space is hyperbolic if there exist $\delta \geq 0$ for which every geodesic triangle is δ-thin, i.e. given any geodesic triangle, the union of the δ neighbourhoods of any two sides of this triangle contain the third side. A finitely generated group G is said to be hyperbolic, if given any finite generating set S, the Cayley graph of G with respect to S is a hyperbolic metric space when equipped with the word metric. We say that a hyperbolic group is elementary if it contains a cyclic subgroup of finite index. We will be exclusively concerned with non-elementary hyperbolic groups.

Hyperbolic groups have nice combinatorial properties that arise due to their strongly Markov structure.

Definition 2.1. A finitely generated group G is strongly Markov if given any generating set S there exists a finite directed graph $\mathcal{G}$ with vertex set V, edge set E (with at most one directed edge between an ordered pair of vertices) and a labeling map $\rho : E \to S$ such that:

(1) there exists an initial vertex $* \in V$ such that no directed edge ends at $*$;
(2) the map taking finite paths in $\mathcal{G}$ starting at $*$ to G that sends a path with concurrent edges $(*, x_1), \ldots, (x_{n-1}, x_n)$ to $\rho(*, x_1)\rho(x_1, x_2)\cdots\rho(x_{n-1}, x_n)$, is a bijection;
(3) the word length of $\rho(*, x_1)\cdots\rho(x_{n-1}, x_n)$ is n.

In [10] Ghys and de le Harpe extended Cannon's work on Kleinian groups [4] and proved that hyperbolic groups are strongly Markov.

Proposition 2.2 ([10], Chapitre 9, Théorème 13)**.** *Any hyperbolic group is strongly Markov.*

Suppose that $\mathcal{G} = (E, V)$ is a directed graph associated to G satisfying the properties in Definition 2.1. We define a transition matrix A, indexed by $V \times V$, by

$$A(v_1, v_2) = \begin{cases} 1 & \text{if } (v_1, v_2) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Using A we define a space

$$\Sigma_A = \{(x_n)_{n=0}^{\infty} : x_n \in V \text{ and } A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}_{\geq 0}\}$$

and $\sigma : \Sigma_A \to \Sigma_A$ by $\sigma((x_n)_{n=0}^{\infty}) = (x_{n+1})_{n=0}^{\infty}$. The system (Σ_A, σ) is known as a subshift of finite type.

Recall that a matrix M with zero-one entries is called irreducible if for each i, j there exists $n(i,j)$ for which $M^{n(i,j)}(i,j) > 0$. This is equivalent to the directed graph $\mathcal{G}$ being connected. We call M aperiodic if there exists n such that every entry of M^n is strictly positive. Due to the $*$ vertex, which forms its own connected component in $\mathcal{G}$, A is never irreducible. However, it is possible that, after removing from A the row and column corresponding to the $*$ state, the resulting matrix is aperiodic. In fact, for the hyperbolic groups and generating sets considered by Sharp in [19], it is always possible to find a corresponding directed graph described by an aperiodic matrix (after removing $*$). This is not true in general and to improve upon the results in [19], we need to exploit geometrical and combinatorial properties of hyperbolic groups to obtain additional structural information about the directed graph $\mathcal{G}$. Throughout the rest of this section we introduce the preliminaries that will allow us to analyse $\#(W_n \cap N)$ for general hyperbolic groups.

As mentioned above, in general, the graph $\mathcal{G}$ may have several connected components. By relabeling the vertex set V, we may assume that A has the form

$$A = \begin{pmatrix} A_{1,1} & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \dots & A_{m,m} \end{pmatrix},$$

where each $A_{j,j}$ is irreducible for $j = 1, ..., m$. We call the $A_{j,j}$ the irreducible components of A.

Let $\lambda > 1$ denote the exponential growth rate of W_n. It is easy to see by Property (2) and (3) in Definition 2.1, that all of the $A_{j,j}$ must have spectral radius at most λ. Furthermore there must be at least one $A_{j,j}$ with spectral radius exactly λ. We call an irreducible component maximal if it has spectral radius λ. We label the maximal components B_j for $j = 1, \dots, m$ and define Σ_{B_j}, $j = 1, \dots, m$ analogously to Σ_A. For each Σ_{B_j} there exists $p_j \geq 1$ such that Σ_{B_j} admits a cyclic decomposition into p_j disjoint sets,

$$\Sigma_{B_j} = \bigsqcup_{k=0}^{p_j - 1} \Sigma_k^j.$$

We call p_j the cyclic period of Σ_{B_j}. The shift map σ sends Σ_k^j into Σ_{k+1}^j where $k, k+1$ are taken modulo p_j. Hence each Σ_k^j is σ^{p_j}-invariant. In fact, each system $\sigma^{p_j} : \Sigma_k^j \to \Sigma_k^j$ is a subshift of finite type with aperiodic transition matrix.

The following key result, that relies on Coornaert's estimates for $\#W_n$, shows that the maximal components B_j do not interact with each other. This result allows us to gain a better understanding of the structure of $\mathcal{G}$.

Proposition 2.3 ([3], Lemma 4.10)**.** *The maximal components of A are disjoint. There does not exist a path in $\mathcal{G}$ that begins in one maximal component and ends in another.*

Proof. For the convenience of the reader, we include a sketch of the proof. Suppose there is a path of length l between maximal components that starts at a vertex x in B_j and end at vertex y in B_k. Then for large n, the number of length n paths that begin in B_j, traverse $m < n - l$ edges in B_j to x, then follow our path to y in B_k and traverse $n - m - l$ edges in B_k, is growing like λ^n. Since we can vary m between 1 and $n - l$, this implies there are at least $Cn\lambda^n$ paths from B_j to B_k for some $C > 0$. This would imply $\#W_n$ grows at least like $n\lambda^n$, contradicting Coornaert's estimates for $\#W_n$ [6]. $\square$

This fact will be useful when counting certain quantities related to relative growth. To further facilitate these counting arguments, we define the following matrices.

Definition 2.4. For each $j = 1, ..., m$, define a matrix C_j by,

$$C_j(u,v) = \begin{cases} 0 & \text{if } u \text{ or } v \text{ belong to a maximal component that is not } B_j, \\ A(u,v) & \text{otherwise.} \end{cases}$$

Now suppose that $N \lhd G$ is a normal subgroup for which $G/N \cong \mathbb{Z}^\nu$ and let $\varphi : G \to G/N \cong \mathbb{Z}^\nu$ be the quotient homomorphism. We define a function $f : \Sigma_A \to \mathbb{Z}^\nu$ by

$$f((x_n)_{n=0}^\infty) = \varphi(\rho(x_0, x_1)),$$

where ρ is the labeling map from Definition 2.1. Since $f((x_n)_{n=0}^\infty)$ depends only on the first two coordinates of $(x_n)_{n=0}^\infty$, we can consider f as a map from the directed edge set of $\mathcal{G}$ to $\mathbb{R}$. We then have that $\varphi(g) = f(*, x_1) + f(x_1, x_2) + \cdots + f(x_{|g|-1}, x_{|g|})$ where $(*, x_1), ..., (x_{|g|-1}, x_{|g|})$ is the unique path associated to g by Property (2) of Definition 2.1. Using f, we weight the matrices C_j componentwise and define, for $t \in \mathbb{R}^\nu$,

$$C_j(t)(u,v) = e^{2\pi i \langle t, f(u,v) \rangle} C_j(u,v).$$

We define the matrices $B_j(t)$ analogously.

3. Proof of Theorem 1.1

Suppose G is a non-elementary hyperbolic group and N a normal subgroup satisfying the hypothesis of Theorem 1.1. Let $\varphi : G \to \mathbb{Z}^\nu$ denote the quotient homomorphism. To study the relative growth of N, we would like to express $\#(W_n \cap N)$ in terms of the matrices $C_j(t)$. Using the orthogonality identity

$$\int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} e^{2\pi i \langle t, \varphi(g) \rangle} \, dt = \begin{cases} 1 & \text{if } \varphi(g) = 0 \\ 0 & \text{otherwise} \end{cases}$$

we can write

$$\#(W_n \cap N) = \sum_{|g|=n} \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} e^{2\pi i \langle t, \varphi(g) \rangle} \, dt = \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} \sum_{|g|=n} e^{2\pi i \langle t, \varphi(g) \rangle} \, dt.$$

The following result will allow us to rewrite $\#(W_n \cap N)$ in terms of the matrices C_j. Let v_* be the vector in $\mathbb{R}^V$ with a one in the coordinate corresponding to the $*$ vertex and zeros elsewhere. Also, let $\mathbf{1} \in \mathbb{R}^\nu$ be the vector with a 1 in each coordinate.

Lemma 3.1. *There exists $\epsilon > 0$ such that for all $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$*

$$\sum_{|g|=n} e^{2\pi i \langle t, \varphi(g) \rangle} = \sum_{j=1}^m \langle C_j^n(t) v_*, \mathbf{1} \rangle + O((\lambda - \epsilon)^n)$$

as $n \to \infty$. The implied constant is independent of t.

Proof. Using the correspondence between G and Σ_A, we can write

$$\left| \sum_{|g|=n} e^{2\pi i \langle t, \varphi(g) \rangle} - \sum_{j=1}^m \langle C_j^n(t) v_*, \mathbf{1} \rangle \right| = (m-1) \left| \sum_{g \in M_n} e^{2\pi i \langle t, \varphi(g) \rangle} \right| \leq (m-1) \, \#M_n,$$

where M_n consists of the elements in G of word length n whose corresponding path in $\mathcal{G}$ does not enter a maximal component. It is clear that $\#M_n = O((\lambda - \epsilon)^n)$ for some $\epsilon > 0$ and so the result follows. ∎

Using this lemma, we see that

$$\#(W_n \cap N) = \sum_{j=1}^{m} \int_{\mathbb{R}^\nu/\mathbb{Z}^\nu} \langle C_j^n(t) v_*, \mathbf{1} \rangle \, dt + O((\lambda - \epsilon)^n).$$

Hence to study the relative growth of N would like to understand the spectral behaviour of the $C_j(t)$ for $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$. From their definitions, it is clear that the matrices C_j each have p_j simple maximal eigenvalues of modulus λ and the rest of the spectrum is contained in a disk of radius strictly smaller than $\lambda - \epsilon$, for some $\epsilon > 0$. We shall be interested in the values of t for which the operators $C_j(t)$ have spectral radius λ. These values of t are characterised by the following lemma.

Lemma 3.2. *For any $t \in \mathbb{R}^\nu$, the operator $C_j(t)$ has spectral radius at most λ. Furthermore, $C_j(t)$ has spectral radius exactly λ if and only if it has p_i simple maximal eigenvalues of the form $e^{2\pi i\theta} e^{2\pi ik/p_i} \lambda$ for $k = 0, \ldots, p_i - 1$ and some $\theta \in \mathbb{R}$. This occurs if and only if $B_j(t) = e^{2\pi i\theta} M B_j M^{-1}$ where M is a diagonal matrix with modulus one diagonal entries. Furthermore, when $C_j(t)$ has p_i simple maximal eigenvalues of modulus λ, the rest of the spectrum is contained in a disk of radius strictly less than λ.*

Proof. When C_j consists of a single component (ignoring the $*$ vertex) and so is the same as B_j, this is Wielandt's Theorem [9]. When this is not the case, we can write the spectrum of $C_j(t)$ as a union of the spectra of the irreducible components making up $C_j(t)$. By definition, each C_j has one component B_j with spectral radius λ and all other components have spectral radius strictly less than λ. Therefore applying Wielandt's Theorem to each component gives the required result. □

We now follow the method presented in [19]. Let $f_j = f|_{\Sigma_{B_i}}$ for $j = 1, \ldots, m$. If a sequence $\gamma = (x_0, x_1, ..., x_n)$ is such that $B_j(x_i, x_{i+1}) = 1$ for $i = 0, \ldots, n$ and $x_0 = x_n$, then we call γ a cycle and define its length as $l(\gamma) = n$. Let $\mathcal{C}_j$ be the collection of all such cycles and note that the length of any cycle in $\mathcal{C}_j$ is a multiple of p_j. Given a cycle $\gamma \in \mathcal{C}_j$, we define its f_j-weight to be

$$w_{f_j}(\gamma) = f_j(x_0, x_1) + \cdots + f_j(x_{n-1}, x_n).$$

Let Γ_j be the subgroup of $\mathbb{Z}^\nu$ generated by $\{w_{f_j}(\gamma) : \gamma \in \mathcal{C}_j\}$. We define Δ_j to be the following subgroup of Γ_{f_j},

$$\Delta_j = \{w_{f_j}(\gamma) - w_{f_j}(\gamma') : \gamma, \gamma' \in \mathcal{C}_j \text{ and } l(\gamma) = l(\gamma')\}.$$

(This is a version of Krieger's Δ-group [14]. For a proof that it is a group, see page 892 of [20].) We now choose two cycles $\gamma, \gamma' \in \mathcal{C}_j$ such that $l(\gamma) - l(\gamma') = p_j$ and set $c_j = w_{f_j}(\gamma) - w_{f_j}(\gamma')$. Applying the results of [15] to the aperiodic shift $(\Sigma_{B_j}, \sigma^{p_j})$, we see that the group Γ_j/Δ_j is cyclic and is generated by the element $c_j + \Delta_j$. Our aim is to show that this group has finite order. To do so, we will use a result of Marcus and Tuncel. For each $j = 1, \ldots, m$, let E_j denote the directed edge set for the graph with transition matrix B_j. Write V_j for the analogously defined vertex sets. We say that a function $g : E_j \to \mathbb{R}$ is cohomologous to a constant if there exists $C \in \mathbb{R}$ and $h : V_j \to \mathbb{R}$ such that $g(x, y) = C + h(y) - h(x)$ for all $(x, y) \in E_j$.

Lemma 3.3 ([15])**.** *If $\langle t, f_j^{p_j} \rangle$ is not cohomologous to a constant for any non-zero $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, then Γ_j/Δ_j has finite order.*

It is clear that, for $t \in \mathbb{R}^\nu$, $\langle t, f_j^{p_j} \rangle$ is cohomologous to a constant if and only if $\langle t, f_j \rangle$ is cohomologous to constant. Using ideas from [5], we will show that the hypothesis of the above lemma is satisfied for each $j = 1, \ldots, m$.

Lemma 3.4. *For non-zero $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$ and for all $j = 1, \ldots, m$, $\langle t, f_j \rangle$ is not cohomologous to a constant.*

Proof. We begin by noting that, since φ is surjective, for any $t \in \mathbb{R}^\nu \backslash \{0\}$ the function $\psi_t := \langle t, \varphi \rangle : G \to \mathbb{R}$ is a non-trivial group homomorphism. Theorem 1.1 and Proposition 7.2 of [5] imply that if $\langle t, f_j \rangle$ (for any $j \in \{1, \dots, m\}$) is cohomologous to a constant, then that constant is given by

$$\lim_{n\to\infty} \frac{1}{\# W_n} \sum_{|g|=n} \frac{\psi_t(g)}{n}.$$

Since our generating set S is symmetric, $|g| = |g^{-1}|$ for all $g \in G$ and so the above limit is 0 by symmetry. Hence we need to show that $\langle t, f_j \rangle$ is not cohomologous to 0. By Livsic's criterion [16], $\langle t, f_j \rangle$ is cohomologous to 0 if and only if $\langle t, w_{f_j}(\gamma) \rangle = 0$ for all loops $\gamma \in C_j$.

Suppose for contradiction that $\langle t, w_{f_j}(\gamma) \rangle = 0$ for all loops $\gamma \in C_j$. Now, for $\gamma = (x_0, \dots, x_n) \in C_j$, $g_\gamma = \rho(x_0, x_1)\rho(x_1, x_2) \dots \rho(x_{n-1}, x_n)$ belongs to the kernel of ψ_t. Furthermore, g_γ has word length n. Also, Property (2) from Definition 2.1 implies that for any two distinct loops $\gamma, \gamma' \in \mathcal{C}_j$, we have $g_\gamma \neq g_{\gamma'}$ whenever γ and γ' have the same initial vertex. Since the number of loops of length np_j in C_j is growing like λ^{np_j}, this implies that there exists $C > 0$ such that

$$\#(W_{np_j} \cap \ker(\psi_t)) \geq C\lambda^{np_j}$$

for $n \geq 1$ and hence that

$$\limsup_{n\to\infty} \frac{\#(W_n \cap \ker(\psi_t))}{\# W_n} > 0.$$

Since $\ker(\psi_t)$ is an infinite index subgroup of G, this contradicts the result of Gouëzel, Matheus and Maucourant [11] written above as (1.1). □

Remark 3.5. Since the above proof relies on the zero density result of Gouëzel, Matheus and Maucourant [11], quantifying the decay rate in (1.1) requires a priori knowledge of the convergence to zero.

Let $D_j = |\Gamma_j / \Delta_j|$ for $j = 1, \dots, m$. From the above discussion, we know that each D_j is finite. We also note that Lemma 3.4 shows that $\text{rank}_{\mathbb{Z}}(\Gamma_j) = \nu$ and so $|\mathbb{Z}^\nu / \Gamma_j|$ is finite for each $j = 1, \dots, m$. Combining this with all of the above work, allows us to state the following result that describes the spectral behaviour of the $C_j(t)$ as t varies. We use the notation $\varrho(M)$ to denote the spectral radius of a matrix M.

Proposition 3.6. *For $t \in \mathbb{R}^\nu/\mathbb{Z}^\nu$, define $\chi_t \in \widehat{\mathbb{Z}^\nu}$ by $\chi_t(x) = e^{2\pi i \langle t, x\rangle}$. Then we have that*

$$\{\chi_t : \varrho(C_j(t)) = \lambda\} = \Delta_{f_j}^\perp,$$

where $\Delta_{f_j}^\perp = \{\chi \in \widehat{\mathbb{Z}^\nu} : \chi(\Delta_{f_j}) = 1\}$. Furthermore, when $\chi_t \in \Delta_{f_j}^\perp$, $C_j(t)$ has p_j simple maximal eigenvalues of the form $e^{2\pi i\theta} e^{2\pi i k/p_j}\lambda$ for some $\theta \in \mathbb{R}$ and $k = 0, \dots, p_j - 1$.

Proof. This is essentially Proposition 4 from [19] which is derived from work in [17]. However, here we need to consider the non-aperiodic matrices $C_j(t)$. To deduce this more general statement, we can apply Proposition 4 from [19] to the maximal component associated to the matrix $C_j^{p_j}(t)$. This is justified since this maximal component is aperiodic. To conclude the proof, we note that the part of the spectrum of $C_j(t)$ coming from $B_j(t)$ is invariant under the rotation $z \mapsto ze^{2\pi i/p_j}$. □

Proposition 3.6 implies that there exist $D_j < \infty$ values of t for which the spectral radius of $C_j(t)$ is maximal and equal to λ. Denote these values by $t = 0, t_1^j \dots, t_{D_j-1}^j$. When t takes one of these values, $C_j(t)$ has p_j simple maximal eigenvalues of the form $e^{2\pi i\theta} e^{2\pi i k/p_j}\lambda$ for $k = 0, \dots, p_j - 1$ and for some $\theta \in \mathbb{R}$. We now choose, for each $j = 1, \dots, m$, a neighbourhood U_0^j of zero and define $U_r^j = U_0^j + t_k^j$ for $k = 0, \dots, D_j - 1$. Results from perturbation theory guarantee that, as long as each U_0^j is sufficiently small, there exists $\epsilon > 0$ such that the following hold for each $j = 1, \dots, m$.

(1) If $t \in \bigcup_{r=0}^{D_j-1} U_r^j$, then the matrices $C_j(t)$ each have p_j simple, maximal eigenvalues of the form $\lambda_j(t)e^{2\pi ik/p_j}$ for $k = 0, \dots, p_j - 1$, where $t \to \lambda_j(t)$ is analytic and independent of $k = 0, \dots, p_j - 1$.
(2) Let $M_\nu(\mathbb{C})$ denote the vector space of $\nu \times \nu$ complex matrices. For each $j = 1, \dots, m$ and $k = 0, \dots, p_j - 1$, there exists an analytic matrix-valued function $Q_{j,k} : \bigcup_{r=0}^{D_j-1} U_r^j \to M_\nu(\mathbb{C})$, where $Q_{j,k}(t)$ is the eigenprojection onto the eigenspace associated to the eigenvalue $\lambda_j(t)e^{2\pi ik/p_j}$ of the matrix $C_j(t)$.
(3) If $t \in (\mathbb{R}^\nu/\mathbb{Z}^\nu) \setminus \bigcup_{r=0}^{Dj-1} U_r^j$ then the spectral radius of each $C_j(t)$ is bounded uniformly above by $\lambda - \epsilon$.

Using this description of the spectrum, we can write

$$\#(W_n \cap N) = \sum_{j=1}^{m} \sum_{r=0}^{D_j-1} \sum_{k=0}^{p_j-1} \int_{U_r^j} \lambda_j(t)e^{2\pi ikn/p_j} \langle Q_{j,k}(t)v_*, \mathbf{1}\rangle \, dt + O((\lambda - \epsilon)^n),$$

for some $\epsilon > 0$. Hence there exists constants $c_{r,k}^j = \langle Q_{j,k}(t_r^j)v_*, \mathbf{1}\rangle$, for $r = 0, \dots, D_j - 1$ and $k = 0, \dots, p_j - 1$, such that $\#(W_n \cap N)$ is equal to

$$\sum_{j=1}^{m} \left(\sum_{r=0}^{D_j-1} \sum_{k=0}^{p_j-1} e^{2\pi in(r/D_j + k/p_j)} c_{r,k}^j \right) \int_{U_0^j} \lambda_j(t)^n \left(1 + O(\|t\|)\right) \, dt + O((\lambda - \epsilon)^n). \tag{3.1}$$

The asymptotics of each

$$a_n^j := \int_{U_0^j} \lambda_j(t)^n \left(1 + O(\|t\|)\right) \, dt$$

were studied in [17], where it was shown that, for each $j = 1, \dots, m$, there exists $\tau_j > 0$ such that

$$a_n^j \sim \frac{\tau_j \lambda^n}{n^{\nu/2}} \tag{3.2}$$

as $n \to \infty$. Applying this along the subsequence Dn, where D is given by the product of all the $p_1, \dots, p_m$ and $D_1, \dots, D_m$, we see that

$$\#(W_{Dn} \cap N) = \frac{\widetilde{C}\lambda^{Dn}}{(Dn)^{\nu/2}} + o\left(\frac{\lambda^{Dn}}{(Dn)^{\nu/2}} \right) \tag{3.3}$$

as $n \to \infty$, where

$$\widetilde{C} = \sum_{j=1}^{m} \tau_j \left(\sum_{r=0}^{D_j-1} \sum_{k=0}^{p_j-1} c_{r,k}^j \right).$$

It is clear that $\widetilde{C} \in \mathbb{R}_{\geq 0}$. However, for (3.3) to be a useful asymptotic expression, we would like that $\widetilde{C}$ is strictly positive. We now show that this is always the case.

Lemma 3.7. *We necessarily have that $\widetilde{C} > 0$.*

Proof. Fix $j \in \{1, \dots, m\}$ and recall that for any loop $\gamma = (x_0, \dots, x_{Dn}) \in \mathcal{C}_j$ with $w_{f_j}(\gamma) = 0$, the group element $g_\gamma = \rho(x_0, x_1)\rho(x_1, x_2) \dots \rho(x_{Dn-1}, x_{Dn})$ belongs to the kernel of φ (or, equivalently, to N) and furthermore, g_γ has word length Dn. Also, for any two distinct loops $\gamma, \gamma' \in \mathcal{C}_j$, we have $g_\gamma \neq g_{\gamma'}$ whenever γ and γ' have the same initial vertex. Combining these observations and applying the pigeonhole principle gives that

$$\#(W_{Dn} \cap N) \geq (\#V_j)^{-1} \#\{\gamma \in \mathcal{C}_j : l(\gamma) = Dn,\, w_{f_j}(\gamma) = 0\}$$

for all $n \geq 1$. Pollicott and Sharp proved in [17] that

$$\#\{\gamma \in \mathcal{C}_j : l(\gamma) = Dn,\, w_{f_j}(\gamma) = 0\} \sim \frac{K\lambda^{Dn}}{(Dn)^{\nu/2}}$$

as $n \to \infty$ for some $K > 0$. Hence

$$\widetilde{C} = \limsup_{n\to\infty} \frac{(Dn)^{\nu/2} \#(W_{Dn} \cap N)}{\lambda^{Dn}} \geq K(\#V_j)^{-1} > 0,$$

as required. □

We can now conclude the proof of our main result.

Proof of Theorem 1.1. Combining (3.1) and (3.2) implies that

$$\#(W_n \cap N) = O\left(\sum_{j=1}^{m} \int_{U_0^j} \lambda_j(t)^n \left(1 + O(\|t\|)\right) dt\right) = O\left(\frac{\lambda^n}{n^{\nu/2}}\right)$$

which proves the first part of Theorem 1.1. The second part follows from (3.3) and the fact that $\widetilde{C} > 0$. □

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