LINEAR ACTIONS OF FREE GROUPS

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Abstract. In this paper we study dynamical properties of linear actions by free groups via the induced action on projective space. This point of view allows us to introduce techniques from Thermodynamic Formalism. In particular, we obtain estimates on the growth of orbits and their limiting distribution on projective space.

0. Introduction

Let $SL(d,\mathbb{R})$ denote the $d\times d$ matrices with real entries and determinant one. We shall consider the case $d \geq 3$. There is a natural linear action $SL(d,\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}^d$ given by matrix multiplication. Given a discrete subgroup $\Gamma \subset SL(d,\mathbb{R})$ and a fixed non-zero vector, it is interesting to consider the orbit $\mathcal{O}(v) = \{Av : A \in \Gamma\} \subset \mathbb{R}^d$.

When Γ is a uniform lattice (i.e., $SL(d,\mathbb{R})/\Gamma$ is compact) Greenberg showed that for a non-zero vector $v \in \mathbb{R}^d$ the set $\mathcal{O}(v) \subset \mathbb{R}^d$ is dense [3]. J. Dani showed that providing v is irrational, the result extends to the case that $\Gamma = SL(d,\mathbb{Z})$ [1]. Under either of these hypotheses the set $\{A \in \Gamma : ||Av|| \leq T\}$ has infinite cardinality. (Here, $||\cdot||$ denotes the standard euclidean 2-norm). However, for groups which are not lattices the orbits need not be dense, and indeed may be quite sparse. For example, $\#\{A \in \Gamma : ||Av|| \leq T\}$ may be finite for certain choices of v. To see that some restriction is necessary, notice that if v is an eigenvector for $A \in \Gamma$ then the same is also true for the matrices A^n, $n \in \mathbb{Z}$. In consequence, $||A^n v|| \leq T$ for infinitely many n. (A similar phenomenon occurs whenever the projectivized vector v lies in the limit set.) Except in these cases, one may ask how this counting function behaves as $T \to \infty$.

In this paper we shall consider the linear actions of a class of free groups $\Gamma \subset SL(d,\mathbb{R})$. Let Γ be freely generated by the (symmetric) set $\Gamma_0 = \{A_1^{\pm 1}, \ldots, A_k^{\pm 1}\}$. We call a generator $A \in \Gamma_0$ *pointed* if it has a unique eigenvalue of maximal modulus and the corresponding eigenspace V_A is one-dimensional. Denote by W_A the direct sum of the (generalized) eigenspaces of the other eigenvalues. We say that the generators are in *general position* if for each $A \in \Gamma_0$ we have $V_A \not\subset \bigcup_{B\in\Gamma_0-\{A\}} W_B$.

It is useful to make two hypotheses.

Hypothesis I. We shall assume that Γ_0 are generators for Γ which are pointed and in general position.

The first author would like to thank the Leverhulme Trust for their support. The second author was supported by an EPSRC Advanced Research Fellowship.

Hypothesis II. We shall assume that there are elements $A_1, A_2 \in \Gamma$ such that the logarithms of the maximal eigenvalues are not rationally related.

These are generic conditions on the generators. The first hypothesis plays a rôle in the proof of the beautiful result of Tits that a subgroup $G \subset SL(d, \mathbb{R})$ is either virtually solvable or it contains a free group on two generators as a subgroup (cf. [12], [4]). In the analogous setting of $SL(2, \mathbb{R})$, the second hypothsis is equivalent to the non-arithmeticity of the length spectrum of the associated Riemann surface or mixing for the corresponding geodesic flow.

Before we state our main result, we shall introduce two pieces of notation. Given $l \geq 1$, we shall denote by $\Gamma^{(l)} \subset \Gamma$ the free subgroup generated by the lth powers, i.e., the elements $\{A^l : A \in \Gamma_0\}$. We denote by

$$U = \mathbb{R}^d - \bigcup_{A \in \Gamma_0} W_A$$

the complement of hyperplanes W_A, $A \in \Gamma_0$.

Our main result is the following.

Theorem 1. *Let $\Gamma \subset SL(d, \mathbb{R})$ be a free group with generators Γ_0 satisfying Hypotheses I and II. Let $v \in U$. Then there exists $l \geq 1$, $C = C(v, l) > 0$ and $p = p(l) > 0$ such that*

$$\#\{A \in \Gamma^{(l)} : ||Av|| \leq T\} \sim CT^p.$$

(In fact, the same conclusion is true for all sufficiently large l).

As a consequence we have the following estimate on matrices counted by their norms.

Corollary. *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 T^p \leq \#\{A \in \Gamma^{(l)} : ||A|| \leq T\} \leq C_2 T^p$$

Proof. Since $||Av|| \leq ||A|| \, ||v||$ we can write

$$\#\{A \in \Gamma^{(l)} : ||A|| \leq T\} \leq \#\{A \in \Gamma^{(l)} : ||Av|| \leq T||v||\}.$$

On the other hand, a simple geometric argument shows that for each $v \in U$ there exists $D > 0$ such that $||A|| \leq D||Av||$, for all $A \in \Gamma$. Hence,

$$\#\{A \in \Gamma^{(l)} : ||A|| \leq T\} \geq \#\{A \in \Gamma^{(l)} : D||Av|| \leq T\}.$$

This completes the proof.

The value p is precisely the abscissa of convergence of the Dirichlet series defined by $\eta(s) = \sum_{A \in \Gamma^{(l)}} ||Av||^{-s}$. We can see that $p > 0$ by the following argument. First observe that there exists $c > 0$ such that $||Av|| \leq e^{c|A|}$, where $|A|$ denotes the word length of A, i.e., the number of generators from Γ_0 used to write A. Recalling that $\#\Gamma_0 = 2k$, we then have the inequality

$$\eta(s) \geq \sum_A e^{-sc|A|} = 1 + \sum_{n=1}^{\infty} 2k(2k-1)^{n-1} e^{-scn},$$

where the Right Hand Side diverges for $s < \log(2k-1)/c$. In particular, we see that $p \geq \log(2k-1)/c > 0$.

It is also an easy observation that p is independent of the choice of v. This follows since given any $v, v' \in U$ there exists $E > 0$ such that

$$\frac{1}{E} \leq \frac{||Av||}{||Av'||} \leq E, \quad \forall A \in \Gamma.$$

Example. For $A \in \Gamma^{(l)}$ write $A = (a_{ij})_{i,j=1}^{d} \in SL(d,\mathbb{R})$. For any $i = 1, \ldots, d$ we can take v to be the ith basis vector. If we suppose that $v \in U$ then there exists a constant $C_i > 0$ such that

$$\#\left\{A \in \Gamma : \sum_{j=1}^{d} a_{ij}^2 \leq T\right\} \sim C_i T^{p/2}, \text{ as } T \to +\infty.$$

As motivation for the proof of Theorem 1, we should consider the classical interpretation of $SL(2,\mathbb{R})$ as isometries of the Poincaré disc $\mathbb{D}^2$. In this case, the corresponding action on the ideal boundary S^1 exhibits hyperbolic-like behaviour. The natural analogue of this for $d \geq 3$ is the projective action on $\mathbb{R}P^{d-1}$. This action on $\mathbb{R}P^{d-1}$ will have sources corresponding to eigenvectors of eigenvalues of modulus smaller than unity. There will also be sinks, corresponding to eigenvectors of eigenvectors with eigenvalues of modulus greater than unity.

We define the *limit set* $\Lambda \subset \mathbb{R}P^{d-1}$ of $\Gamma^{(l)}$ to be the closed set which is the accumulation point of the set $\{Av : A \in \Gamma^{(l)}\}$, where we take any point $v \in [U]$ in the projectivization of U. For each $s > p$ we can define a probability measure m_s on $\mathbb{R}P^{d-1}$ by

$$m_s = \frac{\sum_{A \in \Gamma^{(l)}} \delta_{Av} ||Av||^{-s}}{\sum_{A \in \Gamma^{(l)}} ||Av||^{-s}}.$$

The next theorem describes the distribution of the orbit $\Gamma^{(l)}v$ on $\mathbb{R}P^{d-1}$, and could be viewed as an analogue of the Patterson-Sullivan measure for hyperbolic manifolds [8], [11].

Theorem 2. *Let $\Gamma \subset SL(d,\mathbb{R})$ be a free group with generators Γ_0 satisfying Hypotheses I and II and let $l > 1$ be such that the conclusions of Theorem 1 hold. There exists a probability measure m such that we have the convergence $\lim_{s \to p} m_s = m$ in the weak star topology. Furthermore, m is an ergodic non-atomic measure supported on the limit set of Γ and*

$$\frac{d(A_* m)}{dm}(x) = ||A^{-1}x||^{-p}$$

for all $A \in \Gamma$.

The authors would like to thank the referee for a number of helpful comments.

1. Actions on projective spaces

Assume that $A_1, \ldots, A_k \in SL(d,\mathbb{R})$ generate a free group Γ. Write $\Gamma_0 = \{A_1, \ldots, A_k, A_{k+1}, \ldots, A_{2k}\}$, where $A_{k+i} = A_i^{-1}$, for $i = 1, \ldots, k$.

If A is a concatenation of n generators then we write $|A| = n$. Each element

$$A = \begin{pmatrix} a_{11} & \dots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \dots & a_{dd} \end{pmatrix} \in \Gamma$$

has the standard linear action $A : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$A(z_1, \dots, z_d) = (a_{11}z_1 + \dots + a_{1d}z_d, \dots, a_{d1}z_1 + \dots + a_{dd}z_d).$$

Let $\mathbb{R}P^{d-1} = (\mathbb{R}^d \backslash \{0\})/\sim$ denote the real projective space, where $\sim$ is the equivalence relation $(z_1, \dots, z_d) \sim (\lambda z_1, \dots, \lambda z_d)$ for $\lambda \in \mathbb{R}\backslash\{0\}$. We define a metric D on $\mathbb{R}P^{d-1}$ by

$$D(v, w) = \cos^{-1}\left(\frac{\langle v, w\rangle}{||v||||w||}\right).$$

An element $A \in \Gamma$ induces a projective action $A : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ given by

$$A[z_1, \dots, z_d] = [a_{11}z_1 + \dots + a_{1d}z_d, \dots, a_{d1}z_1 + \dots + a_{dd}z_d].$$

Let us denote by $[V_A] = (V_A - \{0\})/\sim$ and $[W_A] = (W_A - \{0\})/\sim$ the points and hyperplanes in $\mathbb{R}P^{d-1}$ corresponding to the eigenspaces V_A and hyperplanes W_A in $\mathbb{R}^d$.

The contraction property. There exists $0 < \theta < 1$ and a family of closed sets $\mathcal{C}(A)$, $A \in \Gamma_0$, such that

$$||D_z A|| \le \theta, \quad \forall z \in \bigcup_{B \in \Gamma_0 - \{A^{-1}\}} \mathcal{C}(B),$$

where $D_z A$ is the derivative of the projective action of A at the point z. Moreover, for each $A \in \Gamma_0$ and $B \in \Gamma_0 - \{A^{-1}\}$ we have that $A\mathcal{C}(B) \subset \mathcal{C}(A)$.

Example. If we consider the free group generated by the matrices

$$A_1 = \begin{pmatrix} 1000 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{100} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 500.05 & -0.45 & 499.995 \\ 499.95 & 0.055 & 499.995 \\ 499.95 & 0.045 & 500.005 \end{pmatrix}$$

then the contraction property holds. More generally consider for $a > 1$ the free group with the two generators

$$A_1 = \begin{pmatrix} a^3 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{2}(a^3 + 1/a) & \frac{1}{2}(-1/a + 1/a^2) & \frac{1}{2}(a^3 - 1/a^2) \\ \frac{1}{2}(a^3 - 1/a) & \frac{1}{2}(1/a + 1/a^2) & \frac{1}{2}(a^3 - 1/a^2) \\ \frac{1}{2}(a^3 - 1/a) & \frac{1}{2}(1/a - 1/a^2) & \frac{1}{2}(a^3 + 1/a^2) \end{pmatrix}$$

where

$$A_2 = CA_1C^{-1} \text{ with } C = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

The matrix A_1 has the standard bases as eigenvectors, and A_2 has the eigenvectors rotated by the matrix C.

There are natural co-ordinates (θ, ϕ) on $\mathbb{R}P^2$ associated to spherical co-ordinates (i.e. we write $[x, y, z] = [\sin\phi\cos\theta, \cos\phi\cos\theta, \sin\theta]$) Observe that the A_1 image $(x', y', 0)$ of $(x, y, 0)$, for example, satisfies $\tan(x/y) = \lambda^4 \tan(x'/y')$. A simple calculation shows that the projective action of $A_1 : \mathbb{R}P^2 \mapsto \mathbb{R}P^2$ is contracting in the region corresponding to $\cos^2\theta > a^4/(1 + a^4)$. The image of the this region (which we denote by $\mathcal{C}(A_1)$) is then contained in the region with $\tan\theta > a^2$.

Considering $(0, y, z)$, for example, with $\tan\alpha = z/y$ we see that the projective action of $A_1^{-1} : \mathbb{R}P^2 \mapsto \mathbb{R}P^2$ is contracting in the region corresponding to $\cos^2\alpha > a/(1+a)$. The image of the this region, denote by $\mathcal{C}(A_1^{-1})$ is contained in the region with $\tan\alpha > \sqrt{a}$.

Since A_2 is derived from A_1 by a change in the orientation of the eigenvectors, similar estimates hold.

To keep the four regions $(A_1^{\pm 1})$ and $(A_2^{\pm 1})$ disjoint we can ask that $\theta, \alpha < \pi/8$, which requires $a > 5.82843....$ Thus, taking $a = 10$ suffices.

FIGURE 1. The regions $\mathcal{C}(A_i)$ $(i = 1, 2)$

The next lemma shows that Γ contains a free subgroup with the above property.

Lemma 1. *Let Γ be a free group with generating set Γ_0, satisfying Hypothesis I. We can choose $0 < \theta < 1$ and $l \geq 1$ such that the free group $\Gamma^{(l)}$ generated by the elements A^l, $A \in \Gamma_0$, satisfies the contraction property.*

Proof. For each $A \in \Gamma_0$ we can choose disks $\mathcal{C}(A) = \{x \in \mathbb{R}P^{d-1} : D([V_A], x) \leq \epsilon/2\}$, where $\epsilon = \min\{\sup_{B \in \Gamma_0 - \{A^{-1}\}} D([V_A], [W_B]) : A \in \Gamma_0\}$. Then the union $\bigcup_{B \in \Gamma_0 - \{A^{-1}\}} \mathcal{C}(B)$ is contained inside the basin of attraction of $A : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$. In particular, for sufficiently large $l \geq 1$ we have that

$$A^l \left(\bigcup_{B \in \Gamma_0 - \{A^{-1}\}} \mathcal{C}(B) \right) \subset \mathcal{C}(A).$$

Set $\theta_0 = ||D_{[V_A]}A|| < 1$ and fix choices $\theta_0 < \theta_1 < \theta < 1$. Consider the neighbourhood $U = \{z \in \mathbb{R}P^{d-1} : ||D_z A|| \leq \theta_1\}$ of $[V_A]$. We may choose l_0 sufficiently large that for $l \geq l_0$ we have that

$$A^l \left(\bigcup_{B \in \Gamma_0 - \{A^{-1}\}} \mathcal{C}(B) \right) \subset U.$$

Then for $z \in \bigcup_{B \in \Gamma_0 - \{A^{-1}\}} \mathcal{C}(B)$ we have the bound

$$||D_z A^l|| \leq ||D_z A^{l_0}|| \; ||D_{A^{l_0} z} A^{l-l_0}|| \leq C\theta_1^{l-l_0}.$$

To complete the proof, we need only choose l sufficiently large that $C\theta_1^{l-l_0} < \theta$.

We repeat this construction for each $A \in \Gamma_0$, and take θ in the statement of the lemma to be the maximum of the values above.

We can assume, without loss of generality, that $l = 1$ in the sequel.

We shall now describe the relation between $\mathrm{Jac}(A)$ and $||Av||$.

Proposition 1.

(1) *Suppose that $v \in \mathbb{R}^d$ then $Jac(A)(v) = ||v||^{(d-1)}/||Av||^{(d-1)}$.*

(2) *Suppose that $v \in U$ then $||A^m v||^{1/m}$ converges to the maximal eigenvalue for A, as $m \to +\infty$.*

Proof. We begin with the proof of part (1). We can represent a vector $v \in \mathbb{R}^d$ in terms of spherical co-ordinates $\omega = (\omega_1, \ldots, \omega_{d-1})$ on the sphere S^{d-1} and its length $||v|| = r$.

To perform the calculations we shall compare these spherical co-ordinates with standard euclidean coordinates. Let us denote by $\psi_v : U \to V$ a chart from a neighbourhood U of the origin in euclidean space to a neighbourhood V of v in spherical co-ordinates.

Let us denote by $\mathrm{Jac}^{\mathrm{sph}}(A)(v)$ the Jacobian of A at v in terms of spherical coordinates, and by $\mathrm{Jac}^{\mathrm{euc}}(A)(v)$ the Jacobian of A at v in terms of euclidean coordinates. It follows by the chain rule that we can write

$$\mathrm{Jac}^{\mathrm{sph}}(A)(v) = \mathrm{Jac}(\psi_{Av})\mathrm{Jac}^{\mathrm{euc}}(A)(v)\mathrm{Jac}(\psi_v^{-1}).$$

Observe that $\mathrm{Jac}^{\mathrm{euc}}(A)(v) = 1$, since $A \in SL(d, \mathbb{R})$. Moreover, by the standard change of variable from spherical coordinates to euclidean coordinates, we can write that $\mathrm{Jac}^{\mathrm{sph}}(A)(v) = ||v||^{d-1}$. In particular, the above equality reduces to

$$\mathrm{Jac}^{\mathrm{sph}}(A)(v) = \frac{||v||^{d-1}}{||Av||^{d-1}}.$$

Part (2) follows from the spectral radius theorem.

Remark. The linear action on projective space by matrices is also familiar from the work of Birkhoff on the Hilbert metric. An interesting interpretation of the weight for the projectivized action of positive matrices $GL(d, \mathbb{R})$ on the positive quadrant (relative to the Hilbert metric) appears in a paper of Wojtkowski [13].

2. Subshifts of finite type

A sequence $(x_0, \dots, x_{n-1}) \in \{1, \dots, 2k\}^n$ is called *admissible* if $A_{x_{i+1}} \neq A_{x_i}^{-1}$ for $i = 0, \dots, n-2$. Given an admissible sequence $x = (x_0, \dots, x_{n-1})$ we shall write $A_{x,n} = A_{x_0} A_{x_1} \cdots A_{x_{n-1}}$.

Let $X = \{(x_n)_{n=0}^{\infty} \in \{1, \dots, 2k\}^{\mathbb{Z}^+} : A_{x_{n+1}} \neq A_{x_n}^{-1} \quad \forall n \geq 0\}$ be the space of infinite admissible sequences and let $\sigma : X \to X$ be the shift map given by $(\sigma x)_n = x_{n+1}$, $n \geq 0$.

It is convenient to regard finite admissible sequences $(x_0, \dots, x_{n-1})$ as infinite sequences in the alphabet $\{1, \dots, 2k\} \cup \{0\}$ by adjoining an infinite string of zeros to obtain $(x_0, \dots, x_{n-1}, 0, 0, \dots)$. For brevity we shall write this as $(x_0, \dots, x_{n-1}, \dot{0})$. We shall denote the set of finite sequences completed in the above way by X_0 and write $\widehat{X} = X \cup X_0$. Furthermore, we adopt the convention that A_0, the group element associated to the symbol 0, is equal to the identity.

We define a metric on the space $\widehat{X}$ by

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1 - \delta(x_n, y_n)}{(1/\theta)^n}.$$

With this metric, X_0 is a dense subset of $\widehat{X}$.

Lemma 2. *Assume that $(x_0, \dots, x_{n-1})$ is an admissible sequence then*

$$A_{x,n} : \mathcal{C}(A_{x_{n-1}}) \to \mathcal{C}(A_{x_0})$$

is a contraction. In particular, there exists $C > 0$ and $0 < \theta < 1$ such that $diam(A_{x,n}\mathcal{C}(A_{x_{n-1}})) \leq C\theta^n$.

Proof. Let DA denote the derivative of the projective map $A : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$. By the chain rule we can write

$$DA_{x,n}(v) = DA_{x_0}(A_{x_1} \dots A_{x_{n-1}} v) \cdots DA_{x_i}(A_{x_{i+1}} \dots A_{x_{n-1}} v) \cdots DA_{x_{n-1}}(v).$$

However, since $A_{x_{i+1}} \dots A_{x_{n-1}} v \in \mathcal{C}(A_{x_i})$, we see that $||DA_{x_i}(A_{x_{i+1}} \dots A_{x_{n-1}} v)|| \leq \theta$.

Assume for the result of the section that $v \in \mathbb{R}P^{d-1}$ is not on the complementary planes associated to any of the generators from Γ_0. We need the following quantitative estimates.

Lemma 3. *Let $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{n-1})$ be admissible sequences. There exists $C > 0$ such that if $0 \leq m \leq n$ and $x_i = y_i$, for $i = 0, \dots, m-1$, then $D(A_{x,n}v, A_{y,n}v) \leq C\theta^m$.*

Proof. Observe that

$$A_{x,n}v, A_{y,n}v \in A_{x_0} \cdots A_{x_{m-2}} \mathcal{C}(A_{x_{m-1}}) = A_{y_0} \cdots A_{y_{m-2}} \mathcal{C}(A_{y_{m-1}}).$$

Thus, in particular, $D(A_{x,n}v, A_{y,n}v) \leq \mathrm{diam}(A_{x_0} \cdots A_{x_{m-2}} \mathcal{C}(A_{x_{m-1}})) \leq C\theta^m$, as required.

Lemma 4. *For any* $x = (x_n)_{n=0}^{\infty} \in X$*, the limit* $v_x = \lim_{n\to+\infty} A_{x,n}v$ *exists and satisfies the estimate* $D(v_x, A_{x,m}v) \leq C\theta^m$*.*

Proof. By Lemma 2, $A_{x,n}v$, $n \geq 0$, is a Cauchy sequence and so $v_x = \lim_{n\to+\infty} A_{x,n}v$ exists. Letting $n \to +\infty$ in Lemma 2 gives $D(v_x, A_{x,m}v) \leq C\theta^m$.

Using the above lemma, we may define a Hölder continuous surjective map $\pi : X \to \Lambda$ from X to the limit set, by $\pi(x) = v_x$.

We define $\mathcal{F}_\theta = \left\{ f : \widehat{X} \to \mathbb{R} : |f|_\theta < +\infty \right\}$, where

$$|f|_\theta = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y \right\}.$$

This is a Banach space with respect to the norm $||f||_\theta = |f|_\theta + |f|_\infty$. We have the following two useful technical results.

Lemma 5. *There exists* $C > 0$ *such that for* $x, y \in \widehat{X}$*,* $n \geq 1$*, we have*

$$\begin{aligned} &\left| \log \left(\frac{Jac(A_{x,n})(v)}{Jac(A_{\sigma x,n-1})(v)} \right) - \log \left(\frac{Jac(A_{y,n})(v)}{Jac(A_{\sigma y,n-1})(v)} \right) \right| \\ &\leq Cd(x,y). \end{aligned} \tag{2.1}$$

Proof. Assume that $x_i = y_i$, for $i = 0, \ldots, m-1$. If $m \geq n$, then the Left Hand Side of (3.1) vanishes, since $x \mapsto \mathrm{Jac}(A_{x_0})(v)$ is locally constant. On the other hand if $n \geq m$ then by Lemma 3 we have that

$$D(A_{x,n}v, A_{y,n}v) \leq C\theta^m = Cd(x,y).$$

However, since $\mathrm{Jac}(A_{x_0})(\cdot) : \mathbb{R}P^{d-1} \to \mathbb{R}$ is obviously Lipschitz, we see that

$$|\log(\mathrm{Jac}(A_{x_0})(v)) - \log(\mathrm{Jac}(A_{y_0})(v))| \leq C' d(x,y).$$

Finally, since $A_{x,n} = A_{x_0} A_{\sigma x, n-1}$, we can apply the chain rule to write $DA_{x,n}(\cdot) = DA_{x_0}(A_{\sigma x,n-1}\cdot) DA_{\sigma x,n-1}(\cdot)$. Taking determinants and logarithms and evaluating at v gives

$$\begin{aligned} &\log \left(\frac{\mathrm{Jac}(A_{x,n})(v)}{\mathrm{Jac}(A_{\sigma x,n-1})(v)} \right) \\ &= \log \left(\frac{\mathrm{Jac}(A_{x_0})(A_{\sigma x,n-1}v)\mathrm{Jac}(A_{\sigma x,n-1})(v)}{\mathrm{Jac}(A_{\sigma x,n-1})(v)} \right) \\ &= \log(\mathrm{Jac}(A_{x_0})(A_{\sigma x,n-1}v)). \end{aligned} \tag{2.2}$$

Observe that $x_0 = y_0$ and the function $\log \mathrm{Jac}(A_{x_0})(\cdot)$ is analytic. Thus it suffices to observe that since $(\sigma x)_i = (\sigma y)_i$ for $i = 0, \ldots, m-2$ then by Lemma 3, $D(A_{\sigma x,n-1}v, A_{\sigma y,n-1}v) \leq C\theta^{m-1}$

We define a function $r : X_0 \to \mathbb{R}$ by

$$r(x_0, \ldots, x_{n-1}, \dot{0}) = -\frac{1}{d-1} \log \left(\frac{\mathrm{Jac}(A_{x,n})(v)}{\mathrm{Jac}(A_{\sigma x,n-1})(v)} \right).$$

We can extend this to a function on $\widehat{X}$, by the following result.

Proposition 2. *The function $r : X \to \mathbb{R}$ given by*

$$r(x) = -\frac{1}{d-1} \lim_{n\to+\infty} \log \left(\frac{Jac(A_{x,n})(v)}{Jac(A_{\sigma x,n-1})(v)} \right)$$

is both well-defined and an element of $\mathcal{F}_\theta$.

Proof. From the identities (2.1) and (2.2) this immediately follows.

The usefulness of the function $r : \widehat{X} \to \mathbb{R}$ comes from the following identity.

Proposition 3. *Assume that $x = (x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots) \in X_0$, where $x_{n-1} \neq 0$, then we can identify*

$$-\frac{1}{d-1} \log Jac(A_{x,n})(v) = \sum_{i=0}^{n-1} r(\sigma^i x).$$

3. Transfer operators

We denote by $\mathcal{F}_\theta(\mathbb{C}) = \left\{ f : \widehat{X} \to \mathbb{C} : |f|_\theta < +\infty \right\}$, where

$$|f|_\theta = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y \right\}.$$

This is a Banach space with respect to the norm $||f||_\theta = |f|_\theta + |f|_\infty$.

For $f \in \mathcal{F}_\theta(\mathbb{C})$ we can define *Ruelle transfer operators* $L_f : \mathcal{F}_\theta(\mathbb{C}) \to \mathcal{F}_\theta(\mathbb{C})$ by

$$(L_f h)(x) = \sum_{\substack{\sigma y = x \\ y \neq \dot{0}}} e^{f(y)} h(y).$$

Remark. This definition of the Ruelle transfer operator differs from the usual definition in that in the summation over pre-images y of x we exclude the possibility $y = \dot{0}$. However, it agrees with the more familiar definition for all $x \neq \dot{0}$ and its only effect on the spectrum is to exclude an eigenvalue $e^{f(\dot{0})}$ (corresponding to the eigenvector which is the characteristic function for the set $\{\dot{0}\}$).

We can associate to each continuous function $f : \widehat{X} \to \mathbb{R}$ the *pressure* $P(f) \in \mathbb{R}$ defined by

$$P(f) = \limsup_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} e^{f^n(x)} \right).$$

The pressure is also given by the equivalent variational identity

$$P(f) = \sup\{h(\nu) + \int f d\nu : \nu \text{ is an invariant probability}\}.$$

If $f : \widehat{X} \to \mathbb{R}$ is Hölder continuous, then the above supremum is attained at a unique probability measure μ called the equilibrium state for f.

Proposition 4 (Ruelle Operator Theorem [10], [7]**).**

(1) *For $s \in \mathbb{R}$, the spectral radius of the operator L_{-sr} is equal to $e^{P(-sr)}$ and this is a simple eigenvalue of strictly maximal modulus. Furthermore, associated to this eigenvalue, there is a strictly positive eigenfunction h_s, and an eigenmeasure ν_s. (We adopt the normalization $\nu_s(1) = 1$ and $\nu(h_s) = 1$.)*

(2) *For $s \in \mathbb{C}$, in a sufficiently small neighbourhood of $\mathbb{R}$, the operator L_{-sr} continues to have a simple eigenvalue of maximal modulus denoted by $e^{P(-sr)}$. We shall again denote the associated eigenfunction and eigenfunctional by h_s and ν_s, respectively, with the corresponding normalization.*

Since, by the above proposition, $e^{P(-sr)}$ is an isolated eigenvalue we know that $s \mapsto P(-sr)$ is analytic for s in a neighbourhood of $\mathbb{R}$. It is well known that for $t_0 \in \mathbb{R}$, $dP(-tr)/dt|_{t=t_0} = -\int r d\mu_{t_0}$, where μ_{t_0} is the equilibrium state for $-t_0 r$. Thus, in particular, the function $s \mapsto P(-sr)$ on $\mathbb{R}$ is strictly decreasing from $+\infty$ to $-\infty$.

For a bounded linear operator $T : B \to B$ acting on a Banach space B let $\rho(T)$ denote the spectral radius. We define the *essential spectrum* $ess(T)$ to be the subset of the spectrum $spec(T) \subset \mathbb{C}$ of T consisting of those $\lambda \in spec(T)$ such that at least one of the following is true

(1) $Range(\lambda - T)$ is not closed in B;
(2) λ is a limit point of $spec(T)$;
(3) $\cup_{r=1}^{\infty} ker(\lambda - T)^r$ is infinite dimensional.

We define the *essential spectral radius* to be $\rho_e(T) = sup\{|\lambda| : \lambda \in ess(T)\}$. The operator $T : B \to B$ is *quasi-compact* if the essential spectral radius is strictly smaller than the spectral radius.

Proposition 5 [7]**.** *For $s \in \mathbb{C}$, the spectral radius of $L_{-sr} : \mathcal{F}_\theta(\mathbb{C}) \to \mathcal{F}_\theta(\mathbb{C})$ satisfies $\rho(L_{-sr}) \leq \rho(L_{Re(f)})$ and essential spectral radius satisfies $\rho_e(L_{-sr}) \leq \theta\rho(L_{-Re(s)r})$.*

Proposition 5 implies that for any $\epsilon > 0$ we may write

$$L_{-sr}^n = \sum_{\lambda} \mathbb{P}_\lambda L_{-sr}^n + Q L_{-sr}^n, \tag{3.1}$$

where the summation is over eigenvalues λ for L_{-sr} satisfying $\rho_e(L_{-sr}) + \epsilon \leq |\lambda| \leq \rho(L_{-Re(s)r})$, $\mathbb{P}_\lambda$ is the eigenprojection associated to λ and Q is the projection associated to the part of the spectrum in $\{z : |z| \leq \rho_e(L_{-Re(s)r}) + \epsilon\}$ so that, in particular, $\lim_{n\to+\infty} ||QL_{-sr}^n||^{1/n} \leq \rho_e(L_{-Re(s)r}) + \epsilon$.

4. Poincaré series

An important tool that is useful in the proofs of Theorems 1 and 2 is a complex function analogous to the classical Poincaré series in hyperbolic geometry. We define a complex function $\eta(s)$ by

$$\eta(s) = \sum_{A\in\Gamma} ||Av||^{-s}. \tag{4.1}$$

The Dirichlet series (4.1) converges to an analytic function in $s \in \mathbb{C}$ provided $Re(s)$ is sufficiently large. We shall denote its abscissa of convergence by p, which we recall from the introduction, is strictly positive.

We can use Proposition 1(i) and Proposition 3 to rewrite $\eta(s)$ in terms of the transfer operator. More precisely,

$$\begin{aligned} \eta(s) &= 1 + \sum_{n=1}^{\infty} \sum_{\substack{\sigma^n x = \dot{0} \\ x \neq \dot{0}}} e^{-sr^n(x)} \\ &= 1 + \sum_{n=1}^{\infty} L_{-sr}^n 1(\dot{0}), \end{aligned} \tag{4.2}$$

where 1 denotes the constant function taking the value 1.

In order to obtain estimates on $\pi_v(T)$ we require that $\eta(s)$ has an extension to a larger domain than its half-plane of convergence. This is provided by the next proposition.

Proposition 6. *The function $\eta(s)$ has a meromorphic extension to some strip $Re(s) \geq p - \epsilon$, with a simple pole at $s = p$.*

Proof. Substituting (3.1) into (4.2) gives that

$$\begin{aligned} \eta(s) &= 1 + \sum_{\lambda} \left(\sum_{n=1}^{\infty} L_{-sr}^n \mathbb{P}_\lambda(s) 1 \right)(\dot{0}) + \left(\sum_{n=1}^{\infty} L_{-sr}^n Q(s) 1 \right)(\dot{0}) \\ &= 1 + \sum_{\lambda} \left(\sum_{n=0}^{\infty} L_{-sr}^n \mathbb{P}_\lambda(s) L_{-sr} 1 \right)(\dot{0}) + \left(\sum_{n=1}^{\infty} L_{-sr}^n Q(s) 1 \right)(\dot{0}) \\ &= 1 + \sum_{\lambda} \left((I - L_{-sr}^{(\lambda)})^{-1} \mathbb{P}_\lambda(s) L_{-sr} 1 \right)(\dot{0}) + \left(\sum_{n=1}^{\infty} L_{-sr}^n Q(s) 1 \right)(\dot{0}) \end{aligned} \tag{4.3}$$

where $L_{-sr}^{(\lambda)}$ is the restriction of L_{-sr} to the finite dimensional generalized eigenspace associated to λ.

The final term in (4.3) converges to an analytic function when $\rho_e(L_{-Re(s)r}) < 1$, which will be satisfied provided $Re(s) > p - \epsilon$, for some choice of $\epsilon > 0$, by Proposition 5. Moreover, the other term in in (4.3) is meromorphic since $(I - L_{-sr})^{-1}$ can be written in the form

$$\sum_{\lambda} (I - L_{-sr}^{(\lambda)})^{-1} = \sum_{\lambda} \frac{N_\lambda(s)}{\det(I - L_{-sr}^{\lambda})}, \tag{4.4}$$

where $N_\lambda(s)$ are analytic operator valued functions and, furthermore, it is well known that $\det(I - L_{-sr}^{(\lambda)})$ are analytic [5].

Finally, when $s = p$ the operator L_{-pr} has 1 as a simple maximal eigenvalue, by Proposition 4. In particular, in a neighbourhood of p the expression (4.4) implies that

$$\eta(s) = \frac{\phi(s)}{1 - e^{P(-sr)}},$$

where $\phi(s)$ is analytic and $\phi(0) \neq 0$. However,

$$\lim_{s \to p} \frac{s-p}{1-e^{P(-sr)}} = \frac{1}{\int r d\mu_p},$$

which does not vanish since $\int r d\mu_p = (1/n) \int r^n d\mu_p > 0$. From this we deduce that the pole is simple.

The next proposition gives us more information on the location of poles.

Proposition 7. *The pole at $s = p$ is the only pole on $Re(s) = p$*

Proof. If the Poincaré series $\eta(s)$ has a pole at $s = p + it$ then L_{-sr} has unity as an eigenvalue. It then follows by standard arguments based on convexity arguments for finite sums there there exists $M \in C^0(X, 2\pi\mathbb{Z})$ and $u \in C^0(X, \mathbb{R})$ such that $tr(x) = M(x) + u(\sigma x) - u(x)$ [7]. Let us assume for a contradiction that such an identity holds.

For a free group there is a natural bijection between conjugacy classes and periodic orbits for the shift map $\sigma : X \to X$. In particular, given a periodic orbit $\{x, \sigma x, \ldots, \sigma^{n-1} x\}$ we associate the unique conjugacy class $\langle A \rangle$ in Γ, where $A \in \Gamma$ corresponds to the concatenation of the edge labelling around the closed path in the graph corresponding to x.

It is easy to see that $\lim_{m \to \infty} ||A^m v||^{1/m} = e^{r^n(x)}$. Moreover, by Lemma 4, this limit is equal to λ_A, where λ_A is the maximal eigenvalue of A.

By assumption, $e^{tr^n(x)} = e^{M^n(x)} \in \{e^{2\pi k} : k \in \mathbb{Z}\}$. However, it is clear that $\{\lambda_A^t : A \in \Gamma\}$ is not contained in such a multiplicative subgroup of $\mathbb{R}^+$ by virtue of Hypothesis II.

5. Proof of Theorem 1

We now explain how to complete the proof of Theorem 1. Let

$$\pi_v(T) = \#\{A \in \Gamma : ||Av|| \leq T\}.$$

Observe that $\eta(s)$ can be represented by the Stieltjes integral

$$\eta(s) = \int_0^\infty T^{-s} d\pi_v(T). \tag{5.1}$$

The analytical properties of $\eta(s)$ described in Proposition 7 imply asymptotic estimates on $\pi_v(T)$ using the following classical result.

Proposition 8 (Ikehara-Wiener Tauberian Theorem) [2,p.54]**.** *Suppose that $F(s)$ has the following properties.*

(1) *In the half-plane $Re(s) > \delta$ the function has the representation*

$$F(s) = \int_0^\infty T^{-s} dA(T)$$

where $A(T)$ is a positive, monotone increasing function and $\delta > 0$.

(2) *In the region $Re(s) \geq 1$, $s \neq 1$, the function F has the representation*

$$G(s) = F(s) - \frac{C}{s - \delta}$$

where $G(s)$ is continuous on the half-plane $Re(s) \geq \delta$ and $C > 0$.

Then $A(T) \sim CT^\delta$, as $T \to +\infty$.

Applying Proposition 8 to the identity (5.1) we see that $\pi_v(T) \sim CT^p$, where C is the residue of the simple pole for $\eta(s)$ at $s = p$.

6. Proof of Theorem 2

In order to prove Theorem 2 we shall modify the analysis in the preceding sections. We shall consider a Dirichlet series associated to certain cylinder sets, and then employ an approximation argument. More precisely, let $[\underline{i}] = \{x \in \widehat{X} : x_j = i_j, 0 \leq j \leq n-1\}$ denote a cylinder, where $\underline{i} = (i_0, \dots, i_{n-1})$. For $Re(s) > p$, we may write

$$\eta_{\underline{i}}(s) = \sum_{n=0}^{\infty} \sum_{\substack{\sigma^n x = \dot{0} \\ x \neq \dot{0}}} \chi_{[\underline{i}]}(x) e^{-sr^n(x)} = \sum_{n=0}^{\infty} L_{-sr}^n \chi_{[\underline{i}]}(\dot{0})$$

Proposition 9. *The function $\eta_{\underline{i}}(s)$ is analytic in a neighbourhood of $Re(s) \geq p$, except for a simple pole at $s = p$ with residue $\nu_p([\underline{i}]) h_p(\dot{0}) / \int r d\mu_p$. In particular, $\lim_{s \searrow p} \eta_{\underline{i}}(s)/\eta(s) = \nu_p([\underline{i}])$.*

Proof. By Proposition 4 there exists a neighbourhood of $\{s : Re(s) \geq p\} - \{p\}$ in which we have the bound $\limsup_{n \to +\infty} ||L_{-sr}^n||^{1/n} < 1$. In particular, $\eta_{\underline{i}}(s)$ converges uniformly to an analytic function.

For s in a neighbourhood of p, we can use Proposition 4 to write

$$L_{-sr}^n w = e^{nP(-sr)} \nu_s(w) h_s + U_s^n w,$$

where $\limsup_{n \to +\infty} ||U_s^n||^{1/n} < 1$. We then observe that

$$\begin{aligned} \eta_{\underline{i}}(s) &= \sum_{n=0}^{\infty} L_{-sr}^n \chi_{[\underline{i}]}(\dot{0}) \\ &= \sum_{n=0}^{\infty} e^{nP(-sr)} \nu_s(\chi_{[\underline{i}]}) h_s(\dot{0}) + \sum_{n=0}^{\infty} U_s^n \chi_{[\underline{i}]}(\dot{0}) \\ &= \frac{\nu_s(\chi_{[\underline{i}]}) h_s(\dot{0})}{1 - e^{P(-sr)}} + \sum_{n=0}^{\infty} U_s^n \chi_{[\underline{i}]}(\dot{0}). \end{aligned} \tag{6.1}$$

Clearly, the series in the last line converges to an analytic function. The first term can be written

$$\frac{\nu_s(\chi_{[\underline{i}]}) h_s(\dot{0})}{1 - e^{P(-sr)}} = \frac{1}{s-p} \frac{\nu_p(\chi_{[\underline{i}]}) h_p(\dot{0})}{\int r d\mu_p} + \psi(s),$$

where $\psi(s)$ is analytic in a neighbourhood of p.

We shall use this result, together with an approximation argument, to prove the first part of Theorem 2. Given a string $\underline{i}$ we define an associated geometric cylinder $\mathcal{C}(\underline{i}) = A_{i_0} \cdots A_{i_{n-1}} \mathcal{C}(A_{i_n})$. For $n \geq 1$, we define two functions $F_n, G_n : \bigcup_{|\underline{i}|=n} \mathcal{C}(\underline{i}) \to \mathbb{R}$ by $F_n(x) = \inf\{f(y) : y \in C(\underline{i})\}$ and $G_n(x) = \sup\{f(y) : y \in C(\underline{i})\}$ for $x \in C(\underline{i})$.

Given $\epsilon > 0$, we can choose n sufficiently large that $||F_n - G_n||_\infty < \epsilon$. We may then choose m_0 sufficiently large such that whenever $m \geq m_0$ we have that $A_{i_0} \cdots A_{i_m} v \in \bigcup_{|\underline{i}|=n} \mathcal{C}(\underline{i})$. Clearly,

$$\sum_{|A| \geq m} F_n(Av) ||Av||^{-s} \leq \sum_{|A| \geq m} f(Av) ||Av||^{-s} \leq \sum_{|A| \geq m} G_n(Av) ||Av||^{-s}.$$

Observe that

$$\begin{aligned}\limsup_{s\searrow p}\frac{\sum_A f(Av)||Av||^{-s}}{\eta(s)} &= \limsup_{s\searrow p}\frac{\sum_{|A|\geq m} f(Av)||Av||^{-s}}{\eta(s)}\\ &\leq \limsup_{s\searrow p}\frac{\sum_{|A|\geq m} G_n(Av)||Av||^{-s}}{\eta(s)}\\ &= \int G_n d(\pi_*\nu_p),\end{aligned}$$

where the last equality follows from Proposition 9 by writing G_n as a linear combination of indicator functions of cylinders.

Similarly

$$\liminf_{s\searrow p}\frac{\sum_A f(Av)||Av||^{-s}}{\eta(s)} \geq \int F_n d(\pi_*\nu_p).$$

Also we see that $|\int F_n d(\pi_*\nu_p) - \int G_n d(\pi_*\nu_p)| \leq ||F_n - G_n||_\infty \leq \epsilon$ and $\int F_n d(\pi_*\nu_p) \leq \int f d(\pi_*\nu_p) \leq \int G_n d(\pi_*\nu_p)$. Since $\epsilon > 0$ was arbitrary, we deduce that

$$\lim_{s\searrow\delta}\frac{\sum_A f(Av)||Av||^{-s}}{\eta(s)} = \int f dm,$$

where we set $m = \pi_*\nu_p$.

To complete the proof of Theorem 2 we shall show how m behaves under the action of $A \in \Gamma$.

Proposition 10.

$$\frac{d(B_*m)}{dm}(x) = ||B^{-1}x||^{-p}$$

for all $B \in \Gamma$.

Proof. Give a Hölder continuous function $f : \mathbb{R}P^{d-1} \to \mathbb{R}$ and $B \in \Gamma$ we can write

$$\begin{aligned}\int f(Bx)dm(x) &= \lim_{s\searrow p}\left(\frac{\sum_{A\in\Gamma} f(BAv)||Av||^{-s}}{\sum_{A\in\Gamma}||Av||^{-s}}\right)\\ &= \lim_{s\searrow p}\left(\frac{\sum_{A'\in\Gamma} f(A'v)||B^{-1}A'v||^{-s}}{\sum_{A'\in\Gamma}||A'v||^{-s}}\right)\\ &= \lim_{s\searrow p}\left(\frac{\sum_{A'\in\Gamma} f(A'v)\left(\frac{||B^{-1}A'v||}{||A'v||}\right)^{-s}||A'v||^{-s}}{\sum_{A'\in\Gamma}||A'v||^{-s}}\right)\\ &= \int f(x)\rho(x)^{-p}dm(x)\end{aligned}$$

where $\rho(x) = ||B^{-1}x||/||x||$. This completes the proof of the proposition.

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