POINCARÉ SERIES AND COMPARISON THEOREMS FOR VARIABLE NEGATIVE CURVATURE

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0. Introduction

Poincaré series have always played a central rôle in the theory of automorphic functions and harmonic analysis on manifolds of constant negative curvature. The trace formulae approach was introduced by Selberg [20] in 1956 and has become the key to their analysis. However, with the advent of a more geometrical viewpoint (through the work of Patterson [15, 16], Sullivan [21, 22] and others) it becomes natural to broaden the scope to Poincaré series for manifolds of *variable* negative curvature. The purpose of this note is to describe an ergodic theoretic method for analyzing Poincaré series in this greater generality.

Let M be a compact Riemannian manifold (possibly with totally geodesic boundary) with strictly negative variable curvatures. Let $\pi_1(M)$ denote its fundamental group. For any point $p \in M$ we can associate to each homotopy class $\gamma \in \pi_1(M) - \{e\}$ the length $l(\gamma)$ of the shortest geodesic arc in γ from p to itself. We define the associated Poincaré series by

$$\eta(s) = \sum_{\gamma \in \pi_1(M) - \{e\}} e^{-sl(\gamma)}$$

for a complex variable $s \in \mathbb{C}$.

This function converges for $Re(s)$ sufficiently large, and we denote its abscissa of convergence by δ. (Since M has negative curvature it is clear that $\delta > 0$.) Our main result on Poincaré series is the following.

Theorem. *The function $\eta(s)$ has an extension as a meromorphic function to the half-plane $Re(s) > \delta - \epsilon$, for some $\epsilon > 0$.*

A second application of our ideas can be described as follows. In [18] we presented certain comparison theorems for compact manifolds of constant negative curvature, inspired by the results of Milnor. In particular, we showed that certain averages of the ratio of the geometric length to word length converged to a constant. In this note we develop analogues of these results in the broader context of compact manifolds of variable negative curvature.

Key words and phrases. Riemannian manifold, negative curvature, fundamental group, strongly Markov, Poincaré series, thermodynamic formalism, orbit counting function.

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Although there exist analogues of the Selberg trace formulae for manifolds of variable negative curvature (notably by Duistermaat-Guillemin [7, 10] and Colin de Verdiere [5]) they are not appropriate for describing the meromorphic domain of the Poincaré series. The approach we shall develop rests on two main ingredients. The first is the use of geometric group theory to associate to the fundamental group of the manifold a (labelled) directed graph describing the group. This allows us to construct a suitable subshift of finite type in this case where the use of Markov partitions for the associated geodesic flow would not be adequate. The second is the thermodynamical theory of subshifts of finite type, in particular, the theory of the Ruelle transfer operator.

Some elements of this analysis we have already developed in the context of manifolds of *constant* negative curvature in [18], to which we shall refer as appropriate.

Notation. For two functions $a, b : \mathbb{R}^+ \to \mathbb{R}$ with $b(t) > 0$ we write $a(t) \sim b(t)$ if $\lim_{t\to+\infty} \frac{a(t)}{b(t)} = 1$, $a(t) = O(b(t))$ if $\limsup_{t\to+\infty} \frac{|a(t)|}{b(t)} < +\infty$ and $a(t) = o(b(t))$ if $\limsup_{t\to+\infty} \frac{|a(t)|}{b(t)} = 0$.

1. Some Preliminaries

Let Γ be an (infinite) finitely presented group with identity element $e \in \Gamma$. Let $S \subset \Gamma$ be a finite symmetric set of generators for Γ (i.e. S generates Γ and if $\gamma \in S$ then $\gamma^{-1} \in S$). We can define the *word length* of an element $\gamma \in \Gamma$ to be the least number of elements from S whose concatenation equals γ. We denote this value by $|\gamma|$.

We let $(\quad , \quad)$ denote the *Lyndon-Chiswell-Gromov product* in Γ (with respect to S) defined by $(\gamma, \gamma') := \frac{1}{2}\left(|\gamma| + |\gamma'| - |\gamma^{-1}\gamma'|\right)$ [4], [11]. Gromov introduced an analogous definition adapted to (hyperbolic) metric spaces X [9]. For a given base point $x \in X$ this is defined by $(y, w)_x := \left(d(x, y) + d(x, w) - d(y, w)\right)/2$. Given $l > 0$, there exists $a > 0$ depending only on X and l such that for any point $z \in X$ with $d(x, z) \le l$ we have the estimate

$$d([xz], [yw]) - a \le (y, w)_x \le d([xz], [yw]) + a, \tag{1.1}$$

where $[xz]$ and $[yw]$ denote the geodesic arcs joining x to z and y to w, respectively. (This follows, after an elementary calculation, from Lemme 17 in Chapitre 2 of [9].)

If $\Gamma = \pi_1(M)$ is the fundamental group of a compact manifold M (possibly with totally geodesic boundary) with universal cover X then there exists constants $C_1, C_2, K > 0$ such that for all $\gamma, \gamma' \in \Gamma$ we have that

$$C_1(\gamma x, \gamma' x)_x - K \le (\gamma, \gamma') \le C_2(\gamma x, \gamma' x)_x + K. \tag{1.2}$$

An important ingredient in our analysis will be the following lemma for the universal cover X of the manifold M.

Lemma 1. *There exists $\beta > 0$, $K > 0$ and $C > 0$ with the following property. Given four points $x, y, w, z \in X$ with minimum distance $2r > C$ between the geodesic arcs $[xz]$ and $[yw]$ then*

$$|d(x, w) + d(y, z) - d(x, y) - d(z, w)| \le K e^{-\beta r}$$

Proof. It suffices to show that under the hypotheses of the lemma we can choose points $q_1 \in [xy]$, $q_2 \in [yz]$, $q_3 \in [xw]$ and $q_4 \in [zw]$, $p \in X$ and $\beta, A > 0$ such that $\sup_{i=1,\dots,4} d(p, q_i) \leq Ae^{-\beta r}$. This is enough to prove the result since we can write

$$
\begin{aligned}
&|d(x,w) + d(y,z) - d(x,y) - d(z,w)| \\
&= |(d(x,q_3) + d(q_3,w)) + (d(y,q_2) + d(q_2,z)) - (d(x,q_1) + d(q_1,y)) - (d(z,q_4) + d(q_4,w))| \\
&\leq |d(x,q_3) - d(x,q_1)| + |d(q_3,w) - d(q_4,w)| + |d(z,q_2) - d(z,q_4)| + |d(q_2,y) - d(q_1,y)| \\
&\leq d(q_3,q_1) + d(q_3,q_4) + d(q_2,q_4) + d(q_2,q_1) \\
&\leq 4 \sup_{i \neq j} d(q_i,q_j) \\
&\leq 8 \sup_{i=1,\dots,4} d(p,q_i) \\
&\leq 8Ae^{-\beta r}
\end{aligned}
$$

where the first equality is a consequence of the fact that the points q_i ($i = 1, \dots, 4$) lie on appropriate geodesic arcs. The conclusion of the lemma would then follow with $K = 8A$.

To proceed we make two simple observations in the context of trigonometry on manifolds of negative curvature which will be useful in the sequel.

(i) There exists $C_0 > 0$ such that for any geodesic triangle $\triangle(abc)$ with vertices a, b, c satisfying $\angle abc \geq \frac{\pi}{2}$ and $d(a,b) \geq C_0$ then
  (a) $d(a,c) \geq d(a,b)$
  (b) $\angle bac \leq \frac{\pi}{4}$.

(ii) There exists $\beta > 0$ such that for any $\alpha > 0$ we can find a constant $A > 0$ with the following property. For any geodesic triangle $\triangle(def)$ with $\angle def \geq \alpha$ and points $p \in [de]$ and $q \in [df]$ with $S := d(d,p) = d(d,q) \leq \min(d(d,e), d(d,f))$ we have that $d(p,q) \leq \frac{A}{2}e^{-\beta R}$, where $R = d(d,e) - S$.

(Each of these observations are elementary for constant curvature spaces using hyperbolic sine and cosine formulae and carry over to variable curvature using, for example, Théorèmes 9 and 12 from Chapitre 3 of [9].)

We can join $[xz]$ and $[yw]$ by a geodesic arc $[uv]$ of shortest length (with endpoints $u \in [xz]$, $v \in [yw]$). Let $p \in [uv]$ be the midpoint of the geodesic arc $[uv]$ then by definition $d(u,p) = r$.

We shall omit the simpler "degenerate" cases by assuming that neither u nor v is an endpoint of the arcs $[zx]$ or $[wy]$, respectively. The proof in the general case can be easily modified to deal with these cases.

We begin by observing that, since $\angle uvy = \frac{\pi}{2}$, we can bound the angle $\angle vuy \leq \frac{\pi}{2}$ by considering $\triangle(uvy)$. By applying observation (ii) with $\triangle(vuy) = \triangle(edf)$ and $r = R$ we see that the point $q \in [uy]$ with $d(u,q) = r$ satisfies $d(p,q) \leq \frac{A}{2}e^{-\beta r}$.

We next introduce the value s by $s + r = d(u,y)$ i.e. $s = d(q,y)$. Since $\angle(uvy) = \frac{\pi}{2}$ we can apply observation (i) with $\triangle(abc) = \triangle(uvy)$ to see that:
(a) $r + s = d(u,y) \geq 2r$ i.e. $s \geq r$; and
(b) $\angle(vuy) \leq \frac{\pi}{4}$.
From (b) we get that $\angle(xuy) \geq \angle(xuv) - \angle(vuy) \geq \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

To complete the construction of $q_1 \in [xy]$ we need the following observation.

(iii) There exists $C_1 > 0$ such that for all geodesic triangles $\triangle(ghk)$ with $\angle ghk = \frac{\pi}{2}$ and $\angle hgk \geq \frac{\pi}{4}$ the following holds. Let $d(g,k) = R + S$ $(R, S \geq 0)$ and $d(h,k) = T$ then if $R \geq C_1$ we have that $T \geq S$.

The constant C in the statement of the Lemma will be taken to be the maximum of C_0 and C_1.

We let h denote the point on $[ux]$ (or its extension) such that the distance $d(h,y)$ is minimized. Note that the geodesic $[hy]$ meets $[ux]$ (or its extension) at right angles. We now apply (iii) to the right angled triangle $\triangle(uhy)$ to conclude that $d(h,y) \geq s$ and so $d(x,y) \geq s$.

We now define $q_1 \in [xy]$ to be the point with $d(q_1, y) = s$. Applying (ii) to $\triangle(yux)$ we obtain $d(q, q_1) \leq \frac{A}{2} e^{-\beta r}$.

Finally, by the triangle inequality we see that

$$d(p, q_1) \leq d(p,q) + d(q, q_1) \leq \frac{A}{2} e^{-\beta r} + \frac{A}{2} e^{-\beta r} = A e^{-\beta r}.$$

We can repeat the above argument (from after the construction of the point p) three more times with the pair (x,y) replaced by each of the pairs $(y,z), (x,w)$ and (z,w) to find corresponding points q_2, q_3, q_4 with $d(p,q_1), d(p,q_2), d(p,q_3) \leq A e^{-\beta r}$. This completes the proof.

2. Strongly Markov Groups and Thermodynamic Formalism

In this section we show how to describe $\pi_1(M)$ in terms of a subshift of finite type. We introduce a (Hölder continuous) function defined on this subshift which carries the geometric information required to relate the Poincaré series to this construction.

Definition. We call Γ *strongly Markov* if for any finite symmetric set of generators S there exists a finite directed graph G with:

(i) a vertex set V;
(ii) an edge set $E \subset V \times V$;
(iii) a distinguished vertex $* \in V$ such that no edge in E terminates at $*$;
(iv) a labelling of the edges $\lambda : E \to S$,

such that there is a bijection between:

(a) finite paths ω in the graph starting with the distinguished state $*$; and
(b) elements $\gamma \in \Gamma$

which associates to the path ω along concurrent edges $(*, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ the group element $\gamma = \lambda(*, v_1)\lambda(v_1, v_2)\ldots\lambda(v_{n-1}, v_n)$ given by group multiplication of the labelling of the associated edges. Moreover, the word length of γ exactly equals the number of edges in the path (i.e. $|\gamma| = n$). (We always assume that the graph G is minimal in the sense that no subgraph (with the same labelling) satisfies (a) and (b).)

Proposition 2 (Cannon). *For a compact manifold (possibly with totally geodesic boundary) with strictly negative sectional curvatures the fundamental group $\pi_1(M)$ is strongly Markov.*

Given the vertex set V and the edge set E for the directed graph G we want to add an extra state 0 to form a larger vertex set $V' = V \cup \{0\}$. We form a new edge set E' by adding to E an edge from i to 0 for each $i \in V - \{*\}$ and an edge from 0 to itself. We adopt the convention that $\lambda(i, 0) = e$ (the identity element in S) $\forall i \in V' - \{*\}$. We associate a square matrix A with zero-one entries whose columns and rows are indexed by V' and whose entries are determined by:

(1) $A(i, j) = 1$ if $(i, j) \in E'$;
(2) $A(i, j) = 0$ otherwise.

It is important to note that the matrix A is not irreducible and, moreover, the submatrix derived from A by deleting the entries coming from $*$ and 0 will, in general, not be irreducible.

Let

$$X_A = \{x = (x_n) \in \prod_{n \geq 0} V' : A(x_n, x_{n+1}) = 1 \quad \forall n \geq 0\}$$

and let $\sigma : X_A \to X_A$ be the shift defined by $(\sigma x)_n = x_{n+1}$. Note that there is a natural bijection between $\pi_1(M)$ and those elements $x \in X_A$ which end in an infinite string of zeros, i.e., those for which there exists $n_0 \geq 0$ such that $x_n = 0$ for all $n \geq n_0$.

We define a metric on X_A by

$$d(x, y) = \sum_{n=0}^{+\infty} \frac{1 - \delta_{x_n, y_n}}{2^n}$$

where δ_{ij} is the standard Kronecker delta. This makes X_A into a compact metric space and $\sigma : X_A \to X_A$ into a continuous transformation.

Proposition 3. *There exists a Hölder continuous function $r : X_A \to \mathbb{R}$ such that for $x = (*, x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ we have that*

$$r^n(x) := r(x) + r(\sigma x) + \ldots + r(\sigma^{n-1} x) = l(\gamma), \tag{2.1}$$

where $\gamma \in \pi_1(M)$ is given by $\gamma = \lambda(*, x_1)\lambda(x_1, x_2) \ldots \lambda(x_{n-1}, x_n)$.

Proof. We define r on the space X_A^0 of sequences ending in an infinite string of zeros by

$$\begin{aligned} r(x_0, x_1, x_2, \ldots, x_n, 0, 0, \ldots) &= l\left(\lambda(x_0, x_1)\lambda(x_1, x_2) \ldots \lambda(x_{n-1}, x_n)\right) \\ &\quad - l\left(\lambda(x_1, x_2) \ldots \lambda(x_{n-1}, x_n)\right). \end{aligned}$$

With this definition it is clear that r satisfies (2.1). We shall show that $r : X_A^0 \to \mathbb{R}$ is Hölder continuous. Since X_A^0 is a dense subset of X_A^0, it will then follow that r has a unique Hölder continuous extension $r : X_A \to \mathbb{R}$.

In order to show that $r : X_A^0 \to \mathbb{R}$ is Hölder continuous, consider sequences $x = (x_0, x_1, \dots, x_n, 0, 0, \dots)$, $y = (y_0, y_1, \dots, y_m, 0, 0, \dots)$, where $n, m \geq 1$ and suppose that $x_i = y_i$ for $i = 0, \dots, k \leq \min(n, m)$ but $x_{k+1} \neq y_{k+1}$. Writing $\gamma = \lambda(x_0, x_1)\lambda(x_1, x_2) \dots \lambda(x_{n-1}, x_n)$ and $\gamma' = \lambda(y_0, y_1)\lambda(y_1, y_2) \dots \lambda(y_{m-1}, y_m)$ and setting $a = \lambda(x_0, x_1)^{-1} = \lambda(y_0, y_1)^{-1}$ we see that

$$|r(x) - r(y)| = |l(\gamma) - l(a\gamma) - l(\gamma') + l(a\gamma')|.$$

Thus, noting that $k = (\gamma, \gamma')$, it is enough to show that there exists $B > 0$ and $0 < \theta < 1$ such that

$$|l(\gamma) - l(a\gamma) - l(\gamma') + l(a\gamma')| \leq B\theta^{(\gamma,\gamma')}.$$

To prove this we apply Lemma 1 with $x = \tilde{p}$, $y = \gamma'\tilde{p}$, $w = \gamma\tilde{p}$ and $z = a^{-1}\tilde{p}$, where $\tilde{p}$ is any lift of $p \in M$ to the universal cover X and observe that

$$\begin{aligned}
&l(\gamma) - l(a\gamma) - l(\gamma') + l(a\gamma') \\
&= d(\gamma\tilde{p}, \tilde{p}) - d(a\gamma\tilde{p}, \tilde{p}) - d(\gamma'\tilde{p}, \tilde{p}) + d(\gamma'\tilde{p}, a\tilde{p}) \\
&= d(\gamma\tilde{p}, \tilde{p}) - d(\gamma\tilde{p}, a^{-1}\tilde{p}) - d(\gamma'\tilde{p}, \tilde{p}) + d(\gamma'\tilde{p}, a^{-1}\tilde{p}),
\end{aligned}$$

where $d(a\gamma\tilde{p}, \tilde{p}) = d(\gamma\tilde{p}, a^{-1}\tilde{p})$ and $d(a\gamma'\tilde{p}, \tilde{p}) = d(\gamma'\tilde{p}, a^{-1}\tilde{p})$ since $a \in S$ acts as an isometry. In view of (1.1), (1.2) and the fact that $d(\tilde{p}, a\tilde{p}) \leq \max_{b \in S} d(\tilde{p}, b\tilde{p})$, there exist constants $B_1, B_2 > 0$ such that the minimum distance $2r$ between the geodesic arcs $[\tilde{p}, a^{-1}\tilde{p}]$ and the geodesic arcs $[\gamma\tilde{p}, \gamma'\tilde{p}]$ is bounded below by $B_1(\gamma, \gamma') - B_2$. Thus, provided $k = (\gamma, \gamma')$ is sufficiently large, we can apply Lemma 1 and so

$$|l(\gamma) - l(a\gamma) - l(\gamma') + l(a\gamma')| \leq Ke^{-\beta r} \leq (Ke^{\beta B_2})e^{-(B_1\beta)(\gamma,\gamma')}$$

The estimate follows by taking $B = Ke^{\beta B_2}$ and $\theta = e^{-(B_1\beta)}$.

Remark. Although r is not necessarily positive on $X_A - \{(0, 0, \dots)\}$, it is eventually positive in the sense that there exists $n > 0$ such that $r^n(x) > 0$ for all $x \in X_A - \{(0, 0, \dots)\}$.

3. Transfer Operators and Poincaré Series

In this section we introduce a certain class of operators and relate them to Poincaré series. We shall then be in a position to deduce our main result on the meromorphic domain of $\eta(s)$ from the spectral properties enjoyed by these operators.

If $f \in C^\alpha(X_A, \mathbb{C})$, the Banach space of α-Hölder continuous functions, we can define *Ruelle transfer operators* $L_f : C^\alpha(X_A, \mathbb{C}) \to C^\alpha(X_A, \mathbb{C})$ by

$$(L_f h)(x) = \sum_{\substack{\sigma y = x \\ y \neq \dot{0}}} e^{f(y)} h(y),$$

where $\dot{0} = (0, 0, 0, \dots)$.

Remark. This definition of the Ruelle tranfer operator differs from the usual definition in that in the summation over pre-images y of x we exclude the possibility $y = \dot{0}$. However, it agrees with the more familiar definition for all $x \neq \dot{0}$ and its only effect on the spectrum is to exclude an eigenvalue $e^{f(\dot{0})}$ (corresponding to the eigenvector which is the characteristic function for the set $\{\dot{0}\}$). The reason we make this change is so that the spectral radius of L_f only depends on the restriction of f to $X_A - \{\dot{0}\}$, which is necessary for the final statement of Lemma 2, and to avoid overcounting in Lemma 3.

For a bounded linear operator $T : B \to B$ acting on a Banach space B let $\rho(T)$ denote the spectral radius. We define the *essential spectrum* $ess(T)$ to be the subset of the spectrum $spec(T) \subset \mathbb{C}$ of T consisting of those $\lambda \in spec(T)$ such that at least one of the following is true

(1) $Range(\lambda - T)$ is not closed in B;
(2) λ is a limit point of $spec(T)$;
(3) $\cup_{r=1}^{\infty} ker(\lambda - T)^r$ is infinite dimensional.

We define the *essential spectral radius* to be $\rho_e(T) = sup\{|\lambda| : \lambda \in ess(T)\}$. The operator $T : B \to B$ is *quasi-compact* if the essential spectral radius is strictly smaller than the spectral radius.

Lemma 2. *The operator $L_f : C^\alpha(X_A, \mathbb{C}) \to C^\alpha(X_A, \mathbb{C})$ is quasi-compact. Furthermore, the spectral radius satisfies $\rho(L_f) \leq \rho(L_{Re(f)})$ and essential spectral radius satisfies $\rho_e(L_f) \leq \frac{1}{2^\alpha}\rho(L_{Re(f)})$. In addition, if $f, g \in C^\alpha(X_A, \mathbb{R})$ and, for some $n > 0$, $f^n(x) < g^n(x)$, for all $x \in X_A - \{\dot{0}\}$, then $\rho(L_f) < \rho(L_g)$.*

If A is irreducible then the first assertion was proved in [17]. In the general case the proof is given in Lemma 2 of [18]. The other assertions are not difficult.

To take advantage of our embedding of $\pi_1(M)$ into the subshift of finite type X_A, we observe that $\eta(s)$ may also be written in the following way

$$\eta(s) = \sum_{n=1}^{\infty} \sum_{z \in S_n} e^{-sr^n(z\dot{0})} \tag{3.1}$$

where S_n denotes the set of all allowed finite paths $z = z_0 \dots z_n$ of (edge) length n with $z_0 = *$ and $z_i \neq 0$, $i = 1, \dots, n$, and where $z\dot{0} = (z_0, \dots, z_n, 0, 0, \dots)$.

It is possible to write the Right Hand Side of (3.1) in terms of the transfer operator by means of the next lemma. This will allow us to apply the results of the preceding section. We shall use the following simple identity.

Lemma 3. *Define $\chi : X_A \to \mathbb{R}$ by $\chi(x) = 1$ if $x_0 = *$, and 0 otherwise. For any $k \in C(X_A)$ we have the following simple identity*

$$\sum_{z \in S_n} e^{k^n(z\dot{0})} = (L_k^n \chi)(\dot{0}), \qquad \textit{for } n \geq 1.$$

Proof. This is by direct computation. For any $x \in X_A$ and $k \in C(X_A)$ we can write

$$\sum_{\substack{\sigma^n y = x : y_0 = * \\ y_i \neq 0,\ i=1,\dots,n}} e^{k^n(y)} = \sum_{\substack{\sigma^n y = x \\ \sigma^{n-1} y \neq \dot{0}}} e^{k^n(y)} \chi(y) = L_k^n \chi(x).$$

The identity follows upon setting $x = \dot{0}$.

Comparing (3.1) and Lemma 3, we can now write

$$\eta(s) = \sum_{n=1}^{+\infty} (L_{-sr}^n \chi)(\dot{0}) \tag{3.2}$$

for $Re(s) > \delta$.

Since $r^n > 0$ on $X_A - \{\dot{0}\}$, for some $n > 0$, Lemma 2 shows that, for $t \in \mathbb{R}$, the map $t \to \rho(L_{-tr})$ is strictly decreasing, it is clear that $\delta = \inf\{s \in \mathbb{R} : \rho(L_{-sr}) < 1\}$ and so δ is the unique real number such that $\rho(L_{-\delta r}) = 1$.

An important consequence of the spectral theory of the transfer operators described in Lemma 2 and Proposition 3 is the following.

Theorem 1. *Let M be a compact manifold (possibly with totally geodesic boundary) with strictly negative sectional curvatures. Then the function $\eta(s)$ has an extension as a meromorphic function to the half-plane $Re(s) > \delta - \epsilon$, for some $\epsilon > 0$.*

Proof. The proof follows the same general lines as in [18]. Note that for any $\beta > \rho_e(L_{-sr})$ the operator L_{-sr} has only finitely many isolated eigenvalues of finite multiplicity in the region $\rho(L_{-Re(s)r}) \geq |z| \geq \beta$. We can assume without loss of generality that the circle $\beta = |z|$ is disjoint from the spectrum. For each eigenvalue λ lying in this annulus we denote by $\mathbb{P}_\lambda(s) : C^\alpha(X_A, \mathbb{C}) \to C^\alpha(X_A, \mathbb{C})$ the projection onto the generalized eigenspace associated to λ and we denote by $Q(s) : C^\alpha(X_A, \mathbb{C}) \to C^\alpha(X_A, \mathbb{C})$ the projection associated to the part of the spectrum in $|z| < \beta$. The maps $s \to \mathbb{P}_\lambda(s), s \to Q(s)$ are analytic and commute with the operator L_{-sr}.

We then have the following spectral decomposition

$$L_{-sr}^n = \sum_\lambda \mathbb{P}_\lambda L_{-sr}^n + Q L_{-sr}^n \tag{3.3}$$

Substituting (3.3) into (3.2) gives that

$$\begin{aligned} \eta(s) &= \sum_\lambda \left(\sum_{n=0}^{\infty} L_{-sr}^n \mathbb{P}_\lambda(s) L_{-sr} \chi \right)(\dot{0}) + \left(\sum_{n=1}^{\infty} L_{-sr}^n Q(s) \chi \right)(\dot{0}) \\ &= \sum_\lambda \left((I - L_{-sr}^{(\lambda)})^{-1} \mathbb{P}_\lambda(s) L_{-sr} \chi \right)(\dot{0}) + \left(\sum_{n=1}^{\infty} L_{-sr}^n Q(s) \chi \right)(\dot{0}) \end{aligned} \tag{3.4}$$

where $L_{-sr}^{(\lambda)}$ is the restriction of L_{-sr} to the finite dimensional generalized eigenspace associated to λ.

The second term in (3.4) converges to an analytic function when $\beta < 1$. Since we know by Lemma 2 that the essential spectral radius $\rho_e(L_{-sr})$ is bounded above by $\left(\frac{1}{2}\right)^{\alpha} \rho(L_{-Re(s)r})$ and since $t \to \rho(L_{-tr})$ is decreasing there exists $\delta' < \delta$ such that if $Re(s) > \delta'$ then the condition $\beta < 1$ is satisfied.

We next observe that the first term in (3.4) is meromorphic since $(I - L_{-sr})^{-1}$ can be written in the form

$$\sum_{\lambda}(I - L_{-sr}^{(\lambda)})^{-1} = \sum_{\lambda} \frac{N_\lambda(s)}{det(I - L_{-sr}^{\lambda})},$$

where $N_\lambda(s)$ are analytic operator valued functions and, furthermore, it is well known that $det(I - L_{-sr}^{(\lambda)})$ are analytic [19].

Remark. In fact, it is possible to show that there is a simple pole at $s = \delta$. One can see this either from general considerations about transfer operators, or from the fact that the counting function $N(T) = \#\{\gamma \in \pi_1(M) : l(\gamma) \leq T\}$ satisfies the inequalities $C^{-1}e^{\delta T} \leq N(T) \leq Ce^{\delta T}$, for some $C > 1$ [6].

4. A Result of Margulis

In this section we shall use a result of Margulis [12] to show that for compact manifolds without boundary there are no poles on the line $Re(s) = \delta$ except for a simple pole at $s = 1$. We remark that in this case the abscissa of convergence δ is equal to the exponential growth rate of volume in the universal cover X of M, i.e., $\delta = \lim_{T\to\infty} \frac{1}{T} \log Vol(\{y \in X : d(x, y) \leq T\})$, for any $x \in X$.

Proposition 4 (Margulis). *Let M be a compact manifold without boundary with strictly negative sectional curvatures. If $N(T) = \#\{\gamma \in \pi_1(M) : l(\gamma) \leq T\}$ then there exists a constant $C > 0$ such that $N(T) \sim Ce^{\delta T}$, as $T \to +\infty$.*

To make use of Proposition 4, we observe that we can write

$$\eta(s) = \int_0^\infty e^{-st} dN(t) = s \int_0^\infty e^{-st} N(t) dt$$

for $Re(s) > \delta$. We can define $\alpha(T) := N(T) - Ce^{\delta T}$ then by Proposition 4 we see that $\alpha(T) = o(e^{\delta T})$. Suppose that $\eta(s)$ has a pole at $s = \delta + i\tau$ with $\tau \neq 0$. It is then clear that

$$\bar{\alpha}(s) = \int_0^\infty e^{-st} \alpha(t) dt = \eta(s) - \frac{C}{s - \delta}$$

also has a pole at $s = \delta + i\tau$. However, $|\bar{\alpha}(\sigma + i\tau)| \leq \int_0^\infty e^{-\sigma t}|\alpha(t)|dt$. Since $|\alpha(t)| = o(e^{\delta T})$ we have that for any $\epsilon > 0$ we can choose $T > 0$ such that $|\alpha(t)| \leq \epsilon$ for $t \geq T$ and so

$$
\begin{aligned}
|\bar{\alpha}(\sigma + i\tau)| &\leq \int_0^\infty e^{-\sigma t}|\alpha(t)|dt \\
&\leq \int_0^T e^{-\sigma t}|\alpha(t)|dt + \int_T^\infty e^{-\sigma t}|\alpha(t)|dt \\
&\leq Const.T + \epsilon \frac{1}{\sigma - \delta}.
\end{aligned}
$$

In particular, $\limsup_{\sigma \to \delta+} |(\sigma - \delta)\bar{\alpha}(\sigma + i\tau)| \leq \epsilon$, and since ϵ can be chosen arbitrarily small this contradicts the existence of a pole at $s = \delta + i\tau$. This shows that the following result is a corollary to Proposition 4 and Theorem 1.

Proposition 5. *Let M be a compact manifold without boundary with strictly negative sectional curvatures. Then function $\eta(s)$ has no pole on the line $Re(s) = \delta$, apart from a simple pole at $s = \delta$.*

5. Comparison Theorems

In this section we shall apply the analytical theory of Poincaré series to prove certain comparison theorems in the case where M is a compact manifold without boundary. Recall the classical result of Milnor [13] that the ratio $\frac{|\gamma|}{l(\gamma)}$ is bounded away from zero and infinity as γ runs through $\pi_1(M) - \{e\}$. In particular, this shows that the averages $\frac{1}{N(T)} \sum_{l(\gamma) \leq T} \frac{|\gamma|}{l(\gamma)}$ are similarly bounded. In the following theorem we show that in fact the averages converge to a limit as $T \to +\infty$.

Theorem 2. *Let M be a compact manifold without boundary and with strictly negative sectional curvatures. Choose a finite symmetric generating set S for $\pi_1(M)$. Let $w : \pi_1(M) \to \mathbb{R}^+$ be a function satisfying*

(i) *there exist constants $A_1, A_2 > 0$ such that $A_1 l(\gamma) \leq w(\gamma) \leq A_2 l(\gamma)$; and*
(ii) *there exist constants $A > 0$ and $0 < \theta < 1$ such that for all generators $a \in S$ and all $\gamma, \gamma' \in \pi_1(M)$ we have*

$$
|w(\gamma) - w(a\gamma) - w(\gamma') + w(a\gamma')| \leq A\theta^{(\gamma,\gamma')}.
$$

Then the averages

$$
\frac{1}{N(T)} \sum_{\substack{\gamma \in \pi_1(M) - \{e\} \\ l(\gamma) \leq T}} \frac{w(\gamma)}{l(\gamma)}
$$

converge to a positive limit as $T \to +\infty$. In particular, with the special choices

(a) *$w(\gamma) = |\gamma|$;*
(b) *$w(\gamma)$ is the length of the shortest geodesic arc in γ from p to itself with respect to a second Riemannian metric (with negative sectional curvatures) on M*

the above conclusion holds.

Given Proposition 5, the proof of Theorem 2 follows from the arguments in sections 8 and 9 of [18]. We briefly sketch the main steps.

We begin by defining a weighted Poincaré series

$$\eta(s,z) = \sum_{\gamma \in \pi_1(M)-\{e\}} e^{-sl(\gamma)+zw(\gamma)}.$$

This summation converges to an bi-analytic function provided that $Res(s) > \delta$ and $|z|$ is sufficiently small (how small depending on s). Clearly, $\eta(s,0) = \eta(s)$.

Since $w : \pi_1(M) \to \mathbb{R}^+$ satisfies condition (ii) we can, as in Lemma 1, define a Hölder contionuous function $f : X_A \to \mathbb{R}$ with the property that if $x = (*, x_1, \dots, x_n, 0, 0, \dots)$ then $f^n(x) = w(\gamma)$, where $\gamma = \lambda(*, x_1)\lambda(x_1, x_2)\dots\lambda(x_{n-1}, x_n)$.

As in (3.2) we may write

$$\eta(s,z) = \sum_{n=1}^{\infty} (L^n_{-sr+zf}\chi)(\dot{0})$$

for $Re(s) > \delta$ and $|z|$ small. This enables us to study the analytic properties of $\eta(s,z)$ via the spectral properties of L_{-sr+zf}. In particular, given the information on $\eta(s)$ contained in Proposition 5, the arguments of Section 8 of [18] (based on eigenvalue perturbation theory applied to L_{-sr+zf}) go through and we can conclude the following result.

Proposition 6. *Define a function* $\xi(s)$ *by*

$$\xi(s) := \frac{\partial}{\partial z}\eta(s,0).$$

This function is analytic in a neighbourhood of $\{s \in \mathbb{C} : Re(s) \geq \delta\} - \{\delta\}$. *Furthermore, in a neighbourhood of* $s = \delta$, *we may write*

$$\xi(s) = \frac{A}{(s-\delta)^2} + \frac{B}{s-\delta} + U(s),$$

with $A > 0$ *and* $U(s)$ *analytic.*

Proof. This is Corollary 6.1 of [18]. That we have $A > 0$ is a consequence of condition (i) of Theorem 2 and the fact that $Res_{s=\delta}\eta(s) > 0$. Specifically, we compare $\xi(s)$ with $-\eta'(s)$ for $s \in \mathbb{R}$.

To complete the proof of Theorem 2, observe that we can write $\xi(s)$ as a Stieltjes integral:

$$\xi(s) = \int_0^\infty e^{-sT} dM(T),$$

where $M(T) = \sum_{l(\gamma) \leq T} w(\gamma)$. It is an immediate consequence of Proposition 6 and standard Tauberian results [1] that

$$M(T) \sim \frac{A}{6} T e^{\delta T}, \qquad \text{as } T \to +\infty.$$

It is then an elementary deduction that

$$\sum_{l(\gamma)\leq T} \frac{w(\gamma)}{l(\gamma)} \sim \frac{A}{6} e^{\delta T}.$$

Combining this with Proposition 4, that $N(T) \sim Ce^{\delta T}$, gives the desired result.

Remark. Consider a surfaces of constant curvature $\kappa = -1$ with a fixed reference metric g_1. If we use another metric g_2 to give the weighting $w(\gamma) = l_{g_2}$, as in Theorem 2 (a), then a simple argument shows that the limit

$$\mathcal{L}(g_2) := \lim_{T\to+\infty} \frac{1}{N(T)} \sum_{\substack{\gamma\in\pi_1(M)-\{e\} \\ l_{g_1}(\gamma)\leq T}} \frac{l_{g_2}(\gamma)}{l_{g_1}(\gamma)}$$

is proportional to the intersection $i(g_1, g_2)$ for the two metrics [2, 8].

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