GROWTH OF PERIODIC POINTS AND ROTATION VECTORS ON SURFACES

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0. Introduction

In this paper will consider diffeomorphisms $f : M \to M$ of a compact surface M, isotopic to the identity map, and study the growth of their periodic points. An important approach to studying such properties of diffeomorphisms is via the rotation set $\rho(f)$ and the associated rotation vectors. For the particular case of homeomorphisms of tori Franks has shown that if $\rho(f)$ has non-empty interior then every rational point in $\mathrm{int}(\rho(f))$ is represented by a periodic orbit [6]. In a recent paper, Sharp showed that (for diffeomorphisms) under this hypothesis the number of periodic points with prescribed rational rotation vector has exponential growth [18].

Subsequently, Hayakawa studied the analogous problem for compact surfaces of genus at least 2 and obtained a partial generalization of Franks' result [10]. In this paper we shall improve on this result. Furthermore, we shall extend Sharp's result on the exponential growth of periodic points to this setting.

Before we state our main result we briefly describe the idea of the rotation set associated to a point, and the rotation vector associated to a periodic point, for a homeomorphsism $f : M \to M$ isotopic to the identity map on a compact surface M. The rotation set $\rho_x(f) \subset H_1(M, \mathbb{R})$ associated to a point $x \in M$ can be heuristically interpreted as describing the "asymptotic drift" in homology of the orbit of x. In the special case that $f^n(x) = x$ is a periodic point then $\rho_x(f)$ is a single vector. A precise definition is given in the next section.

We now turn to the statement of our main result. In the study of diffeomorphisms of surfaces of higher genus the natural generalisation is not necessarily true without additional hypotheses (cf. Matsumoto's example in section 1). We shall consider maps satisfying the following condition, introduced by Hayakawa:

Condition (*). Let b denote the first Betti number of M (i.e. the dimension of the first homology group). There exist $b+1$ periodic points $f^{n_i}(x_i) = x_i$, $i = 1, \ldots, b+1$, such that the convex hull $\mathrm{co}\{\rho_{x_1}(f), \ldots, \rho_{x_{b+1}}(f)\}$ of the rotation vectors $\rho_{x_1}(f), \ldots, \rho_{x_{b+1}}(f)$ has non-empty interior. There exists an additional point z which is a fixed point under f^m (for some $m \geq 0$, where m is a multiple of $n_1, \ldots, n_{b+1}$) with non-zero index and such that $\rho_z(f) \in \mathrm{int}\left(\mathrm{co}\{\rho_{x_1}(f), \ldots, \rho_{x_{b+1}}(f)\}\right)$.

For background on the indices of fixed points, we refer the reader to [4, VII.5].

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Our main result is the following.

Theorem 1. *Let M be a compact surface (possibly with boundary) of genus at least 2. Suppose that $f : M \to M$ is a diffeomorphism which is isotopic to the identity map and which satisfies Condition (*). Then for any vector $\xi \in int\big(co\{\rho_{x_1}(f), \dots, \rho_{x_{b+1}}(f)\}\big) \cap H_1(M, \mathbb{Q})$ there exists $d \geq 1$ such that*

$$\liminf_{n \to +\infty} \frac{1}{dn} \log \mathrm{Card}\{f^{dn}(x) = x : \rho_x(f) = \xi\} > 0.$$

In particular, there exists a periodic point x with $\rho_x(f) = \xi$.

Remarks.

(i) In [10], Hayakawa showed that the existence of a periodic point x with $\rho_x(f) = \xi$ holds for ξ in a *dense* subset of int $\big(\mathrm{co}\,\big(\rho_{x_1}(f), \dots, \rho_{x_{b+1}}(f)\big)\big)$.

(ii) The proof actually yields the stronger statement that for ξ in a compact subset of int $\big(\mathrm{co}\{\rho_{x_1}(f), \dots, \rho_{x_{b+1}}(f)\}\big)$ the quantity

$$\liminf_{n \to +\infty} \frac{1}{dn} \log \mathrm{Card}\{f^{dn}(x) = x : \rho_x(f) = \xi\}$$

is uniformly bounded away from zero.

We briefly summarize the contents of the paper. In section one we recall the definition and basic properties of rotation sets and rotation vectors for periodic points.

Our method makes use of the Thurston Classification Theorem for isotopy classes of surface homeomorphisms in an essential way. A similar approach has been been employed by, for example, Handel [9], Llibre and MacKay [11]. In section two we present the necessary ideas from Thurston's work.

Section three presents work contained in Hayakawa's paper [10]. We consider an iterate $f^m : M \to M$ so that the periodic points $x_1, \dots, x_{b+1}$ and z become fixed. We "blow up" these fixed points to obtain a new surface (with boundary) N and an associated homeomorphism $F : N \to N$. We next consider the canonical representative of the isotopy class of F guaranteed by Thurston's theorem. We denote this homeomorphism $G : N \to N$ and show that it is pseudo-Anosov. (We shall also denote by $g : M \to M$ the map obtained from $G : N \to N$ by collapsing the boundary components of N back to points.)

In section four we use Markov partitions and symbolic dynamics for the pseudo-Anosov homeomorphsism to give an estimate from below on the number of periodic points with a given rotation vector. A key ingredient is an earlier result of the authors estimating periodic points for a subshift of finite type [16].

1. Rotation Vectors

In this section we shall give a precise definition of the set of rotation vectors and their properties. The treatment we give is taken from [15].

Let M be a compact surface (possibly with boundary) of genus at least 2 and let $f : M \to M$ be a homeomorphism isotopic to the identity map. We introduce the associated mapping torus $M^f = M \times [0,1]/\sim$ where we identify $(x,1) \sim (f(x),0)$. Since f is isotopic to the identity, M^f is homeomorphic to $M \times S^1$. We define an associated flow $f_t : M^f \to M^f$ by $f_t(x,u) = (x, t+u)$, subject to the identifications.

We define the integer Bruschlinsky cohomology $H^1(M,\mathbb{Z})$ to be the isotopy classes of continuous maps $k : M \to K$, where K denotes the unit circle. We can then introduce the real Bruschlinsky cohomology $H^1(M,\mathbb{R})$ by $H^1(M,\mathbb{R}) = H^1(M,\mathbb{R}) \otimes_{\mathbb{Z}} \mathbb{R}$. (This is well known to be isomorphic to the usual simplicial cohomology for M [2], [17].)

Given a point $v \in M$ and $T > 0$ we can define a linear functional $\Lambda_{v,T} : C^0(M^f, K) \to \mathbb{R}$ by

$$\Lambda_{v,T}(k) = \frac{1}{T}\int_0^T \frac{d}{dt}\arg(k(f_t v))dt$$

when $k \in C^0(M^f, K)$ is continuously differentible along the orbits (and we write $k(v) = e^{it\arg(k(v))}$) and this definition extends to all functions in $C^0(M^f, K)$ by uniform continuity.

The following result is due to Schwartzmann [17].

Lemma 1. *For any $v \in M^f$ the set of functionals $\Lambda_{v,T}$ is equicontinuous in the weak star topology. Moreover, the limit points are constant on isotopy classes and so correspond to linear functionals on $H^1(M^f,\mathbb{Z})$, i.e., elements of $H_1(M^f,\mathbb{R})$.*

Since M has genus at least 2 then there is a canonical identification $H_1(M^f,\mathbb{R}) = H_1(M,\mathbb{R}) \oplus \mathbb{R}$. This is because, by a result of Hamstrom [8] the identification (via homeomorphism) of M^f with $M \times S^1$ is unique up to homotopy. Under this identification, all of the element of $H_1(M^f,\mathbb{R})$ which occur as limit points are of the form $\Lambda_v = (\Lambda_v^{(1)}, \Lambda_v^{(2)})$, where $\Lambda_v^{(1)} \in H_1(M,\mathbb{R})$ and $\Lambda_v^{(2)} \in \mathbb{R}$. Moreover, by construction we always have that $\Lambda_v^{(2)} = 1$. We also have that any element $v \in M^f$ is of the form $v = (x,u)$ where $x \in M$ and $u \in [0,1]$. The functional $\Lambda_v^{(1)}$ depends only on x and, for clarity, we shall denote it by Λ_x. Let $\mathcal{F}_x$ denote the set of all functionals Λ_x which arise from functionals $\Lambda_{(x,u),T}$ in this way.

Definition. We define the *set of rotation vectors* $\rho_x(f) \subset H_1(M,\mathbb{R})$ associated to a point $x \in M$ to be the set of all functionals $\Lambda_x \in \mathcal{F}_x$, where Λ_x is interpreted as an element of homology. If $\rho_x(f)$ has a single element then we call this element the *rotation vector* and also denote it by $\rho_x(f)$.

The following is an easy consequence of the definitions.

Lemma 2.

(i) *If $n \geq 1$ then $\rho_x(f^n) = n\rho_x(f)$; and*

(ii) *If $f^n(x) = x$ is a periodic point then $\rho_x(f)$ is a single rational vector. In fact, if n is the least period of x then*

$$\rho_x(f) = \frac{1}{n}\left[\{f_t x : 0 \leq t \leq n\}\right]_1$$

where $[\gamma]_1$ is the first component of $[\gamma] \in H_1(M^f,\mathbb{Z}) = H_1(M,\mathbb{Z}) \oplus \mathbb{Z}$

The construction can be carried out in the case where M is equal to the two dimensional torus $\mathbb{T}^2$. However, $\rho_x(f)$ is then only defined up to translation by an integer vector. The more usual definition in this case is to choose a lift $\tilde{f}$ of f to the Universal Cover $\mathbb{R}^2$ and to set $\rho_x(f)$ to be the set of limit points of $(\tilde{f}^n\tilde{x}-\tilde{x})/n$ (where $\tilde{x}$ is any lift of x to $\mathbb{R}^2$). The ambiguity in $\rho_x(f)$ is due to the freedom of choice of lift.

In the context of tori it is perhaps more natural to consider the (global) rotation set $\rho(f)$ for f, introduced by Misiurewicz and Ziemian [14]. This is defined to be the set of limit points of the sequences

$$\frac{\tilde{f}^{n_i}x_i - x_i}{n_i}, \quad x_i \in \mathbb{R}^2, \quad n_i \to +\infty$$

and is convex and compact. For surfaces of higher genus one might define a global rotation set by $\rho(f) = \cup_{x\in M}\rho_x(f)$, but it is far from clear that it has any particularly desirable properties (cf.[15]). For this reason, we content ourselves with working with the rotation vectors.

Remark. For surfaces of arbitrary genus there is an equivalent definition of rotation vectors due to Franks [7]. Fix a base point x_0 and associate to every point x a measurable family of curves γ_x from x_0 to x. Since f is isotopic to the identity we can choose a continuous family of homeomorphisms $h_t : M \to M$ $(0 \le t \le 1)$ with $h_0(x) = x$ and $h_1(x) = h(x)$ and write $\alpha_x(t) = h_t(x)$. One then associates to x the closed curve based at x_0 given by $\gamma_{f(x)}^{-1} \circ \alpha_x \circ \gamma_x$ and denote its homology class by $\mathcal{R}_x(f) = [\gamma_{f(x)}^{-1} \circ \alpha_x \circ \gamma_x]$. We then let $\rho_x(f)$ be the limit points of $\frac{1}{n}\left(\mathcal{R}_x(f) + \mathcal{R}_{fx}(f) + \ldots + \mathcal{R}_{f^{n-1}x}(f)\right)$.

Example. The following simple example due to Matsumoto [13] shows that we may not weaken the hypotheses of Theorem 1 by removing the condition requiring the existence of an additional periodic point z with non-zero index and with $\rho_z(f) \in \mathrm{int}\left(\mathrm{co}\{\rho_{x_1}(f), \ldots, \rho_{x_{b+1}}(f)\}\right)$. Let M be a compact surface of genus 2 and let α be a simple closed curve which cuts M into two copies of a punctured torus M_1 and M_2. On each M_i, choose simple closed curves γ_i and δ_i such that γ_1, γ_2, δ_1, δ_2 generate $H_1(M, \mathbb{Z})$ and such that each pair γ_i, δ_i has a unique poin of intersection x_i. Let C_i, D_i be annular neighbourhoods of γ_i, δ_i, respectively, chosen so that they do not intersect α. Define diffeomorphisms $\phi_i : M \to M$ to be the identity map on $M - C_i$ and on C_i to be the isotopy linking the identity on ∂C_i to a rotation through 2π on γ_i. Similarly, define $\psi_i : M \to M$ with D_i replacing C_i. Finally, define a diffeomorphism $f : M \to M$ by $f = \phi_1 \circ \phi_2 \circ \psi_1 \circ \psi_2$.

It is easy to see that f has fixed points with rotation vectors $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1) \in H_1(M, \mathbb{R}) \cong \mathbb{R}^4$. Furthermore, x_1 and x_2 are fixed points with rotation vectors $(1, 1, 0, 0)$, $(0, 0, 1, 1)$, respectively. In particular, it is possible to find five periodic points such that the convex hull of their rotation vectors has non-empty interior. On the other hand, every periodic orbit lies wholly in M_1 or in M_2 and so the associated rotation vectors must lie in the union of codimension two subspaces $H_1(M_1, \mathbb{R}) \cup H_1(M_2, \mathbb{R})$. Thus, the conclusion of Theorem 1 does not hold.

One of the most important ingredients in our proof is Thurston's theorem on the classification of surface homeomorphisms up to isotopy. We begin by recalling the statement of this result. There are two special types of homeomorphism of surfaces which serve as building blocks for the theory.

A homeomorphism $F : N \to N$ on a compact surface N is called *periodic* if there exists $n \geq 1$ such that $F^n = id$. A homeomorphism $F : N \to N$ is called *pseudo Anosov* if there exists two measurable foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ with one dimensional leaves (possibly with prongs) with transverse measures m^+ and m^- and a constant $\lambda > 1$ such that $F_* m^+ = \lambda m^+$ and $F_* m^- = \lambda^{-1} m^-$ (cf. [3], [5]). We shall use the following important result of Thurston [19].

Proposition 1 (Thurston's Classification Theorem). *Let N be a compact connected oriented surface (possibly with boundary) with Euler characterisitic $\chi(N) < 0$. Any surface homeomorphism $F : N \to N$ can be isotopied to a unique homeomorphism $G : N \to N$ such that exactly one of the following holds:*

(i) *$G : N \to N$ is periodic;*
(ii) *$G : N \to N$ is pseudo-Anosov;*
(iii) *$G : N \to N$ is reducible: G leaves invariant a finite family of simple closed curves $\{\gamma_1, \dots, \gamma_n\}$ such that no pair of curves are homotopic and no curve is homotopic to a point or to a boundary component of N. Each γ_i has an open annular neighbourhood U_i (which are pairwise disjoint) such that G leaves $U = \cup_{i=1}^n U_i$ invariant. For each component N_j of $N - U$ there is some (least) $n_j > 0$ such that $G^{n_j}(N_j) = N_j$ and $G^{n_j}|N_j$ is either of finite order or pseudo-Anosov. Each U_i is left invariant by some power of G and the induced mapping on U_i is a generalized twist.*

We call $G : N \to N$ the *Thurston Canonical Form* of $F : N \to N$ and if G is reducible we call $G_j = G^{n_j} : N_j \to N_j$ the components of the canonical form.

Pseudo-Anosov homeomorphisms minimize the topological entropy within their isotopy class.

Proposition 2. *Suppose that $G : N \to N$ is pseudo-Anosov and that $F : N \to N$ is isotopic to G. Then $h(F) \geq h(G) = \log \lambda > 0$, where $\lambda > 1$ is the constant occuring in the definition of pseudo-Anosov homeomorphisms.*

Remark. The value $h(G)$ can also be characterized as the rate of growth of the induced action of G (or F) on $\pi_1 N$.

Another important property of pseudo-Anosov homeomorpisms is that their periodic points persist under isotopy in a precise sense. Let $\tilde{N}$ denote the Universal Cover cover of N and let $\tilde{G} : \tilde{N} \to \tilde{N}$ be a lift of $G : N \to N$. Suppose that $F : N \to N$ is isotopic to G. Then there is a unique lift of F to $\tilde{N}$, determined by the isotopy from F to G, which is isotopic to $\tilde{G}$. If x and y are fixed points of G^n and F^n, respectively, then (G^n, x) and (F^n, y) are said to be Nielsen equivalent if there are lifts $\tilde{x}, \tilde{y}$ of x, y and a deck translation $\gamma \in \pi_1 N$ such that $\tilde{G}^n \tilde{x} = \gamma \tilde{x}$ and $\tilde{F}^n \tilde{y} = \gamma \tilde{y}$.

Proposition 3 (Thurston). *Suppose that $G : N \to N$ is pseudo-Anosov and that $F : N \to N$ is isotopic to G. If $G^n(x) = x$ then there exists $y \in N$ with $F^n(y) = y$ such that (G^n, x) is Nielsen equivlent to (F^n, y).*

A feature of pseudo-Anosov homeomorphisms that will be crucial to our analysis is that they admit Markov partitions and hence can be modelled by subshifts of finite type. We recall the definitions. Let $G : N \to N$ be a pseudo-Anosov map with associated foliations $\mathcal{F}^+$ and $\mathcal{F}^-$. A closed set $R \subset M$ is called a *rectangle* if for any pair of points $x, y \in R$ the leaf from $\mathcal{F}^+$ containing x and the leaf from $\mathcal{F}^-$ containing y intersect in a single point. A rectangles is called *proper* if $R = \text{clos}(\text{int}R)$. A *Markov partition* for $G : N \to N$ consists of a finite family of rectangles $\{R_i\}_{i=1}^k$ in N such that

(a) $\text{int}(R_i) \cap \text{int}(R_j) \neq \emptyset$ for $i \neq j$;
(b) $N = \cup_{i=1}^N R_i$; and
(c) If $x = (x_n)_{n=-\infty}^{+\infty} \in \prod_{n\in\mathbb{Z}}\{1, \dots, k\}$ satisfies $\cap_{n=0}^{\infty} G^{-n}\text{int}(R_{x_n}) \neq \emptyset$ and $\cap_{n=-\infty}^{0} G^{-n}\text{int}(R_{x_n}) \neq \emptyset$ then $\cap_{n=-\infty}^{\infty} G^{-n}\text{int}(R_{x_n}) \neq \emptyset$ and consists of a single point denoted $\pi(x)$.

Proposition 4 ([3], [5]). *A pseudo-Anosov homeomorphism admits Markov partitions of arbitrarily small size.*

Remark. The construction of the rectangles in the Markov partition is such that the prongs for the foliations occur in the boundaries.

Define a $k \times k$ zero-one matrix A by

$$A(i,j) = \begin{cases} 1 \text{ if } G(\text{int}R_i) \cap \text{int}R_j \neq \emptyset \\ 0 \text{ otherwise.} \end{cases}$$

We may suppose that A is aperiodic, i.e., that there exists $N \geq 1$ such that all of the entries of A^N are positive. If we denote

$$X_A = \{x = (x_n)_{n=-\infty}^{+\infty} \in \prod_{n=-\infty}^{\infty} \{1, \dots, k\} : A(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z}\}$$

then the *subshift of finite type* $\sigma : X_A \to X_A$ is defined by $(\sigma x)_n = x_{n+1}$. Property (c) above can be used to define a map $\pi : X_A \to N$.

Proposition 5. *If $G : N \to N$ is a pseudo-Anosov homeomorphism then $\pi : X_A \to N$ is continuous, surjective and satisfies $G \circ \pi = \pi \circ \sigma$. Moreover,*

(i) $h(G) = h(\sigma)$
(ii) *π is a bijection on periodic points, with at most finitely many exceptions (lying on the boundaries of the rectangles).*

3. Application of Thurston's Theory

In this section we return to the consideration of a diffeomorphism $f : M \to M$ satisfying Condition (*). We shall obtain a new surface N by "blowing up" the points $x_1, \dots, x_{b+1}$ and a homeomorphism $F : N \to N$ which is identical to f^m away from the blown up points, for some $m > 0$. We shall apply the Thurston Classification Theorem to F and show that its Thurston Canonical Form is pseudo-Anosov.

Recall that we can choose $m > 0$ so that z is a fixed point of f^m with non-zero index, where m is a mutiple of $n_1, \dots, n_{b+1}$. Thus the points $x_1, \dots, x_{b+1}$ are all fixed points of f^m. A new surface N is defined by removing $x_1, \dots, x_{b+1}$ and replacing them with (small) boundary circles $C_1. \dots, C_{b+1}$. A homeomorphism $F : N \to N$ is defined to be f^m on $N - \cup_{i=1}^{b+1} C_i$ and by the projective action of the derivative of f^m on the boundary circles C_i. (This is the only point at which we use that f is a diffeomorphism.) Let $\tau : N \to M$ denote the map from N to M which collapses the boundary circles C_i to points.

Lemma 3. *The map $\tau : N \to M$ satisfies $\tau \circ F = f^m \circ \tau$ and is a bijection from $N - \cup_{i=1}^{b+1} C_i$ to $M - \{x_1, \dots, x_{b+1}\}$ such that*

(i) *τ semi-conjugates f^m and h;*
(ii) *collapses each boundary component C_i to the point x_i ($i = 0, \dots, b+1$);*
(iii) *is a bijection except on the boundary components C_i ($i = 1, \dots, b+1$).*

We want to apply Proposition 2 to the map $F : N \to N$. The following result showing that the Thurston canonical form is pseudo-Anosov is based on ideas from [10] and [15].

Proposition 6 (Hayakawa). *If $f : M \to M$ satisfies Condition (*) then the Thurston canonical form $G : N \to N$ for $F : N \to N$ is pseudo-Anosov.*

Proof. Observe that if f satisfies Condition (*) then the same is true of $f^m : M \to M$. We begin by showing that the canonical form $G : N \to N$ for $F : N \to N$ has a single component. Assume for a contradiction that the canonical form has more than one component and let γ be one of the reducing curves.

We first remark that $\tau(\gamma)$ is not contractible as a curve on the original surface M. As observed in [15, p.890] if this were not the case and $\tau(\gamma)$ were contractible on M then it must contain at least two of the fixed points for f^m from $\{x_1, \dots, x_{b+1}\}$, since one of the restraints on a reducing curve γ is that is should not be homotopic to a boundary curve on N. However, if $\tau(\gamma)$ contains two fixed points then this would imply that their rotation vectors were the same. To see this we may assume (taking an iterative if necessary) that $\tau(\gamma)$ is f^m-invariant. This implies that the associated periodic orbits for f_t in M^f are homotopic. In particular, they must have the same rotation vector. This gives a contradiction to Condition (*).

We next claim that $\tau(\gamma)$ must be separating curve, i.e., $M - \tau(\gamma)$ has two connected components M_1, M_2, say. As in [15, 890-91] we observe that if we were to assume for a contraction that $\tau(\gamma)$ were not separating (and we already know it cannot be contractible in M, by the argument above) then we can assume without loss of generality that it is a a meridian curve of a handle of M. (Otherwise, by a suitable homeomorphism of the surface we can map $\tau(\gamma)$ to a meridian curve of a handle of M. This simply induces a change in the basis in homology.) The isotopy of $F : N \to N$ to $G : N \to N$ induces an isotopy of $f^m : M \to M$ to $g : M \to M$ relative to the fixed poits $\{x_1, \dots, x_{b+1}\}$, say, and where, in particular, $g \circ \tau = F$. Moreover, taking an iterate if necessary, we may assume that g preserves the curve $\tau(\gamma)$. In particular, we can deduce that the rotation set $\rho(g)$ for $g : M \to M$ is constrained to lie in a co-dimension one hyperplane (not containing the other homology basis element corresponding to the handle). However, the periodic points $x_1, \dots, x_{b+1}$ for f are fixed points for g and $\rho_{x_i}(g) = m\rho_{x_i}(f)$. In particular, since f satisfies Condition (*) the same is true of g which gives a contradiction.

Having now established that $\tau(\gamma)$ is a separating curve we may write

$$\rho(g) = \rho(g|M_1) \cup \rho(g|M_2)$$

In particular, for any point $x \in M_i$ $(i = 1, 2)$ the orbits $g^n(x)$, $n \geq 1$ are constrained to that component M_i and never enter the complementary component. If we assume, without loss of generality that M has a homology basis of closed curves disjoint from γ, then it is easy to see that $H_1(M, \mathbb{R}) = H_1(M_1, \mathbb{R}) \oplus H_1(M_1, \mathbb{R})$ with $\rho(g|M_i) \subset H_1(M_i, \mathbb{R})$ $(i = 1, 2)$.

To complete the proof, we consider the additional periodic point z for f. Since this has non-zero index as a fixed point of f^m, there is a corresponding Nielsen equivalent fixed point y for $g : M \to M$ [1]. Clearly, the rotation vector will satisfy $\rho_y(g) = \rho_z(g) = m\rho_z(f)$. Since $\rho_z(f) \in \text{int}\,(\text{co}(\{\rho_{x_1}(f)), \dots, \rho_{x_{b+1}}(f))\}))$, it follows that

$$\begin{aligned} \rho_y(g) &\in \text{int}(\text{co}(m\rho_{x_1}(f), \dots, m\rho_{x_{b+1}}(f))) \\ &= \text{int}(\text{co}(\rho_{x_1}(g), \dots, \rho_{x_{b+1}}(g))), \end{aligned}$$

contradicting the fact that $\rho_y(g) \in H_1(M_1, \mathbb{R}) \cup H_1(M_2, \mathbb{R})$.

Now that we have establised that the Thurston Canonical Form of F is pseudo-Anosov, we may use Proposition 3 to obtain a lower bound which turns the counting problem of Theorem 1 into one concerning periodic points of pseudo-Anosov maps. More specifically, given a periodic point of G then there exists a Nielsen equivalent periodic point x of F. Projecting to M, these give periodic points τy for g and τx for f^m, respectively, satisfying

$$\rho_{\tau y}(g) = \rho_{\tau x}(f^m) = m\rho_{\tau x}(f).$$

Thus we obtain the lower bound

$$\text{Card}\{g^n y = y : \rho_{\tau y}(g) = \eta\} \leq \text{Card}\{f^{nm} x = x : \rho_{\tau x}(f) = \xi\} \tag{3.1}$$

where $\eta = m\xi$.

4. Estimates on periodic points

In this section we shall use a method applied by Sharp in [18]. We shall use a Markov partition and symbolic dynamics to reduce this to a symbolic problem. We shall then apply a result of the authors on periodic orbits of $\mathbb{Z}^q$ extensions of subshifts of finite type [16].

By Propostion 4 the Thurston Canonical Form $G : N \to N$ of $F : N \to N$ is pseudo-Anosov. We let $\{R_i\}_{i=1}^k$ denote a Markov partition for G and let X_A denote the associated subshift of finite type. We need to encode information about rotation vectors in terms of a function on X_A. To do this, we first fix a base point $y \in M$ and, for $i = 1, \ldots, k$, choose curves c_i joining y to $\tau(R_i)$. For pairs (i, j) with $A(i, j) = 1$, we also choose a point $y_{ij} \in \tau(R_i)$ which is mapped to $\tau(R_j)$ by g, and use the fact that g is homotopic to the identity map to obtain a curve c_{ij} from y_{ij} to $g(y_{ij})$. If necessary, adjust the curves slightly so that the endpoints match up.

Whenever $A(i, j) = 1$ we can associate the loop $C_{i,j} = c_i^{-1} \circ c_{ij} \circ c_j \subset M$ based at y. This defines an element of homology $[C_{i,j}] \in H_1(M, \mathbb{R})$. Providing that the rectangles in the Markov Partition are all chosen sufficiently small, this can be done in a well-defined way.

By Proposition 5 we can deduce that if $\sigma^n x = x$ is a periodic point for σ then

$$\rho_{\tau\pi x}(g) = \frac{1}{n}\left([C_{x_0,x_1}] + [C_{x_1,x_2}] + \ldots + [C_{x_{n-1},x_0}]\right) \in H_1(M, \mathbb{R})$$

(and $m\rho_{\tau\pi x}(f) = \rho_{\tau\pi x}(g)$).

We can define a function $r : X_A \to H_1(M, \mathbb{R})$ by $r(x) = [C_{x_0,x_1}]$. If $\sigma^n x = x$ then $\rho_{\pi x}(g) = \frac{1}{n} r^n(x)$, where $r^n(x) = r(x) + r(\sigma x) + \ldots + r(\sigma^{n-1}x)$. Thus for $\xi \in \mathrm{int}\left(\mathrm{co}\{\rho_{x_1}(f))), \ldots, \rho_{x_{b+1}}(f)\}\right) \cap H_1(M, \mathbb{Q})$, we have, by Proposition 5 (ii), that for all sufficiently large n our counting function $\mathrm{Card}\{g^n y = y : \rho_y(f) = \eta\}$ satisfies

$$\mathrm{Card}\{g^n y = y : \rho_y(f) = \eta\} = \mathrm{Card}\{\sigma^n x = x : \frac{1}{n} r^n(x) = \eta\}, \tag{4.1}$$

where $\eta = m\xi$.

We want to apply the results from [16]. To conform with the notation of that paper, we introduce a function $\psi := qr - p$, where $\eta = p/q$ (with $p = (p_1, \ldots, p_b) \in \mathbb{Z}^b$, $q \in \mathbb{N}$ and where $p_1, \ldots, p_b, q$ are coprime) and recall the following conditions.

(A1) The group Γ generated by the set $\{\psi^n(x) : \sigma^n x = x, n \geq 1\}$ is a finite index subgroup of $\mathbb{Z}^b$.

(A2) There exists a fully supported σ-invariant probability measure μ on X_A with $\int \psi d\mu = 0$ (or, equivalently, $\int r d\mu = \eta$).

Lemma 4. *Conditions (A1) and (A2) hold.*

Proof. Clearly, (A1) is equivalent to the statement that the set $\{r^n(x) : \sigma^n x = x, n \geq 1\}$ generates a finite index subgroup of $\mathbb{Z}^b$. By Condition (*) the family $\{\rho_{x_1}(g), \ldots, \rho_{x_{b+1}}(g)\} = \{m\rho_{x_1}(f), \ldots, m\rho_{x_{b+1}}(f)\}$ must contain a basis for $H_1(M, \mathbb{R}) = \mathbb{R}^b$. The periodic points $x_1, \ldots, x_{b+1}$ correspond to the boundary components of N. Each boundary component contains a periodic point for the pseudo-Anosiv map $G : N \to N$ (as can be seen by considering the foliations [5]) and consequently gives rise to a periodic point for $\sigma : X_A \to X_A$. Moreover, if $\sigma^{k_i} z_i = z_i$ is the periodic point associted to the vector $\rho_{x_i}(g)$ $(i = 1, \ldots, b+1)$ then $\rho_{x_i}(g) = \frac{1}{k_i} r^{k_i}(z_i)$. Condition (A1) now follows directly.

To establish condition (A2) we shall use the *weight-per-symbol set* WPS_r introduced in [12]. This is defined to be

$$\mathrm{WPS}_r = \left\{ \frac{1}{n} r^n(z) : \sigma^n z = z, n \geq 1 \right\}$$

and, by our observations above, $\rho_{x_i}(g) \in \mathrm{WPS}_r$. If $\eta \in \mathrm{int}\left(\mathrm{co}\{\rho_{x_1}(g), \ldots, \rho_{x_{b+1}}(g)\}\right)$ then $\eta \in \mathrm{int}\,(\mathrm{coWPS}_r)$ It follows from [12] that there exists a (fully supported) Markov measure μ such that $\int r d\mu = \eta$.

The validity of conditions (A1) and (A2) established in the above Lemma is important because it allows us to apply the following result of the authors. (We shall use the notation that given two functions $A, B : \mathbb{R} \to \mathbb{R}$ we write $A(t) \sim B(t)$ if $\lim_{t \to +\infty} A(t)/B(t)$.)

Proposition 7 ([16]). *If $\sigma : X_A \to X_A$ and $r : X_A \to \mathbb{Z}$ satisfy (A1) and (A2) then there exists $d \geq 1$, $0 < \delta \leq h(\sigma)$ and $C > 0$ such that*

$$\mathrm{Card}\left\{x : \sigma^{dn} x = x, \psi^{dn}(x) = 0\right\} \sim \frac{Ce^{\delta dn}}{n^{b/2}},$$

for $n \geq 0$. Furthermore, the set $\left\{x : \sigma^k x = x, \psi^k(x) = 0\right\} = \emptyset$ if d does not divide k.

Corollary 7.1. *Then the exists $d \geq 1$ and $0 < \delta \leq h(\sigma)$ such that*

$$\liminf_{n \to +\infty} \frac{1}{dn} \log \mathrm{Card}\{\sigma^{dn}(x) = x : \psi^{dn}(x) = 0\} = \delta.$$

In view of the inequalities (3.1) and (4.1) we have now completed the proof of Theorem 1.

Remark. From [16], the growth rate $\delta = \delta(\eta)$ may be characterized by

$$\delta(\eta) = \sup \left\{ h_\mu(\sigma) : \mu \text{ is a } \sigma\text{-invariant probability measure with } \int r d\mu = \eta \right\}$$

where $h_\mu(\sigma)$ denotes the measure theoretic entropy. If ξ varies over a compact subset $\Delta \subset \mathrm{int}\left(\mathrm{co}(\rho_{x_1}(f), \ldots, \rho_{x_{b+1}}(f))\right)$ then η varies over the compact set $m\Delta \subset \mathrm{int}(\mathrm{WPS}_r)$. To achieve a uniform positive lower bound for $\delta(\eta)$ on $m\Delta$ it suffices to observe that this quantity is bounded below by a similar expression where the supremum is taken over the subset of Markov measures. This reduces the problem to a finite dimensional situation where the dependence of the entropy on the measure leads to the result.

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