CONVERGENCE OF ZETA FUNCTIONS FOR AMENABLE GROUP EXTENSIONS OF SHIFTS

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Dedicated to the memory of Sergiy Kolyada

Abstract. The aim of this paper is show how zeta functions for amenable group extensions of subshifts of finite type may be approximated by rescaled zeta functions for a sequence of subshifts of finite type whose states are determined by a Følner exhaustion of the group. In particular, this recovers a result of Guido, Isola and Lapidus for graphs, and, by using weighted zeta functions, extends it to metric graphs.

1. Introduction

A classical object attached to a finite graph X is its *Ihara zeta function*

$$\zeta_X(z) = \prod_\gamma (1 - z^{|\gamma|})^{-1}, \tag{1.1}$$

where γ runs over the prime closed geodesics of X and $|\gamma|$ denotes the length of the geodesic. (Here, a closed geodesic is a closed path in X, with no backtracking or tail, modulo cyclic permutation. It is prime if it traverses its image exactly once. Its length is the number of edges forming the path.) The product converges for $|z|$ sufficiently small and extends to a rational function, which may be expressed as a determinant in various ways. See [21] for an account of this theory.

Recently, there has also been considerable interest in the analogous theory for infinite graphs, particularly those which occur as covers of finite graphs [3, 4, 5, 9, 10, 11, 15]. It is clear that, in this case, a naïve definition analogous to (1.1) does not work, since, for example, if one considers an infinite regular cover of a finite graph then any prime closed geodesic will have infinitely many translates with the same length. It is these covers that we will be concerned with and a natural way to proceed is to define a zeta function relative to the cover, i.e. to take a product over prime closed geodesics in the cover modulo the action of the covering group. An alternative approach, discussed in [9], is to define the zeta function for the cover as a limit of normalised zeta functions for finite graphs. For example, if the covering group is residually finite, then one might consider the zeta functions of finite subcovers. On the other hand, if the covering group is amenable, then one might consider finite graphs associated to an exhausting Følner sequence. This second situation is studied in [11], where it is shown that, in an appropriate disk, the normalised zeta functions for a Følner sequence converge to the zeta function for the infinite graph relative to the covering. The aim of this note is to extend this result to zeta functions for certain dynamical systems; precisely, amenable group extensions of subshifts of finite type. We are also able to consider weighted zeta functions and hence, as an application, zeta functions for metric graphs.

We will now describe our set-up in more detail. Let $\mathcal{S}$ be a finite set of states with the discrete topology and let A be a $\#\mathcal{S} \times \#\mathcal{S}$ zero-one matrix indexed by $\mathcal{S}$.

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We then define

$$\Sigma = \left\{ (x_n)_{n=0}^\infty \in \mathcal{S}^{\mathbb{Z}^+} \,:\, A(x_n, x_{n+1}) = 1 \ \forall n \geq 0 \right\}$$

with the subspace topology, where $\mathcal{S}^{\mathbb{Z}^+}$ has the product topology, making Σ totally disconnected. If we let $\sigma : \Sigma \to \Sigma$ denote the left shift map then the pair (Σ, σ) is called a subshift of finite type and we call A the transition matrix. (Strictly speaking, (Σ, σ) is a *one-sided* subshift of finite type and there is a corresponding two-sided subshift of finite type defined on the corresponding subset of $\mathcal{S}^{\mathbb{Z}}$. However, as we are only interested in periodic orbits, it is no restriction to limit ourselves to the one-sided case.) We will also assume that $\sigma : \Sigma \to \Sigma$ is topologically transitive, which is equivalent to the requirement that, for each $(i,j) \in \mathcal{S} \times \mathcal{S}$, there is an $n = n(i,j) \geq 1$ such that $A^n(i,j) > 0$.

We say that a periodic orbit $\tau = \{x, \sigma x, \ldots, \sigma^{n-1} x\}$, with $\sigma^n x = x$, is *prime* if $\sigma^m x \neq x$ for $1 \leq m < n$ and write $|\tau| = n$. Let $\mathcal{P}(\sigma)$ denote the set of prime periodic orbits of σ. We define the *Artin–Mazur zeta function* for σ to be the function

$$\zeta_\sigma(z) := \prod_{\tau \in \mathcal{P}(\sigma)} (1 - z^{|\tau|})^{-1} = \exp \sum_{n=1}^\infty \frac{\#\{x \in \Sigma \,:\, \sigma^n x = x\}}{n} z^n. \tag{1.2}$$

(The equality of the two expressions is a standard result.) This function converges for $|z| < \exp(-h(\sigma))$, where $h(\sigma)$ is the topological entropy of σ, and has a rational extension to the whole complex plane given by

$$\zeta_\sigma(z) = \frac{1}{\det(I - zA)}. \tag{1.3}$$

We now consider extensions by countable amenable groups. (In fact, we will only consider finitely generated groups.) There are numerous equivalent definitions of amenability. We shall use the convenient characterisation, due to Følner [8], that a countable group G is amenable if for every finite set $F \subset G$ and every $\epsilon > 0$, there exists a finite set $K \subset G$ such that $\#(F \triangle Fg) < \epsilon \# F$, for all $g \in F$.

Let G be a finitely generated amenable group (with the discrete topology) and suppose that we are given a continuous function $\psi : \Sigma \to G$. Since G is discrete, the continuity of ψ implies that ψ is locally constant, i.e. that there exists $N \geq 1$ such that $\psi(x) = \psi(x_0, x_1, \ldots, x_{N-1})$, for $x = (x_i)_{i=0}^\infty \in \Sigma$. The following standard recoding allows us to assume the $N = 2$. We can define a new subshift of finite type (Σ', σ') with state set

$$\mathcal{S}' := \{(i_0, i_1, \ldots, i_{N-2}) \in \mathcal{S}^{N-1} \,:\, A(i_n, i_{n+1}) = 1, \, n = 0, \ldots, N-3\}$$

and transition matrix A' given by

$$A'((i_0, \ldots, i_{N-2}), (j_0, \ldots, j_{N-2})) = \begin{cases} 1 & \text{if } j_n = i_{n+1}, \, n = 0, \ldots, N-3, \\ & \text{and } A(i_{N-2}, j_{N-2}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the map $((i_0, \ldots, i_{N-2}), (i_1, \ldots, i_{N-1}), \ldots) \mapsto (i_0, i_1, \ldots)$ from Σ' to Σ is a topological conjugacy and, in particular, we may identify the two sets of periodic orbits. Furthermore, the conjugacy takesf ψ to a function on Σ' that depends on two co-ordinates. Therefore, in what follows, it is no loss of generality to suppose that $\psi : \Sigma \to G$ depends on only two co-ordinates.

The function ψ defines a skew product extension $T_\psi : \Sigma \times G \to \Sigma \times G$ by

$$T_\psi(x, g) = (\sigma x, g\psi(x)).$$

We will always assume that T_ψ is topologically transitive, which, in particular, implies that $\Psi = \{\psi(i,j) \,:\, i, j \in \mathcal{S}, \ A(i,j) = 1\}$ is a generating set for G, i.e. that every element of G may be written as a product of elements of Ψ and their inverses.

The map $T_\psi : \Sigma \times G \to \Sigma \times G$ is itself a countable state subshift with infinite transition matrix $\mathbb{A}$, indexed by $\mathcal{S} \times G$, defined by

$$\mathbb{A}((i,g),(j,h)) = \begin{cases} 1 & \text{if } A(i,j) = 1 \text{ and } h = g\psi(i,j), \\ 0 & \text{otherwise.} \end{cases}$$

We note that $T_\psi^n(x,g) = (x,g)$ if and only if $\sigma^n x = x$ and

$$\psi^n(x) := \psi(x)\psi(\sigma x)\cdots\psi(\sigma^{n-1}x) = e,$$

the identity in G. In particular, $T_\psi^n(x,g) = (x,g)$ for some $g \in G$ if and only if $T_\psi^n(x,g) = (x,g)$ for all $g \in G$.

As for σ, we write $\mathcal{P}(T_\psi)$ for the set of prime T_ψ-periodic orbits. (Note that if $T_\psi^n(x,g) = (x,g)$ then $\sigma^n x = x$ but that, if G has torsion, the periodic orbit $\{x, \sigma x, \ldots, \sigma^{n-1}\}$ is not necessarily prime.) The group G acts transitively on $\mathcal{P}(T_\psi)$ and we write $\mathcal{P}_G(T_\psi) = \mathcal{P}(T_\psi)/G$. For $[\tau] \in \mathcal{P}_G(T_\psi)$, we write $|[\tau]| = |\tau|$, for any $\tau \in [\tau]$; clearly, this is well-defined. The (Artin–Mazur) zeta function for T_ψ is defined in this setting to be

(1.4)

$$\zeta_{T_\psi}(z) = \prod_{[\tau] \in \mathcal{P}_G(T_\psi)} (1 - z^{|[\tau]|})^{-1} = \exp \sum_{n=1}^{\infty} \frac{\#\{x \in \Sigma : \sigma^n x = x, \ \psi^n(x) = e\}}{n} z^n.$$

An alternative definition for the zeta function of an infinite graph appears in the work of Chinta, Jorgenson and Karlsson [2]. This is obtained by restricting to closed paths based at a given vertex. The natural analogue for T_ψ is to set, for $a \in \mathcal{S}$,

$$\zeta_{T_\psi}^{(a)}(z) = \exp \sum_{n=1}^{\infty} \frac{\#\{x \in \Sigma : \sigma^n x = x, \ x_0 = a, \ \psi^n(x) = e\}}{n} z^n.$$

The radius of convergence of all these functions may be given in terms of the *Gurevič entropy* $h_{\mathrm{Gur}}(T_\psi)$ of T_ψ [12]. This is defined by

$$h_{\mathrm{Gur}}(T_\psi) = \limsup_{n \to \infty} \frac{1}{n} \#\{x \in \Sigma : \sigma^n x = x, \ x_0 = a, \ \psi^n(x) = e\},$$

(which is independent of the choice of a by transitivity) and so we see that each $\zeta_{T_\psi}^{(a)}(z)$ has radius of convergence $\exp(-h_{\mathrm{Gur}}(T_\psi))$. Since

$$\#\{x \in \Sigma : \sigma^n x = x, \ x_0 = a, \ \psi^n(x) = e\} \le \#\{x \in \Sigma : \sigma^n x = x, \ \psi^n(x) = e\}$$
$$\le \sum_{a \in \mathcal{S}} \#\{x \in \Sigma : \sigma^n x = x, \ x_0 = a, \ \psi^n(x) = e\},$$

$\zeta_{T_\psi}(z)$ has the same radius of convergence.

Remark 1.1. It is interesting to compare the radii of convergence of $\zeta_\sigma(z)$ and $\zeta_{T_\psi}(z)$. As a direct consequence of the definitions of topological entropy and Gurevič entropy, it is clear that $h_{\mathrm{Gur}}(T_\psi) \le h(\sigma)$, and so the radius of convergence of $\zeta_{T_\psi}(z)$ is at least as large as the radius of convergence of $\zeta_\sigma(z)$. The question of when we have equality reduces to a question about the skew product induced by T_ψ on $\Sigma \times G^{\mathrm{ab}}$, where $G^{\mathrm{ab}} := G/[G,G]$ is the abelianisation of G. More precisely, let $\pi : G \to G^{\mathrm{ab}}$ be the natural projection and define $\varphi : \Sigma \to G^{\mathrm{ab}}$ by $\varphi = \pi \circ \psi$. This induces a skew product $T_\varphi : \Sigma \times G^{\mathrm{ab}} \to \Sigma \times G^{\mathrm{ab}}$. Since G is amenable, $h_{\mathrm{Gur}}(T_\psi) = h_{\mathrm{Gur}}(T_\varphi)$ [7]. This means we need to compare $h(\sigma)$ with $h_{\mathrm{Gur}}(T_\varphi)$. If G^{ab} is finite then it is trivial that $h_{\mathrm{Gur}}(T_\varphi) = h(\sigma)$. On the other hand, if G^{ab} is infinite then it is a finite extension of $\mathbb{Z}^d$, for some $d \ge 1$, and we can factor out the finite group to get a function $\varphi_0 : \Sigma \to \mathbb{Z}^d$. This in turn gives a skew product $T_{\varphi_0} : \Sigma \times \mathbb{Z}^d \to \Sigma \times \mathbb{Z}^d$ with $h_{\mathrm{Gur}}(T_{\varphi_0}) = h_{\mathrm{Gur}}(T_\varphi)$. Finally, it follows from the results of [17] that $h_{\mathrm{Gur}}(T_{\varphi_0}) = h(\sigma)$ if and only if $\int \varphi_0 \, d\mu_0 = 0$, where μ_0 is the measure of maximal entropy for σ.

We now wish to consider finite approximations to T_ψ, obtained by restricting to a (large) finite subset of G. More precisely, if K is a finite subset of G then we may obtain a subshift of finite type by restricting $\mathbb{A}$ to $\mathcal{S} \times K$, i.e. we consider the finite matrix A_K, indexed by $\mathcal{S} \times K$ and defined by $A_K((i,g),(j,h)) = 1$ if and only if $\mathbb{A}((i,g),(j,h)) = 1$. We write $\sigma_K : \Sigma_K \to \Sigma_K$ for the subshift of finite type with transition matrix A_K and $\zeta_{\sigma_K}(z)$ for the associated zeta function.

We say that a sequence of finite sets $K_n \subset G$, $n \geq 1$, is a Følner exhaustion of G if

(FE1) $\bigcup_{n=1}^\infty K_n = G$;
(FE2) $K_n \subset K_{n+1}$, for all $n \geq 1$; and
(FE3)
$$\lim_{n\to\infty} \frac{\#(K_n \triangle K_n g)}{\# K_n} = 0,$$
for all $g \in G$.

It is easy to see that amenability is equivalent to the existence of a sequence satisfying (FE2) and (FE3). Moreover, it is well-known that the sequence can also be chosen to satisfy (FE1) (see, for example, Theorem 6.2 of [11], applied to the action of G on its own Cayley graph, for a proof). Thus G is amenable if and only if it admits a Følner exhaustion.

We have the following convergence result.

Theorem 1.2. *Let $\sigma : \Sigma \to \Sigma$ be a subshift of finite type and let $T_\psi : \Sigma \to G \to \Sigma \times G$ be a topologically transitive skew product extension, where G is a finitely generated amenable group. Suppose that K_n, $n \geq 1$ is a Følner exhaustion of G. Then we have*
$$\zeta_{T_\psi}(z) = \lim_{n\to\infty} \zeta_{\sigma_{K_n}}(z)^{1/(\# K_n)},$$
uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < (2\|\mathbb{A}\|)^{-1}\}$, where $\|\mathbb{A}\|$ is the operator norm of $\mathbb{A}$ acting on $\ell^2(\mathcal{S} \times G)$.

Remark 1.3. We note that $\|\mathbb{A}\| \leq \sqrt{d_r d_c}$, where
$$d_r = \max_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} A(i,j) \quad \text{and} \quad d_c = \max_{j \in \mathcal{S}} \sum_{i \in \mathcal{S}} A(i,j),$$
i.e. d_r and d_c are, respectively, the maximum row and column sums of A. See Lemma 4.1 for a proof.

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2. Graphs

In this section, we show how zeta functions for graphs may be interpreted in the framework of the preceding section and compare our results with those of [11]. Let X be a finite connected graph with vertex set $V(X)$ and edge set $E(X)$. (We allow loops and multiple edges.) A closed geodesic is a closed path in X, with no backtracking or tail, modulo cyclic permutation, and a closed geodesic is prime if it traverses its image exactly once. The Ihara zeta function $\zeta_X(z)$ is defined by (1.1) and extends to a rational function via the determinant formula of Bass [1] (see also [14, 21]):

$$\zeta_X(z) = (1 - z^2)^{\chi(X)} \det(I - zM + z^2(D - I))^{-1}, \tag{2.1}$$

where $\chi(X)$ is the Euler characteristic of X, M is the adjacency matrix (i.e. for vertices u, v, $M(u,v)$ is the number of oriented edges from u to v), and D is the diagonal matrix with entries given by the degrees of the vertices.

Another approach to $\zeta_X(z)$ is through subshifts of finite type, as follows. Each edge in $E(X)$ comes with two orientations and we will write $E(X)^o$ for the set of oriented edges. If $e \in E(X)^o$ then we will write $\mathfrak{o}(e)$ and $\mathfrak{t}(e)$, respectively, for the initial and terminal vertices of e, and $\overline{e}$ for the edge with the opposite orientation. Now consider the space

$$\Sigma(X) = \left\{(e_n)_{n=0}^{\infty} \in (E(X)^o)^{\mathbb{Z}^+} : \mathfrak{t}(e_n) = \mathfrak{o}(e_{n+1}),\ e_{n+1} \neq \overline{e}_n \ \forall n \geq 0\right\}$$

together with the shift map $\sigma : \Sigma(X) \to \Sigma(X)$ defined by $\sigma((e_n)_{n=0}^{\infty}) = (e_{n+1})_{n=0}^{\infty}$. Clearly, $\sigma : \Sigma(X) \to \Sigma(X)$ is a subshift of finite type with state set $\mathcal{S} = E(X)^o$.

The map $\kappa e = \overline{e}$ is a fixed point free involution on $\mathcal{S}$. Furthermore, if A denotes the transition matrix for this subshift then $A(i,j) = 1$ if and only if $A(\kappa j, \kappa i) = 1$. Using this symmetry, it is easy to see that $d_r = d_c$. Moreover, we have

$$d_r = \max_{v \in V(X)} \deg(v) - 1.$$

It is clear that closed geodesics of length n in X correspond exactly to periodic orbits of period n for σ and, hence, $\zeta_X(z) = \zeta_\sigma(z)$. This gives an alternative expression for $\zeta_X(z)$ as a determinant, given by (1.3) above.

Now suppose that we have an infinite graph Y, with vertices $V(Y)$ and edges $E(Y)$, which is a regular cover of X with covering group G. (Since G is a quotient of the fundamental group of X, which is a finitely generated free group, G is automatically finitely generated.) As a natural extension of the finite case, one can define a zeta function

$$\zeta_Y(z) = \prod_{[\gamma]} (1 - z^{|\gamma|})^{-1},$$

where now $[\gamma]$ runs over equivalence classes prime closed geodesics of Y modulo the G-action and $|\gamma|$ denotes the length of any geodesic in $[\gamma]$. (This is the function defined in Definition 2.1 of [11]. The situation there is actually a little more general, as they allow covers which are not regular but require that vertex stabilisers are finite.) We claim that this zeta function is equal to $\zeta_{T_\psi}(z)$, for some skew product $T_\psi : \Sigma(X) \times G \to \Sigma(X) \times G$.

The skewing function ψ is defined in the following way. For each vertex $v \in V(X)$, choose a fixed lift $\tilde{v} \in V(Y)$. We then have the following lemma.

Lemma 2.1. *For each $e \in E(X)^o$, there is a unique $g = g(e) \in G$ such that, if $\tilde{e}$ is a lift of e with $\mathfrak{o}(\tilde{e}) = \widetilde{\mathfrak{o}(e)} \cdot h$, then $\mathfrak{t}(\tilde{e}) = \widetilde{\mathfrak{t}(e)} \cdot gh$.*

Proof. First, let $\tilde{e}$ be the lift of e with $\mathfrak{o}(\tilde{e}) = \widetilde{\mathfrak{o}(e)}$ and define $g \in G$ by $\mathfrak{t}(\tilde{e}) = \widetilde{\mathfrak{t}(e)} \cdot g$. If $\tilde{e}'$ is another lift of e then it is the translate of $\tilde{e}$ by some $h \in G$ and we have $\mathfrak{o}(\tilde{e}') = \widetilde{\mathfrak{o}(e)} \cdot h$ and $\mathfrak{t}(\tilde{e}') = \widetilde{\mathfrak{t}(e)} \cdot gh$. □

We now define $\psi : \Sigma(X) \to G$ by $\psi((e_n)_{n=0}^{\infty}) = g(e_0)$.

Lemma 2.2. *A closed geodesic γ in X, corresponding to a periodic σ-orbit $\tau = \{x, \sigma x, \ldots, \sigma^{n-1}x\}$, lifts to a closed geodesic in Y if and only if $\psi^n(x) = e$.*

Proof. Suppose $x = (e_n)_{n=0}^{\infty}$. We will treat the vertex $v := \mathfrak{o}(e_0)$ as the initial point of γ. Let $\tilde{\gamma}$ be the lift of γ that starts at $\tilde{v}$. By Lemma 2.1, $\tilde{\gamma}$ ends at $\tilde{v} \cdot \psi^n(x)$ and is thus closed if and only if $\psi^n(x) = e$. □

We conclude from this that $\zeta_Y(z) = \zeta_{T_\psi}(s)$ and hence the conclusion of Theorem 1.2 holds for $\zeta_Y(z)$, provided G is amenable. More precisely, we have the following.

Theorem 2.3. *Let X be a finite connected graph and let Y be an infinite regular cover of X with amenable covering group G. Suppose that K_n be a Følner exhaustion of G and let $F \subset Y$ be a fundamental domain for the G-action. Let X_n denote the finite graph*

$$X_n = \bigcup_{g \in K_n} F \cdot g.$$

Then

$$\zeta_Y(z) = \lim_{n\to\infty} \zeta_{X_n}(z)^{1/(\# K_n)},$$

uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < (2\|\mathbb{A}\|)^{-1}\}$.

This result already appeared as Theorem 6.6 of [11] except that there the convergence takes place for

$$|z| < \frac{1}{\mathfrak{d} + \sqrt{\mathfrak{d}^2 + 2(\mathfrak{d}-1)}},$$

where $\mathfrak{d} := \max_{v \in V(X)} \deg(v)$. Since

$$\frac{1}{\mathfrak{d} + \sqrt{\mathfrak{d}^2 + 2(\mathfrak{d}-1)}} \le \frac{1}{2\mathfrak{d}} = \frac{1}{2(d_r+1)} < \frac{1}{2d_r} \le \frac{1}{2\|\mathbb{A}\|},$$

where we have used $\|\mathbb{A}\| \le \sqrt{d_r d_c} = d_r$, we see that our approach gives a slightly larger disk of convergence.

Remark 2.4. We end the section by noting two other different approaches to zeta function for infinite graphs. In [6], Deitmar considers the following zeta function. Let Y be an infinite graph with edge set $E(Y)$. Let $w : E(Y) \to \mathbb{R}^+$ satisfy $w \in \ell^1(E(Y))$. For a closed geodesic $\gamma = e_0 e_1 \cdots e_{n-1}$, set $w(\gamma) = w(e_0)w(e_1)\cdots w(e_{n-1})$ Then one can define

$$\zeta_{Y,w}(z) = \prod_\gamma \left(1 - w(\gamma) z^{|\gamma|}\right)^{-1},$$

where the product is taken over prime closed geodesics. This weighting gives rise to a trace class operator on $\ell^2(E^o)$ and hence to an expression for $\zeta_{Y,w}(z)$ as a determinant (Theorem 1.6 of [6]). The special case where w is the indicator function of a finite set $K \subset G$ gives ζ_{σ_K}. It is interesting to ask whether one can obtain convergence results analogous to Theorem 1.2 for as sequence $w_n \in \ell^1(E(Y))$ increasing pointwise to 1 (i.e. for each $e \in E(Y)$, $w_n(e)$ is a sequence increasing to 1).

The recent paper of Lenz, Pogorzelski and Schmidt [15] gives a very general approach based on noncommutative geometry.

3. Weighted Zeta Functions and Metric Graphs

In this section, we will introduce generalisations of the zeta functions $\zeta_\sigma(z)$ and $\zeta_{T_\psi}(z)$ which involve a weighting. This will give functions of two variables and we shall show that convergence results similar to Theorem 1.2 hold for these more general objects. As an application, this will give a convergence result for *metric* graphs, analogous to Theorem 2.3.

We start with generalising the zeta function $\zeta_\sigma(z)$ by introducing a weighting. Let $f : \Sigma_A \to \mathbb{R}$ be a strictly positive function satisfying $f(x) = f(x_0, x_1)$. We can then consider a zeta function depending on two complex variables:

$$\zeta_{\sigma,f}(z,s) = \exp \sum_{n=1}^\infty \frac{z^n}{n} \sum_{\sigma^n x = x} e^{-s f^n(x)}.$$

(This is just a special case of the generalised zeta functions introduced by Ruelle and much studied in dynamics; see, for example, [16, 18, 19].) The series converges to give an analytic function for

$$|z| < \exp(-P(-\mathrm{Re}(s)f,\sigma)),$$

where $P(\cdot,\sigma)$ is the standard pressure function [16, 19]. Furthermore, $\zeta_{\sigma,f}(z,s)$ extends to $\mathbb{C}^2$ by the formula

$$\zeta_{\sigma,f}(z,s) = \frac{1}{\det(I - zA_s)},$$

where A_s is the matrix

$$A_s(i,j) = A(i,j)e^{-sf(i,j)}.$$

We may generalise $\zeta_{T_\psi}(z)$ in the same way and define

$$\zeta_{T_\psi,f}(z,s) = \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{\substack{\sigma^n x = x\\ \psi^n(x)=e}} e^{-sf^n(x)}. \tag{3.1}$$

The domain of convergence of this function may be given in terms of the Gurevič pressure introduced by Sarig [20]. If we induce a function $\tilde{f} : \Sigma \times G \to \mathbb{R}$ by $\tilde{f}(x,g) = f(x)$ then its Gurevič pressure $P_{\mathrm{Gur}}(\tilde{f}, T_\psi)$ is defined by

$$P_{\mathrm{Gur}}(\tilde{f}, T_\psi) = \limsup_{n\to\infty}\frac{1}{n}\log\sum_{\substack{\sigma^n x = x\\ \psi^n(x)=e}} e^{f^n(x)}.$$

(This definition does not require f to be positive.) Then $\zeta_{T_\psi,f}(z,s)$ converges to an analytic function in (z,s) for

$$|z| < \exp(-P_{\mathrm{Gur}}(-\mathrm{Re}(s)\tilde{f}, T_\psi)).$$

Let $\mathbb{A}$ be the infinite matrix

$$\mathbb{A}_s((i,g),(j,h)) = \mathbb{A}((i,g),(j,h))e^{-sf(i,j)},$$

which also acts as a bounded operator on $\ell^2(\mathcal{S}\times G)$. (The proof that $\mathbb{A}_s$ is bounded is given below as part of the proof of Lemma 4.1.) If K is a finite subset of G then we can define a finite matrix $A_{K,s}$ and a zeta function $\zeta_{\sigma_{K_n},f}(z,s)$ in the obvious way.

We have the following convergence result, which includes Theorem 1.2 as a special case.

Theorem 3.1. *Let $\sigma : \Sigma \to \Sigma$ be a subshift of finite type and let $T_\psi : \Sigma \to G \to \Sigma \times G$ be a topologically transitive skew product extension, where G is a finitely generated amenable group. Suppose that K_n, $n \geq 1$ is a Følner exhaustion of G. Then we have*

$$\zeta_{T_\psi,f}(z,s) = \lim_{n\to\infty}\zeta_{\sigma_{K_n},f}(z,s)^{1/(\#K_n)},$$

uniformly on compact subsets of $\{(z,s)\in\mathbb{C}^2 : |z| < (2\|\mathbb{A}_s\|)^{-1}\}$.

As a corollary of this theorem, we have a similar convergence result for the zeta function of a metric graph. Let X be a finite graph, as the previous section, but we now suppose that each edge $e \in E(X)$ is given a positive real length $l(e)$. Thinking of the lengths as a function $l : E(X) \to \mathbb{R}^+$, we call the resulting object a metric graph (X,l). If $\gamma = e_0,\ldots,e_{n-1}$ is a closed geodesic then its length is

$$l(\gamma) = l(e_0) + \cdots + l(e_{n-1})$$

(where the length of an oriented geodesic is the length of the corresponding unoriented edge). We can then define the zeta function

$$\zeta_{(X,l)}(s) = \prod_{\gamma}\left(1 - e^{-sl(\gamma)}\right)^{-1},$$

where again the product is taken over prime closed geodesics. Now let Y be a regular G cover of X, where G is an (infinite) amenable group. The lengths on X lift to Y to give a metric graph (Y, l) and we define the zeta function $\zeta_{(Y,l)}(s)$ by using the prime closed geodesics on Y modulo the G-action. We can write these zeta functions in terms of zeta functions for the shift and the skew product ψ as in the previous section, with the lengths corresponding to a weight f defined by $f(e_0, e_1) = l(e_0)$. We then have the following result.

Theorem 3.2. *Let (X, l) be a finite connected metric graph and let (Y, l) be an infinite regular cover of (X, l) with amenable covering group G. Suppose that K_n is a Følner exhaustion of G and let $F \subset Y$ be a fundamental domain for the G-action. Let X_n denote the finite graph*

$$X_n = \bigcup_{g \in K_n} F \cdot g.$$

Then

$$\zeta_{(Y,l)}(s) = \lim_{n \to \infty} \zeta_{(X_n,i)}(s)^{1/(\#K_n)},$$

uniformly on compact subsets of $\{s \in \mathbb{C} : \|\mathbb{A}_s\| < 1/2\}$.

4. Traces

Let Tr and det denote the usual trace and determinant of a finite matrix. It is easy to see that

$$\#\{x \in \Sigma : \sigma^n x = x\} = \mathrm{Tr}(A^n)$$

and hence we have the standard results that

$$\begin{aligned}\log \zeta_\sigma(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}(A^n) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\lambda \in \mathrm{spec}(A)} (\lambda z)^n \\ &= - \sum_{\lambda \in \mathrm{spec}(A)} \log(1 - z\lambda)\end{aligned}$$

and

$$\zeta_\sigma(z) = \frac{1}{\det(I - zA)}.$$

Similar formulae hold for $\zeta_{\sigma_K}(z)$ for any finite subset $K \subset G$.

Consider the Hilbert space

$$H = \ell^2(\mathcal{S} \times G) = \left\{ u : \mathcal{S} \times G \to \mathbb{C} : \sum_{(i,g) \in \mathcal{S} \times G} |u(i,g)|^2 < \infty \right\},$$

with the inner product

$$\langle u, v \rangle = \sum_{(i,g) \in \mathcal{S} \times G} u(i,g) \overline{v(i,g)},$$

and its space of bounded linear operators $B(H)$. We will write $\delta_{(i,g)}$ for the element of H which is equal to one at (i, g) and zero elsewhere.

We need to extend the notion of trace to this setting. Let $T \in B(H)$ be compact. As usual, $|T| = \sqrt{T^* T}$ and T is said to be trace class if

$$\mathrm{Tr}(|T|) := \sum_{(i,g) \in \mathcal{S} \times G} \langle |T| \delta_{(i,g)}, \delta_{(i,g)} \rangle$$

is finite. In this case, we can define the trace of T to be

$$\mathrm{Tr}(T) = \sum_{(i,g) \in \mathcal{S} \times G} \langle T \delta_{(i,g)}, \delta_{(i,g)} \rangle$$

and we have

$$|\mathrm{Tr}(T)| \le \mathrm{Tr}(|T|). \tag{4.1}$$

Furthermore, we have that the trace class operators form a two-sided ideal in $B(H)$ and that, for $S_1, S_2, T \in B(H)$ with T trace class, we have

$$|\mathrm{Tr}(S_1 T S_2)| \le \|S_1\|\|S_2\|\mathrm{Tr}(|T|). \tag{4.2}$$

There is a natural representation of $\lambda : G \to B(H)$ given by $\lambda(g)u(i,h) = u(i, g^{-1}h)$. Clearly,

$$\langle \lambda(g)u, \lambda(g)v\rangle = \langle u, v\rangle,$$

for all $g \in G$ and $u, v \in \ell^2(\mathcal{S} \times G)$. Let $\mathcal{A} = \{\lambda(g) : g \in G\}'$, i.e. the elements of $B(H)$ which commute with the representation.

Lemma 4.1. *For each $s \in \mathbb{C}$, $\mathbb{A}_s \in \mathcal{A}$ and $\|\mathbb{A}_s\| \le \sqrt{d_r d_c} e^{-\mathrm{Re}(s)\eta(s)}$, where*

$$\eta(s) = \begin{cases} \min\{f(i,j) : A(i,j) = 1\} & \text{if } \mathrm{Re}(s) \ge 0 \\ \max\{f(i,j) : A(i,j) = 1\} & \text{if } \mathrm{Re}(s) < 0. \end{cases}$$

Proof. For $v \in \ell^2(\mathcal{S} \times G)$,

$$\begin{aligned}
\|\mathbb{A}_s v\|_2^2 &= \sum_{(i,g)\in\mathcal{S}\times G} |\mathbb{A}_s v(i,g)|^2 = \sum_{(i,g)\in\mathcal{S}\times G} \left| \sum_{(j,h)\in\mathcal{S}\times G} e^{-sf(i,j)} \mathbb{A}((i,g),(j,h)) v(j,h)\right|^2 \\
&\le e^{-2\mathrm{Re}(s)\eta(s)} \sum_{(i,g)\in\mathcal{S}\times G} \left| \sum_{(j,h)\in\mathcal{S}\times G} \mathbb{A}((i,g),(j,h)) v(j,h)\right|^2 \\
&= e^{-2\mathrm{Re}(s)\eta(s)} \sum_{(i,g)\in\mathcal{S}\times G} \left| \sum_{j\in\mathcal{S}} A(i,j) v(j, g\psi(i,j))\right|^2 \\
&\le e^{-2\mathrm{Re}(s)\eta(s)} \sum_{(i,g)\in\mathcal{S}\times G} \left(\sum_{j\in\mathcal{S}} A(i,j)\right) \left(\sum_{j\in\mathcal{S}} A(i,j) |v(j, g\psi(i,j))|^2\right) \\
&\le d_r e^{-2\mathrm{Re}(s)\eta(s)} \sum_{i\in S}\sum_{j\in S}\sum_{g\in G} A(i,j)|v(j,g\psi(i,j))|^2 \\
&\le d_r d_c e^{-2\mathrm{Re}(s)\eta(s)} \|v\|_2^2,
\end{aligned}$$

where we have used that $A(i,j)^2 = A(i,j)$, which shows that $\mathbb{A}_s$ is bounded and gives the estimate on the norm.

Also

$$\begin{aligned}
(\mathbb{A}\lambda(g))v(i,h) &= \sum_{(j,h')\in\mathcal{S}\times G} \mathbb{A}((i,h),(j,h'))(\lambda(g)v)(j,h') \\
&= \sum_{(j,h')\in\mathcal{S}\times G} A((i,h),(j,h')) v(j, g^{-1}h') = (\lambda(g)\mathbb{A})v(i,h),
\end{aligned}$$

so $\mathbb{A}_s \in \mathcal{A}$. ∎

The algebra $\mathcal{A}$ admits a finite trace Tr_G defined by

$$\mathrm{Tr}_G(T) = \sum_{i\in S} \langle T\delta_{i,e}, \delta_{i,e}\rangle.$$

For any set $K \subset G$, write $\Pi(K) \in B(H)$ for the orthogonal projection onto the subspace $\ell^2(\mathcal{S} \times K)$ and $\Pi(K)^\perp = \Pi(G \setminus K)$ for the projection onto the orthogonal complement. We note the following simple lemma.

Lemma 4.2. *Let K be a finite subset of G. Then, for any $T \in \mathcal{A}$, we have*

$$\mathrm{Tr}(\Pi(K)T\Pi(K)) = \#K\,\mathrm{Tr}_G(T).$$

Proof. The result follows from a direct calculation. We have,

$$\begin{aligned}
\mathrm{Tr}(\Pi(K)T\Pi(K)) &= \sum_{(i,g)\in S\times G} \left\langle \Pi(K)T\Pi(K)\delta_{(i,g)}, \delta_{(i,g)}\right\rangle \\
&= \sum_{(i,g)\in S\times K} \left\langle \Pi(K)T\delta_{(i,g)}, \delta_{(i,g)}\right\rangle \\
&= \sum_{(i,g)\in S\times K} \left\langle \Pi(K)\left(\sum_{(i',g')\in S\times G} \left\langle T\delta_{(i',g')}, \delta_{(i,g)}\right\rangle \delta_{(i',g')}\right), \delta_{(i,g)}\right\rangle \\
&= \sum_{(i,g)\in S\times K} \left\langle \left(\sum_{(i',g')\in S\times K} \left\langle T\delta_{(i',g')}, \delta_{(i,g)}\right\rangle \delta_{(i',g')}\right), \delta_{(i,g)}\right\rangle \\
&= \sum_{(i,g)\in S\times K} \left\langle T\delta_{(i,g)}, \delta_{(i,g)}\right\rangle \\
&= \sum_{(i,g)\in S\times K} \left\langle T\lambda(g)\delta_{(i,e)}, \lambda(g)\delta_{(i,e)}\right\rangle \\
&= \sum_{(i,g)\in S\times K} \left\langle \lambda(g)T\delta_{(i,e)}, \lambda(g)\delta_{(i,e)}\right\rangle \\
&= \#K \sum_{i\in S} \left\langle T\delta_{(i,e)}, \delta_{(i,e)}\right\rangle = \#K\,\mathrm{Tr}_G(T),
\end{aligned}$$

where the penultimate line uses that $T \in \mathcal{A}$. □

5. Proof of Theorem 1.2

In this final section, we complete the proof of Theorem 1.2. Let K_n, $n \geq 1$, be a Følner exhaustion of G. By Lemma 4.2, we have

$$\log \zeta_{T_\psi,f}(z,s) = \sum_{k=1}^{\infty} \frac{z^k}{k} \mathrm{Tr}_G(\mathbb{A}_s^k) = \sum_{k=1}^{\infty} \frac{z^k}{k} \frac{\mathrm{Tr}(\Pi(K_n)\mathbb{A}_s^k\Pi(K_n))}{\#K_n},$$

for any $n \geq 1$. We observe that

$$\mathrm{Tr}(A_{K_n,s}^k) = \mathrm{Tr}(\Pi(K_n)\mathbb{A}_s\Pi(K_n))^k)$$

and so we want to estimate

$$\mathrm{Tr}(\Pi(K_n)\mathbb{A}_s^k\Pi(K_n)) - \mathrm{Tr}((\Pi(K_n)\mathbb{A}_s\Pi(K_n))^k).$$

To simplify formulae, write $\Pi_n = \Pi(K_n)$ and $\Pi_n^\perp = \Pi(G \setminus K_n)$. Then, for $k \geq 2$,

$$\begin{aligned}
\mathrm{Tr}(\Pi_n\mathbb{A}_s^k\Pi_n) &= \mathrm{Tr}\left(\Pi_n(\mathbb{A}_s(\Pi_n + \Pi_n^\perp))^k\Pi_n\right) \\
&= \mathrm{Tr}((\Pi_n\mathbb{A}_s\Pi_n)^k) + \sum_{\substack{\sigma\in\{\perp,1\}^{k-1} \\ \sigma\neq(1,1,\ldots,1)}} \mathrm{Tr}\left(\Pi_n \prod_{j=1}^{k-1}(\mathbb{A}_s\Pi_n^{\sigma_j})\mathbb{A}_s\Pi_n\right).
\end{aligned}$$

Consider the terms in the sum on the right-hand side and noting that, since $\sigma \neq (1,1,\ldots,1)$, the product

$$\Pi_n \prod_{j=1}^{k-1}(\mathbb{A}_s\Pi_n^{\sigma_j})\mathbb{A}_s\Pi_n$$

contains at least one term of the form $\Pi_n \mathbb{A}_s \Pi_n^\perp$, using (4.2). we have the estimate

$$\left| \operatorname{Tr} \left(\Pi_n \prod_{j=1}^{k-1} (\mathbb{A}_s \Pi_n^{\sigma_j}) \mathbb{A}_s \Pi_n \right) \right| \leq \|\mathbb{A}_s\|^{k-1} \operatorname{Tr}(|\Pi_m \mathbb{A}_s \Pi_n^\perp|).$$

Furthermore, from the definition of $\mathbb{A}_s$,

$$\Pi(K_n)\mathbb{A}_s\Pi(G \setminus K_n) = \Pi(K_n)\mathbb{A}_s\Pi(\Omega_n),$$

where

$$\Omega_n \subset \bigcup_{A(i,j)=1} K_n \triangle K_n \psi(i,j),$$

and so

$$\operatorname{Tr}(|\Pi_n \mathbb{A}_s \Pi_n^\perp|) = \operatorname{Tr}(|\Pi_n \mathbb{A}_s \Pi(\Omega_n)|) \leq \|\mathbb{A}_s\| \operatorname{Tr}(\Pi(\Omega_n)) = \|\mathbb{A}_s\| \#\Omega_n.$$

Combining these estimates, we have the following.

Lemma 5.1. *For any $k \geq 2$ and $n \geq 1$, we have*

$$|\operatorname{Tr}(\Pi(K_n)\mathbb{A}_s^k\Pi(K_n)) - \operatorname{Tr}((\Pi(K_n)\mathbb{A}_s\Pi(K_n))^k)| \leq (2^{k-1} - 1)\|\mathbb{A}_s\|^k \#\Omega_n.$$

We now complete the proof of Theorem 1.2. By Lemma 5.1, we see that

$$\begin{aligned}
&\left| \frac{1}{\#K_n} \log \zeta_{\sigma_{K_n}}(z) - \log \zeta_{T_\psi}(z) \right| \\
&= \left| \sum_{k=1}^\infty \frac{z^k}{k} \frac{\operatorname{Tr}((\Pi(K_n)\mathbb{A}_s\Pi(K_n))^k)}{\#K_n} - \sum_{k=1}^\infty \frac{z^k}{k} \frac{\operatorname{Tr}(\Pi(K_n)\mathbb{A}_s^k\Pi(K_n))}{\#K_n} \right| \\
&\leq \left(\sum_{k=1}^\infty \frac{2^{k-1} z^k \|\mathbb{A}_s\|^k}{k} \right) \frac{\#\Omega_n}{\#K_n}.
\end{aligned}$$

The series in k converges for

$$|z| < \frac{1}{2\|\mathbb{A}_s\|}$$

and $\lim_{n\to\infty} \#\Omega_n/\#K_n = 0$ by (FE3). Hence we have $\zeta_{\sigma_{K_n},f}(z,s)^{1/\#K_n}$ converges to $\zeta_{T_\psi,f}(z,s)$ uniformly on compact subsets of this disk. This completes the proof.

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