DESCRIPTIONS OF COMPACT SETS IN C

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Introduction

The study of the dynamics of complex polynomials requires the description of various sets in $\mathbf{C}$ or $\mathbf{C}^n$: for instance, the set of points having a certain dynamical property for a given polynomial $f: \mathbf{C} \to \mathbf{C}$, or the set of polynomials of degree d having a certain global property (a polynomial is represented by its coefficients, which define a point in $\mathbf{C}^{d+1}$).

What do we mean by "describing" a set? Personally, I am mainly interested in looking for a topological description, even though other aspects (measure theory, Hausdorff dimension) are of great interest. As a tool, I sometimes need the use of potential theoretic properties.

We are able to tell what we mean by a *topological description* of a set K when it has the following four properties:

(D0) K is a compact subset of $\mathbf{C}$;
(D1) K is full (i.e., $\mathbf{C} - K$ is connected);
(D2) K is connected;
(D3) K is locally connected.

For such a set K, we shall present here two different strategies for obtaining a topological model of K, i.e., for constructing, from some combinatorial data, a topological space homeomorphic to the given set.

The first one will describe K as a *pinched disc*, i.e., as a quotient of a closed disc by some equivalence relation: using Caratheodory's theorem we obtain a continuous map γ_K of $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ onto the boundary of K; this map defines an equivalence relation on $\mathbf{T}$ and this relation is extended in a natural way to the closed unit disc $\overline{\mathbf{D}}$.

The second strategy will describe K as a projective limit of a sequence of finite disked trees K_n, which are also subspaces of K; each K_n is obtained by adding some topological closed discs to a tree H_n, and H_n is obtained by adding to H_{n-1} some arcs, called *veins*, which lead to points having an external argument of the form $p/2^n$.

To a given polynomial $f: \mathbf{C} \to \mathbf{C}$ of degree $d \geq 2$, one can associate its *filled Julia set* $K(f)$: this is the set of points whose orbit under f is bounded (the actual Julia set $J(f)$ is the boundary of $K(f)$). The set $K(f)$ always has properties (D0) and (D1); for (D2) and (D3), it depends on the polynomial.

The set of degree d polynomials $a_d z^d + \cdots + a_0$ can be identified with $\mathbf{C}^{d+1}$, but any polynomial can be affinely conjugated to a monic centered polynomial (i.e., one with $a_d = 1$, $a_{d-1} = 0$), so we can restrict our study to these, and their set can be identified with $\mathbf{C}^{d-1}$.

In the parameter space $\mathbf{C}^{d-1}$, one can consider the *connectedness locus* $\mathcal{C}_d$: this is the set of polynomials f such that $K(f)$ is connected, or, equivalently, such that each critical point of f has a bounded orbit under f.

The quadratic connectedness locus $M = \mathcal{C}_2$ is known as the *Mandelbrot set* (though it was first introduced by Brooks and Matelski). It satisfies properties (D0), (D1), (D2), and it is conjectured that it satisfies (D3): that is the (MLC) conjecture.

For $d > 2$, the space $\mathcal{C}_d$ is a subset of $\mathbf{C}^{d-1}$, so it does not satisfies (D0), and if we want to get a description along these lines, we have to restrict to subfamilies. Milnor has observed that $\mathcal{C}_d$ is not locally connected for $d > 2$.

One can try to apply the above mentioned strategies to M, assuming (MLC). This boils down to establishing certain combinatorial statements. Actually these can be stated and proved without the use of (MLC) ([DH]) and it is possible to construct an abstract model M_{abs} of M, together with a natural map $\chi: M \to M_{\text{abs}}$ ([Th]). In order to do this we have to make use of specific properties of M listed in section IV. The space M_{abs} can be obtained either as a pinched disc or as a projective limit of disked trees. It is compact, connected and locally connected. The map χ is continuous and surjective, and (MLC) is equivalent to its injectivity. The *generic hyperbolicity conjecture* (that hyperbolic quadratic polynomials form a dense open set) is equivalent to the weaker statement that the pre-image in M of any point in M_{abs} is a set with empty interior.

Important progress has been achieved recently towards (MLC), mainly by Jean-Christophe Yoccoz.

He first proved some years ago that a point which is on the boundary of a hyperbolic component of M is alone in its fiber for the map $\chi: M \to M_{\text{abs}}$. For each $c_0 \in M$ such that 0 is periodic under f_{c_0}, there is a topological copy $c_0 \perp M$ of M in M (in which c_0 corresponds to 0). We call it the *tuning copy* centered at c_0. A point $c \in M$ is said to be *tuned* or *renormalizable* if it belongs to a tuning copy of M in M other than M itself. Recently, J.-C. Yoccoz has proved that any point $c \in M$ which is not tuned is alone in its fiber. The method consists in constructing a sequence of disjoint annuli around c, and proving that the sum of their moduli is infinite (a procedure inaugurated in [BH], whose context we shall describe). Yoccoz' proofs are unpublished, but the reader will find a detailed sketch in Hubbard's paper in these Proceedings. The proof extends to all points which are not infinitely tuned, i.e., which are not in the intersection of a strictly decreasing sequence of tuning copies of M in M (there is also a sketch of this extension in Hubbard's paper). So that now, in order to prove (MLC), it would suffice to prove that the intersection of such a sequence is reduced to a point.

In the setting of the second strategy, the following question arises naturally: without knowing (MLC), can we show that the disked trees which appear in the description of M_{abs} can be imbedded in M? In essence: can we realize the veins of M_{abs} as arcs in M?

In [BD], we proved this for the vein leading to the point $\gamma_M(1/4)$, using "holomorphic surgery". The proof has a combinatorial part and an analytical part. It seemed that most of the difficulty was in the analytical part. However we have not been able to adapt the combinatorial part so as to reach any point of the form $\gamma_M(p/2^n)$. Very recently Jeremy Kahn, a student of McMullen, has proposed a sketch of a proof which should lead to this result by a different route: he uses the method of Yoccoz.

I. The pinched disc model

I.1 External arguments. Let K be a space satisfying (D0), (D1), (D2), i.e., a compact set in $\mathbf{C}$, connected and full. By the Riemann mapping theorem applied to $\overline{\mathbf{C}} - K$, one can find in a unique way a radius $r_K \geq 0$ and a C-analytic isomorphism $\varphi_K : \mathbf{C} - K \to \mathbf{C} - \overline{\mathbf{D}}(r_K)$, such that $\varphi_K(z)/z \to 1$ as $z \to \infty$. The function $G_K = \operatorname{Log}|\varphi_K|$ is the *potential* of K: it is harmonic on $\mathbf{C} - K$, and can be extended into a continuous function on $\mathbf{C}$ by making it constant with value $\operatorname{Log}(r_K)$ on K.

The level lines of G_K are called the *equipotentials*, and we call the lines orthogonal to the equipotentials the *external rays*. For $t \in \mathbf{T}$, the external ray $\mathcal{R}(K, t)$ of argument t is $\varphi_K^{-1}(\{r \cdot \mathrm{e}(t)\}_{r_K<r<\infty})$. (We denote by e the map $t \mapsto e^{2\pi i t}$ from $\mathbf{T}$ to $\mathbf{C}$, and we use the convention that angles are measured taking the whole turn, not the radian, as a unit).

We suppose now that K satisfies also (D3), i.e., that it is moreover locally connected. Then by a theorem of Caratheodory the map $\psi_K = (\varphi_K)^{-1}$ admits a continuous extension $\mathbf{C} - \mathbf{D}(r_K) \to \mathbf{C} - \mathring{K}$, still denoted by ψ_K.

This induces a continuous map $\gamma_K : t \mapsto \psi_K(r_K \cdot \mathbf{e}(t))$ of $\mathbf{T}$ onto the boundary ∂K of K. We call this map the *Caratheodory loop* of K. For $z \in \partial K$, we call the points of $\gamma_K^{-1}(z)$ in $\mathbf{T}$ the *external arguments* of z.

We define the equivalence $\sim_K$ on $\mathbf{T}$ by $t \sim_K t' \iff \gamma_K(t) = \gamma_K(t')$. This equivalence relation is closed and unlinked with totally disconnected classes, i.e., it satisfies:

(E1) the graph is a closed set in $\mathbf{T} \times \mathbf{T}$;
(E2) if $t_1 \sim t_2 \not\sim t_3 \sim t_4$, then the cross ratio $\beta(\mathbf{e}(t_1), \mathbf{e}(t_2); \mathbf{e}(t_3), \mathbf{e}(t_4))$ is positive;
(E3) each class is totally disconnected.

I.2 Pinched discs. If $\sim$ is an equivalence relation on $\mathbf{T}$ closed and unlinked (i.e., satisfying (E1) and (E2)), we define its natural extension $\simeq$ to $\overline{\mathbf{D}}$ in the following way: a class of $\simeq$ is either the convex hull (for the Poincaré metric) of a class of $\sim$ (transported to $S^1 \subset \mathbf{C}$ by $\mathbf{e}$), or a point of $\overline{\mathbf{D}} - L$, where L is the union of the convex hulls of classes of $\sim$.

We extend $\sim$ to an equivalence relation $\simeq$ on $\mathbf{C}$ or $\overline{\mathbf{C}}$ by declaring that a point in $\overline{\mathbf{C}} - \overline{\mathbf{D}}$ is equivalent only to itself.

The quotient space $\overline{\mathbf{D}}/\simeq$ is the *pinched disc* defined by $\sim$. It is imbedded in the *pinched plane* $\mathbf{C}/\simeq$; we call the pair $(\mathbf{C}/\simeq, \overline{\mathbf{D}}/\simeq)$ a *pinched pair.*

Remarks: (1) In this construction, it is not important to take convex hulls for the Poincaré metric: the space we would obtain taking convex hulls for the euclidean metric would be homeomorphic.

(2) According to a theorem of Moore, the pinched plane $\mathbf{C}/\simeq$ is homeomorphic to $\mathbf{R}^2$. But we shall not make use of this fact.

(3) The condition (E2) implies that there are only countably many classes of $\sim$ with more than 2 points. Indeed such a class defines a class of $\simeq$ with non-empty interior, and these classes are disjoint.

Let $X = \overline{\mathbf{D}}/\simeq$ be a pinched disc; denote by χ_X the natural map $\overline{\mathbf{D}} \to X$, and by L the union of the convex hulls of the classes of $\sim$ (Fig. I.1). Then $\chi_X(\partial \mathbf{D}) = \chi_X(L)$ is the boundary ∂X of X in $\mathbf{C}/\simeq$, and χ_X induces a homeomorphism of $\overline{\mathbf{D}} - L$ onto the interior $\mathring{X}$ of X.

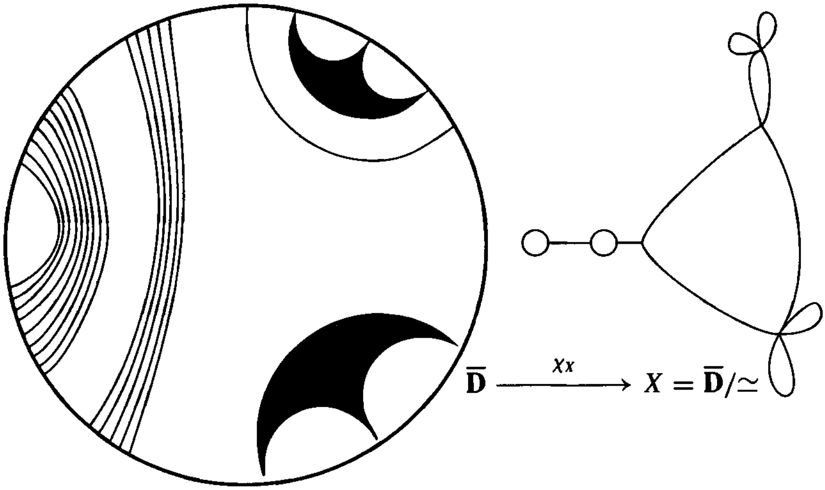


Figure I.1. A pinched disk.

If K is a compact set in $\mathbf{C}$ satisfying (D0) to (D3), we call $\overline{\mathbf{D}}/\simeq_K$ the *pinched disc model* of K, and $(\mathbf{C}/\simeq_K, \overline{\mathbf{D}}/\simeq_K)$ the *pinched pair model* of $(\mathbf{C}, K)$ (Fig. I.2).

Under these hypotheses, the space K is homeomorphic to its pinched disc model. Even better:

THEOREM 1. If K satisfies (D0) to (D3), there is a homeomorphism of the pair $(\mathbf{C}, K)$ onto its pinched pair model which induces φ_K on $\mathbf{C} - K$.

This result is more or less classical. The proof involves the following steps:

PROPOSITION 1. Let U be a connected component of $\mathring{K}$. Then its closure $\overline{U}$ is a topological disc.

This makes use of the following lemma:

LEMMA. The quotient of $\overline{\mathbf{D}}$ by a closed equivalence relation whose classes are points in $\overline{\mathbf{D}}$ or closed arcs in S^1 is homeomorphic to $\overline{\mathbf{D}}$.

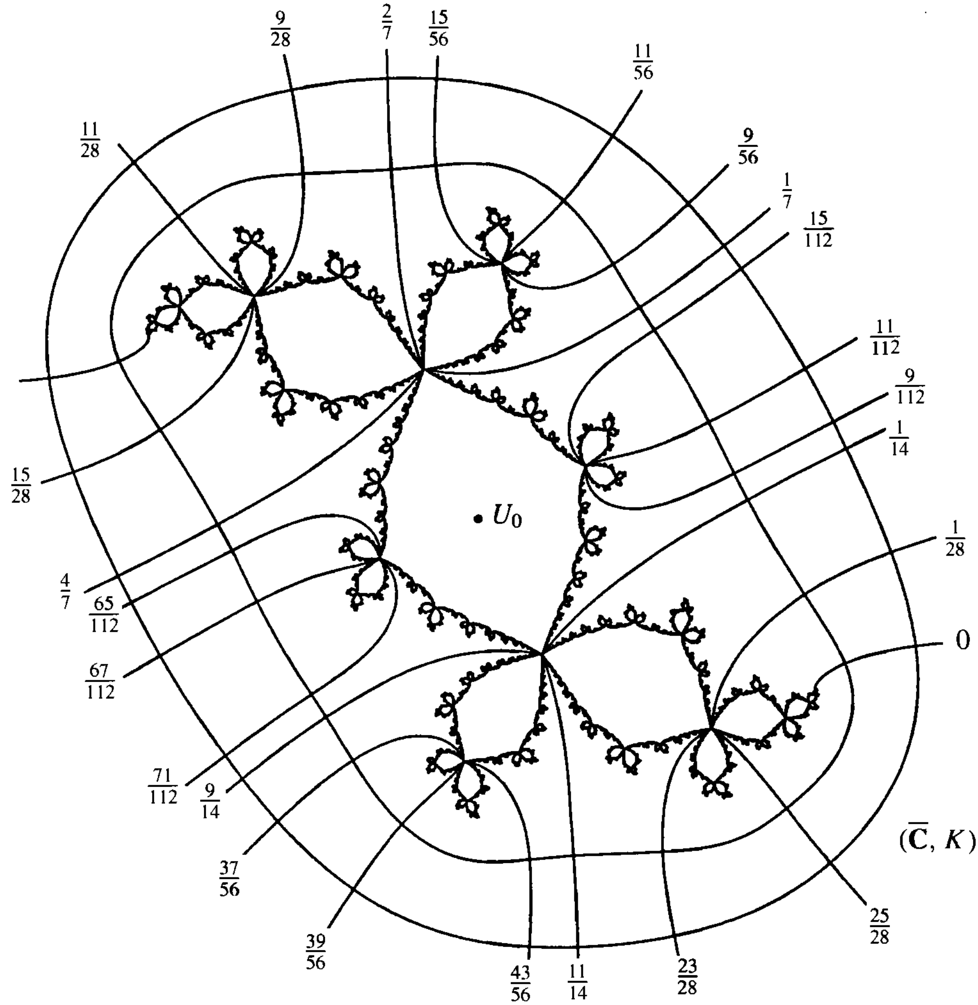


Figure I.2a. The "Rabbit": filled Julia set of a quadratic polynomial $f_c\colon z \mapsto z^2 + c$ such that 0 is periodic of period 3 and $\operatorname{Im}(c) > 0$.

PROPOSITION 2. Let us equip the pinched disc X with any metric compatible with its topology. Then, for any $\epsilon > 0$, there are only finitely many connected components U_i of $\overset{\circ}{X}$ such that $\operatorname{diam}(U_i) > \epsilon$.

I.3 Remarks and questions. (1) It is not true that any equivalence relation on $\mathbf{T}$ satisfying (E1) to (E3) is of the form $\sim_K$ with K satisfying (D0) to (D3):

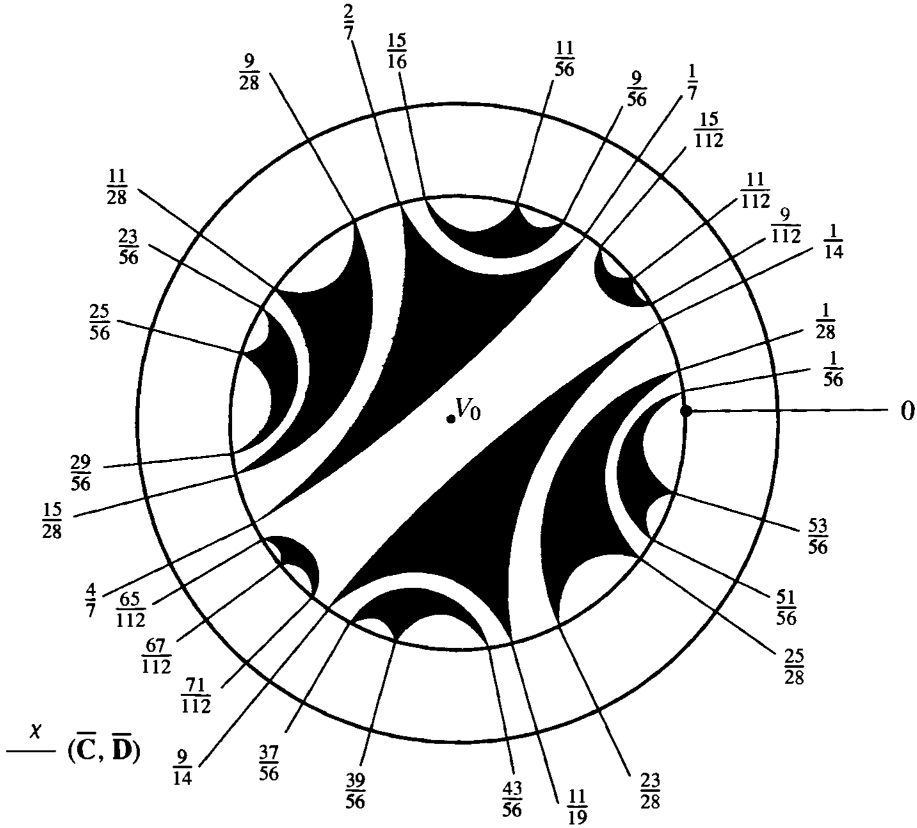


Figure I.2b. The equivalence relation defined by the Rabbit.

Example. Consider in $\mathbf{C} = \mathbf{R}^2$ the quadruple comb (Fig. I.3) $\Xi = \big([-1, +1] \times \{0\}\big) \cup \big(A \times [-1, +1]\big)$, where A contains $1/n$ and $-1/n$ for $n \in \mathbf{N}$, and 0. This set is compact, full and connected, but not locally connected. However every external ray lands, so it defines a map γ_Ξ (which is not continuous) and an equivalence relation $\sim_\Xi$ on $\mathbf{T}$. This equivalence relation is not closed: the external rays of argument $1/4$ and $3/4$ land at different points. One can show that its closure (obtained by adding $1/4 \sim 3/4$) cannot be realized by a compact set satisfying (D0) to (D3).

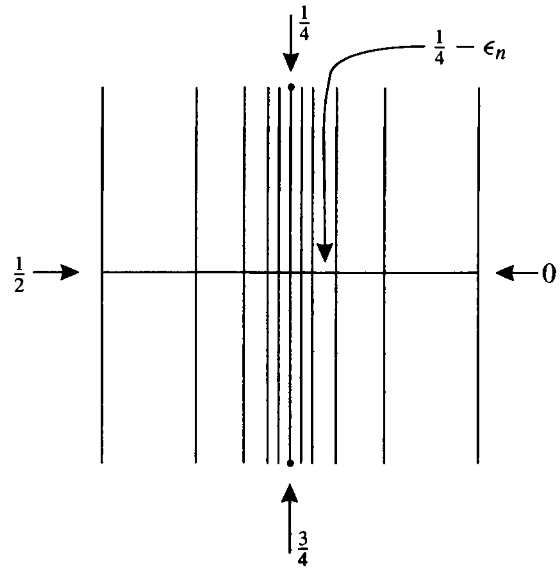


Figure I.3. The quadruple comb Ξ. Here we have $1/4 \not\sim_\Xi 3/4$, but $1/4 - \epsilon_n \sim_\Xi 3/4 + \epsilon_n$ for some sequence (ϵ_n) tending to 0.

I know no characterization of the equivalence relations which are realizable by such a compact set.

(2) A compact set K in $\mathbf{C}$ is called *holomorphically removable* iff any homeomorphism $\mathbf{C} \to \mathbf{C}$ which is holomorphic on $\mathbf{C} - K$ is holomorphic on $\mathbf{C}$ (and thus affine).

Suppose that K satisfies (D0) to (D3). Then K is holomorphically removable iff any compact set K' satisfying (D0) to (D3) and $\sim_{K'} = \sim_K$ is of the form $A(K)$, where A is an affine map with real positive coefficient.

It follows from the Measurable Riemann Mapping Theorem of Morrey-Ahlfors-Bers that, if K has positive measure, then K is not removable. This holds in particular if K has a non-empty interior.

Question: Take a compact set K satisfying (D0) to (D3), with positive measure. Then the set of quasi-conformal imbeddings of K in $\mathbf{C}$ compatible with a fixed Caratheodory loop can be identified with the open unit ball in $L^\infty(K; \mathbf{C})$ (this makes use of a lemma by Rickman). Is it true that, for a non-holomorphically removable set K satisfying (D0) to (D3), the set of all topological imbeddings of K in $\mathbf{C}$ compatible with a fixed Caratheodory loop is always connected? Contractible?

(3) *Question*: For any equivalence relation $\sim$ on $\mathbf{T}$ satisfying (E1) to (E3), can one find a homeomorphism $g: \mathbf{T} \to \mathbf{T}$ transforming $\sim$ into an equivalence of the form $\sim_K$?

II. The vein description

The pinched disc model of a compact K satisfying (D0) to (D3) gives a description of it "from outside". We shall now explore it "from inside", drawing many arcs in it. The new description starts from the pinched disc description. It can actually be achieved in two settings: we can do it for a compact set K in $\mathbf{C}$ satisfying (D0) to (D3), or slightly more generally for a pinched disc $X = \overline{\mathbf{D}}/\simeq$, not necessarily coming from an actual compact set.

We shall work in this more general setting, because we have in view the "abstract Mandelbrot set", which is defined only as a pinched disc so long as we don't know that M satisfies (D3). We denote by γ_X the map $\chi_X \circ \mathbf{e}: \mathbf{T} \to \partial X$, and we say that t is an external argument of x if $\gamma_X(t) = x$. We set $\partial X = \gamma_X(\mathbf{T})$ and $\overset{\circ}{X} = X - \partial X$ (they are the boundary and the interior of X in $\mathbf{C}/\simeq$).

II.1 Legal arcs. Let $X = \overline{\mathbf{D}}/\simeq$ be a pinched disc, denote by χ the natural map $\overline{\mathbf{D}} \to X$ and by γ_X the loop $t \mapsto \chi(\mathbf{e}(t))$. The space X is compact, metrizable, connected and locally connected, and therefore arcwise connected. It is also simply connected. If $\overset{\circ}{X} = \emptyset$, given two points x_0, x_1 in X, there is a unique arc $[x_0, x_1]_X$ in X with extremities x_0, x_1 (we use the convention that a single point is an arc). If $\overset{\circ}{X} \neq \emptyset$, there may be several; we shall give a rule to select one.

We equip X with the following structure: for each connected component U of $\overset{\circ}{X}$, we chose a homeomorphism $\varphi_U: \overline{U} \to \overline{\mathbf{D}}$. The point $\varphi_U^{-1}(0)$ is the *center* of U. A *ray* of U is the inverse image by φ_U of a ray of D, i.e., of a segment $[0, \mathbf{e}(t)]$.

Definition. An arc η in X is called a *legal arc* iff, for any connected component U of $\overset{\circ}{X}$, the set $\eta \cap \overline{U}$ is contained in the union of two rays of U.

Given two points x_0, x_1 in X, there is a unique legal arc in X having x_0, x_1 as its extremities; we denote it by $[x_0, x_1]_X$.

Remark: In the case of a compact set K in $\mathbf{C}$, it would be more natural to cross a connected component U of K along the Poincaré geodesic, but then we would not get trees, and trees are essential in the sequel.

Given n points $x_1, \ldots x_n$ in X, the union of the arcs $[x_i, x_j]_X$ is a finite topological tree that we denote by $[x_1, \ldots, x_n]_X$ and that we call the *legal hull* of $\{x_1, \ldots, x_n\}$. A set obtained this way is a *legal tree.* If $Y \subset X$ is a legal tree and $x \in X$, there is a unique $y \in Y$ such that $[x, y]_X \cap Y = \{y\}$. We call y the *legal projection* of x onto Y and denote it by $\pi_Y(x)$.

The *root point* of a component U of $\mathring{X}$ with center x is the unique point y in $\partial U \cap [\gamma_X(0), x]_X$. It is often convenient to adjust φ_U so that $\varphi_U^{-1}(1)$ is the root point of U.

II.2 Veins. Let us set $A_n = \{p/2^n\}_{p=1,\ldots,2^n}$. We define the n^{th} *approximating tree* T_n of X as the legal hull in X of $\gamma_X(A_n)$.

The tree T_n is the union of T_{n-1} with the legal arcs $N_\tau = [\alpha_\tau, \gamma_X(\tau)]_X$ with $\tau = p/2^n$, p odd, where $\alpha_\tau = \pi_{T_{n-1}}(\gamma_X(\tau))$. The arc N_τ is called the *vein* of X of argument τ; its *origin* is α_τ, and $\gamma_X(\tau)$ is its *extremity.*

Remark: The above choice of A_n is adapted to compact sets arising in the study of quadratic polynomials. For polynomials of degree d, it would be more convenient to take $A_n = \{p/d^n\}$. Then it would no longer be true that there is only one point of A_n between two consecutive points of A_{n-1}. As a consequence, two different veins might have more than one point in common.

In the sequel, we shall stick to the choice $A_n = \{p/2^n\}$.

Let x be a point in ∂X. We can detect if x is on a given vein N_τ in the following way:

If I is an arc in $\mathbf{T}$, or an arc in $\mathbf{R}$ of length < 1, define the *leading point* of I as the dyadic point $p/2^k$ in I with the smallest possible k.

Then $x \in \mathbf{N}_\tau$ iff there are two external arguments t, t' of x in $[0, 1]$ such that τ is the leading point of $[t, t']$.

Suppose now that x is the center of a connected component U of $\mathring{X}$. Set $V = \chi_X^{-1}(U)$. We say that t is an external argument *associated* to x (or to U) iff $\mathbf{e}(t) \in \partial V$. This is equivalent to the condition that $y = \gamma_X(t) \in \partial U$ and t is adjacent to U in $\gamma_X^{-1}(y)$, i.e., t is one of the boundary points of the connected component of $\mathbf{T} - \gamma_X^{-1}(y')$ which contains $\gamma_X^{-1}(y')$ for $y' \in \partial U$, $y' \neq y$. The *root arguments* of U are the two arguments of the root point of U associated to U.

Example. In Fig. I.2, there is a Cantor set C of arguments associated to the central component U_0. The end points of a component of $\mathbf{T} - C$ are equivalent. The arguments $\frac{9}{112}$ and $\frac{15}{112}$ are associated to U_0; $\frac{11}{112}$ is not. The root arguments of U_0 are $\frac{1}{14}$ and $\frac{9}{14}$.

With these definitions, $x \in N_\tau$ iff there are two external arguments $t, t' \in [0, 1]$ associated to x, such that τ is the leading point of $[t, t']$ (the closed interval $[t, t']$ should be replaced by $]t, t']$, $[t, t'[$ or $]t, t'[$ if t, t' or both is a root argument).

In both cases, we can also detect if x is the origin of the vein N_τ: The point x is the origin of N_τ iff $x \in N_\tau$ and $x \in N_{\tau'}$ for some dyadic angle $\tau' \neq \tau$ of smaller order.

Every point having ≥ 3 external arguments, and every center of a component, is the origin of some vein (cf. Remark in II.3 below).

II.3 The approximating disked trees. The set $\pi_0(\overset{\circ}{X})$ of connected components of $\overset{\circ}{X}$ is countable (i.e., finite or infinite countable). Let (B_n) be an increasing sequence of finite subsets of $\pi_0(\overset{\circ}{X})$ having the following properties:

(I.1) for $U \in B_n$, $U \cap T_n \neq \emptyset$;
(I.2) a component of $\overset{\circ}{X}$ containing a branch point of T_n is in B_n.

For instance one can take for B_n the set of components of $\overset{\circ}{X}$ which contain a branch point of T_n.

Remark: Condition (I.2) implies that $\bigcup B_n = \pi_0(\overset{\circ}{X})$. Indeed, let U be a component of $\overset{\circ}{X}$, take six points $a_1, \ldots, a_6$ on ∂U, let $t_1, \ldots, t_6$ be external arguments for them respectively, let τ_1, τ_2, τ_3 be three dyadic numbers such that $t_1, \tau_1, t_2, t_3, \tau_2, t_4, t_5, \tau_3, t_6$ are in this cyclic order on $\mathbf{T}$, and let 2^n be a common denominator for τ_1, τ_2, τ_3. Then the center of U is a branch point of T_n and $U \in B_n$.

We now define the n^{th} approximating disked tree X_n by

$$X_n = T_n \cup \bigcup_{U \in B_n} \overline{U}.$$

One can check that it is actually a disked tree. The union of these disked trees is dense in X. Indeed this union contains $\overset{\circ}{X}$ because of the above remark, and it contains $\gamma_X(A_\infty)$ which is dense in ∂X.

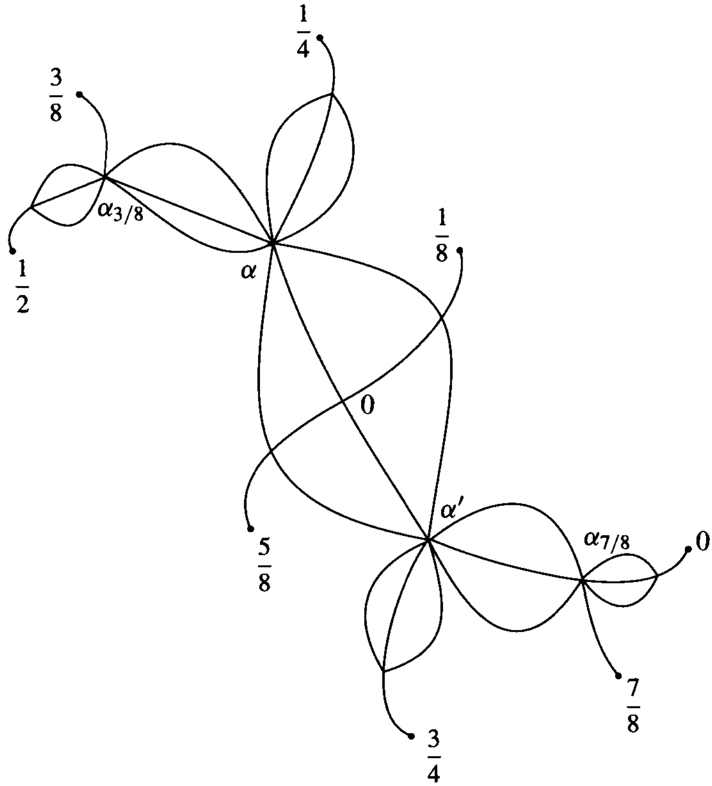


Figure II.1. The disked tree $X_3(K)$ for the rabbit. $\alpha = \alpha_{1/4}$ is the origin of $N_{1/4}$; $\alpha' = \alpha_{3/4}$ is the origin of $N_{3/4}$; 0 is the origin of $N_{1/8}$ and $N_{5/8}$.

Each disked tree X_n is an object which can be described topologically with a finite amount of information. In case X comes from a compact set K in $\mathbf{C}$ satisfying (D0) to (D3), the disked trees X_n can be identified with subsets K_n of K, thus of $\mathbf{C}$, and each pair $(\mathbf{C}, K_n)$ is also, topologically, a combinatorial object admitting a finite description.

The space X can be viewed as the closure, or, if you prefer, as the completion of $\bigcup X_n$. That makes it also the limit of the sets X_n for the Gibbs-Hausdorff distance. But this is actually rather poor information concerning the structure of X: remember that any compact set in $\mathbf{C}$ is the limit of an increasing sequence of finite sets.

We shall see now that X can be viewed also as a projective limit of the sets X_n, and that carries much more information.

II.4 The projective limit theorem. For each n, there is a continuous retraction $\rho_n : X_n \to X_{n-1}$. For an appropriate choice of these retractions, we have:

THEOREM 2. The space X can be identified with the projective limit of the system

$$\cdots \to X_n \xrightarrow{\rho_n} X_{n-1} \to \cdots .$$

The proof makes use of the following lemma. For $x \in X$, define the projection $x_n = \pi_{X_n}(x)$ by: $[X_n \cup \{x\}]_X = X_n \cup [x_n, x]_X$ and $X_n \cap [x_n, x]_X = \{x_n\}$.

LEMMA. For any choice of x in X, the sequence (x_n) converges to x.

Remarks: (1) Let $(Y_n, f_n : Y_n \to Y_{n-1})$ be a projective system of topological spaces, and let Y_∞ be the projective limit. If each Y_n is compact, connected and locally connected, and if moreover the maps f_n have connected fibers, then Y_∞ is compact, connected and locally connected.

(2) The above theorem is a big improvement over the description of X as a closure of a union of well-described sets, but still it does not provide a complete description of X. Let (Y_n, f_n) and (Z_n, g_n) be two projective systems of disked trees with continuous retractions. Suppose one can find for each n a homeomorphism $h_n : Y_n \to Z_n$ in such a way that $g_n \circ h_n$ is isotopic to $h_{n-1} \circ f_n$ for each n (if we want to restrict ourself to combinatorial information, we cannot require more). Does this imply that the projective limits Y_∞ and Z_∞ are homeomorphic?

The answer is: No!

For a counterexample, consider the standard Cantor set C in $I = [0, 1]$, and the compact set K which is the union of I with all the closed discs having as a diameter a connected component of $I - C$ (Fig. II.2). A point a in C divides K into two compact sets K_1, K_2. Take $l > 0$ and set $K' =$

Figure II.2.

$K_1 \cup [a, a+l] \cup \tau_l(K_2)$, where τ_l is the translation by l. Take for Y_n (resp., Z_n) the union in K (resp., K') of the interval with the discs of diameter $\geq 3^{-n}$.

III. Description of filled Julia sets

III.1 Critically finite polynomials. A polynomial $f: \mathbf{C} \to \mathbf{C}$ is said to be *critically finite* iff it satisfies the Thurston condition:

(CF) *Each critical point of* f *has a finite orbit under* f.

We consider here monic centered quadratic polynomials $f = f_c: z \to z^2 + c$ which are critically finite. There is one critical point $x_0 = 0$; let us set $x_n = f^n(x_0)$. There are two cases:

- 0 is periodic—its orbit has k points $x_0 = x_k, x_1, \ldots, x_{k-1}$;
- 0 is strictly preperiodic—its orbits has $l + k$ points $x_0, \ldots, x_{l-1}, x_l = x_{l+k}, \ldots, x_{l+k-1}$ (with $l \geq 2$ since x_1 has only one pre-image).

In both cases $K_c = K(f_c)$ satisfies (D0) to (D3) . If 0 is strictly preperiodic, the interior of K_c is empty. If 0 is fixed, i.e., if $c = 0$, the set K_c is a closed disc. If 0 is periodic with period $k > 1$, the interior of K_c has infinitely many connected components, and $x_0, \ldots, x_{k-1}$ belong to distinct components $U_0, \ldots, U_{k-1}$. There is a unique homeomorphism $\varphi_0: \overline{U}_0 \to \overline{\mathbf{D}}$ conjugating f^k to $z \mapsto z^2$. For each connected component U of the interior of K_c, there is a unique n such that f^n induces a homeomorphism $\overline{U} \to \overline{U}_0$. We choose $\varphi_U = \varphi_0 \circ f^n$ in the definition of legal arcs (section II.1). Then the image by f of a legal arc avoiding 0 or having 0 as an extremity is again a legal arc.

III.2 Hubbard trees and variants. In order to obtain the vein description of K_c, we need a finite amount of information. This information is concentrated in the *Hubbard tree.* We first define the Hubbard tree and its variants. We use the notation $(x_0, \ldots, x_l, \ldots, x_{l+k-1})$ in both cases, with $l = 0$ in the periodic case.

The *Hubbard tree* H_c is the legal hull of the orbit of the critical point 0:

$$H_c = [x_0, \ldots, x_{l+k-1}]_{K_c}$$

The *extended Hubbard tree* is

$$\hat{H}_c = H_c \cup [\beta, \beta']_{K_c} = [\beta, \beta', x_0, \ldots, x_{l+k-1}]_{K_c}$$

where $\beta = \gamma_{K_c}(0)$, $\beta' = \gamma_{K_c}(1/2)$.

In the case 0 is periodic, we define also the *disked Hubbard tree*

$$A_c = H_c \cup \overline{U}_0 \cup \cdots \cup \overline{U}_{k-1}$$

and the *extended disked Hubbard tree* $\hat{A}_c = A_c \cup [\beta, \beta']_{K_c}$. If 0 is strictly preperiodic, i.e., if $l > 0$, then $\overset{\circ}{K}_c = \emptyset$ and we set $A_c = H_c$, $\hat{A}_c = \hat{H}_c$. The sets H_c, $\hat{H}_c$, A_c, $\hat{A}_c$ are forward invariant under f_c.

Denote by $\nu(i)$ (resp., $\hat{\nu}(i)$) the number of branches of H_c (resp., of $\hat{H}_c$) at x_i. One has $\nu(i) \leq \nu(i+1)$ for $i > 0$, and $\nu(0) \leq 2 \cdot \nu(1)$. If H_c is not reduced to x_0, the tree H_c has at least two extremal points, which are of the form x_i, and necessarily $\nu(1) = 1$. So $\nu(0) = 1$ or 2, and the point x_0 cuts H_c into two parts H' and H'' (possibly reduced to x_0).

III.3 Abstract Hubbard trees. The tree H_c, with the structure we want to consider, is an object H which satisfies:

(H1) the space H is a topological finite tree equipped with an embedding class in $\mathbf{C} = \mathbf{R}^2$ (or equivalently a cyclic order on branches at branch points) and with distinct marked points $x_0, \ldots, x_{l+k-1}$;

(H2) each extremity of H is a marked point (branch points are not necessarily marked);

(H3) there are at most 2 branches at x_0, so that $H = H' \cup H''$ with H' and H'' connected, $H' \cap H'' = \{x_0\}$;

(H4) there exists a continuous map $f\colon H \to H$, such that:

- (i) $f(x_i) = x_{i+1}$, $f(x_{k+l-1}) = x_l$;
- (ii) $f|_{H'}$ and $f|_{H''}$ are injective;
- (iii) f preserves the cyclic order on branches at branch points.

Indeed one can take $f = f_c$ (I don't want to consider f as part of the structure, only its existence as an axiom). For the periodic case, the map $c \mapsto H_c$ defines a bijection between values of c such that 0 is periodic under f_c and isomorphy classes of objects satisfying (H1) to (H4) with $l = 0$. Considering f_c only as a topological object, one gets also a bijection between those values of c and Thurston classes of maps $\mathbf{R}^2 \to \mathbf{R}^2$ which are degree 2 ramified coverings with periodic critical point (Thurston's equivalence relation is generated by conjugacy under orientation preserving homeomorphisms and isotopy relative to postcritical points) ([Th], [SL]).

For the strictly preperiodic case, the corresponding map is injective. The classes of trees which can be realized are those for which one can choose f expanding for some metric on H.

So, when we want to speak specifically of some critically finite polynomials, it is often better to label them by their Hubbard trees than by the values of c.

III.4 The vein description of K_c. In this section we explain how we can reconstruct the disked trees approximating the Julia set K_c, as combinatorial objects, when we know the Hubbard tree H_c with the structure described above. See Fig. III.1.

When we know H_c, it is easy to construct $\hat{H}_c$. Indeed β (resp., β') is attached to the last x_i which is an extremity of the tree H_c (resp., to the one before last). It is also easy to construct A_c and $\hat{A}_c$ in the periodic case: we just have to add for each $i = 0, \ldots, k-1$ a closed disc Δ_i centered at x_i. The only question is: how far to extend them? The point x_1 is an extremity; let a be the branch point or marked point closest to x in H_c. If a is a periodic branch point of period p with ν branches, and $p \cdot \nu = k$, then the disc Δ_1 should be extended all the way to a and Δ_i all the way to $f^{i-1}(a)$. If k is even, say $k = 2 \cdot k'$, and $a = x_{k'+1}$, then Δ_i and $\Delta_{k'+i}$ should touch each other for $i = 0, \ldots, k'-1$. In all other cases, one can take a small disc for Δ_i (so small that taking a smaller disc would not change the space obtained).

Once we have $\hat{A}_c$, it is easy to describe by induction on n the disked tree $\hat{A}_c(n) = f^{-n}(\hat{A}_c)$. Indeed $\hat{A}_c(n+1)$, together with the map $f_n^{n+1}\colon \hat{A}_c(n+1) \to \hat{A}_c(n)$ induced by f, is a 2-sheeted covering space ramified over x_1. The new x_0 is the ramification point, the new β and β' are the two inverse images of β, the new x_i is the inverse image of x_{i+1} which is on the side of β or β' according to the position of x_i in $\hat{A}_c(n)$. One defines the maps $f_{n+1}\colon \hat{A}_c(n+1) \to \hat{A}_c(n+1)$ and $\iota_n\colon \hat{A}_c(n) \to \hat{A}_c(n+1)$ so as to make the diagram

$$\begin{tikzcd}
\hat{A}_c(n+1) \arrow[r, "f_{n+1}"] \arrow[d, "f_n^{n+1}"'] & \hat{A}_c(n+1) \arrow[d, "f_n^{n+1}"] \\
\hat{A}_c(n) \arrow[r, "f_n"] \arrow[ur, "\iota_n"] & \hat{A}_c(n)
\end{tikzcd}$$

commutative.

So we get a combinatorial description of $\hat{A}_c(n)$ for each n, together with $\iota_n\colon \hat{A}_c(n) \to \hat{A}_c(n+1)$ up to isotopy. In order to get a topological description of K_c, we have to imbed the spaces $\hat{A}_c(n)$ in $\mathbf{C}$ in an appropriate way, and take their projective limit (cf. Remark 2 in section II.4).

This description is adapted to the dynamic. It does not coincide with the vein description defined in section II (in a "static" way). But the latter can be extracted from it: it may be given by

$$K_c(n) = [\{\gamma(p/2^n)\}]_{K_c} \cup \bigcup_{i \le n} f^{-i}(\overline{U}_0)$$

Then $K_c(n)$ is a sub-disked-tree of $\hat{A}_c(n)$.

III.5 The pinched disc description of K_c. In order to describe the equivalence relation $\sim_c = \sim_{K_c}$, we need the following data:

- if 0 is strictly preperiodic: an external argument of $c = x_1$;
- if 0 is periodic: one of the two root arguments of the connected component U_1 of $\overset{\circ}{K}_c$ containing $c = x_1$

This information can be given when specifying the polynomial. It can also be easily obtained when we know $\hat{A}_c$ by the following recipe:

Imbed $\hat{A}_c$ in the plane $\mathbf{C} = \mathbf{R}^2$, so that the *spine* $[\beta, \beta']_{\hat{A}_c}$ is a horizontal segment, with β on the right. Then, if x is a remarkable point of $\hat{A}_c$ (a marked point, or a branch point, or a point of $\partial U \cap \hat{A}_c$, where U is a connected component of the interior), choose an access ξ to x. Then one can determine for each n the point $f^n(x)$ and the access $f^n(\xi)$. There is one external ray of K_c which lands at x in the access ξ, and its external argument θ has expansion in base 2:

$$\theta = .\epsilon_1\epsilon_2 \ldots \epsilon_n \ldots$$

with $\epsilon_n = 0$ if $f^{n-1}(\xi)$ is above $[\beta', \beta]$, and 1 if it is below (Fig. III.3).

Knowing the required datum, the equivalence relation $\sim_c$ is determined in the following way: Let $\chi : \mathbf{R} \to \mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the canonical map. We first define a partition $\mathcal{J} = (J_0, J_1)$ of $\mathbf{T}$. If 0 is strictly preperiodic and θ is an external argument of c, let θ^* be the representative of θ in $[0, 1[$, and set

$$J_1 = \chi\left(\left[\frac{\theta^*}{2}, \frac{\theta^*+1}{2}\right[\right), \quad J_0 = \chi\left(\left[\frac{\theta^*+1}{2}, \frac{\theta^*+2}{2}\right[\right)$$

or

$$J_1 = \chi\left(\left]\frac{\theta^*}{2}, \frac{\theta^*+1}{2}\right]\right), \quad J_0 = \chi\left(\left]\frac{\theta^*+1}{2}, \frac{\theta^*+2}{2}\right]\right).$$

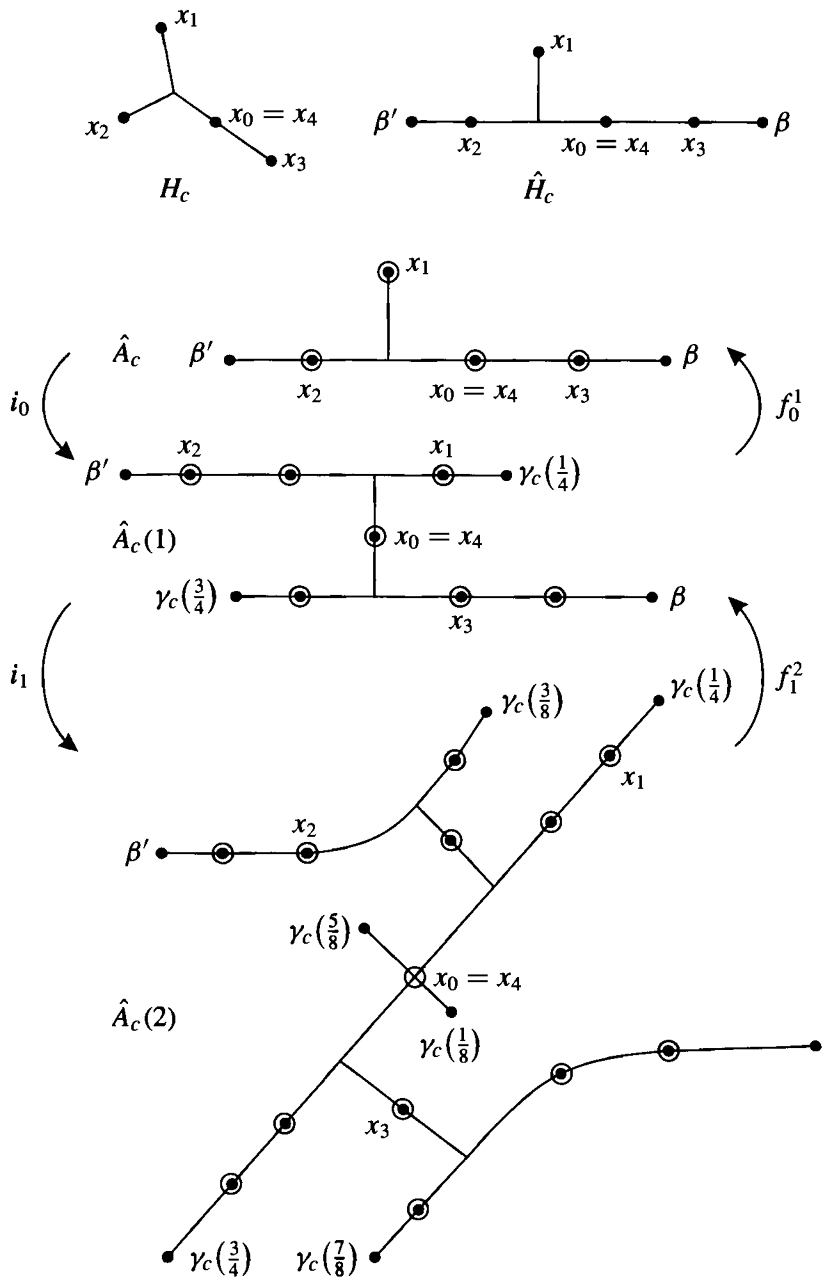


Figure III.1. Example of the construction described in III.4 (in this case $\hat{A}_c(n) = K_c(n+1)$).

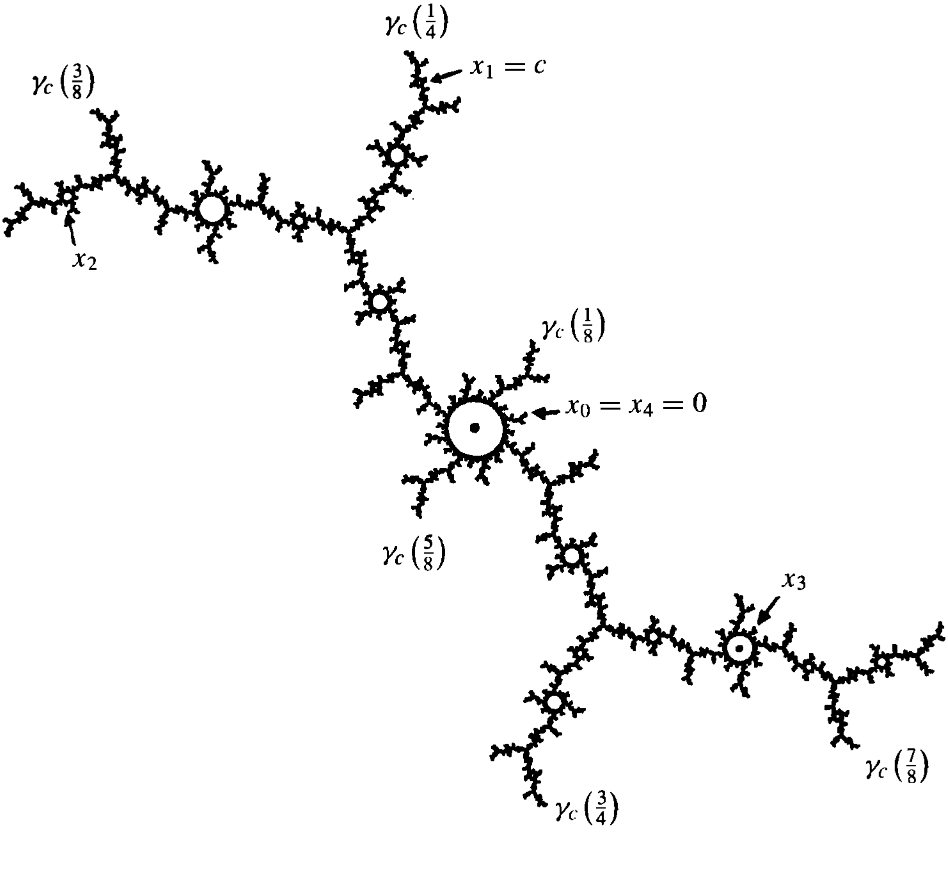


Figure III.2. The actual Julia set corresponding to the combinatorial model in Fig. III.1.

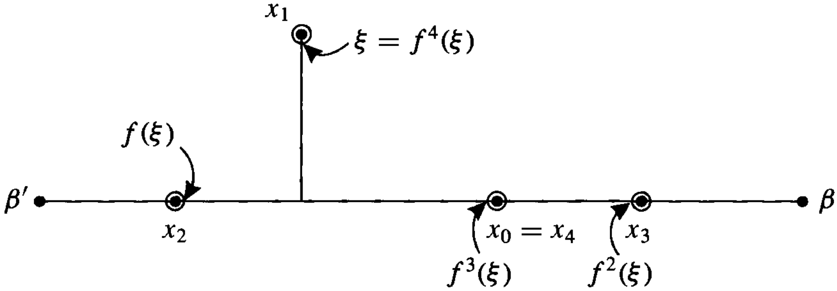


Figure III.3. $\theta = .\overline{0011} = \frac{3}{5} = \frac{1}{5}$.

If 0 is periodic, let θ_- and θ_+ be the two root arguments of U_1, labeled so that they have representatives θ_-^* and θ_+^* satisfying $0 \leq \theta_-^* < \theta_+^* \leq 1$. Then set

$$J_1 = \chi\left(\left]\frac{\theta_-^*}{2}, \frac{\theta_-^*+1}{2}\right]\right), \quad J_0 = \chi\left(\left]\frac{\theta_-^*+1}{2}, \frac{\theta_-^*+2}{2}\right]\right)$$

or

$$J_1 = \chi\left(\left[\frac{\theta_+^*}{2}, \frac{\theta_+^*+1}{2}\right[\right), \quad J_0 = \chi\left(\left[\frac{\theta_+^*+1}{2}, \frac{\theta_+^*+2J}{2}\right[\right)$$

(see Fig. III.4).

For $t \in \mathbf{T}$, the *itinerary* of t with respect to the partition $\mathcal{J}$ is the sequence $s(t) = (s_n(t))_{n\in\mathbf{N}}$ defined by $2^n \cdot t \in J_{s_n}(t)$. In the periodic case, $t \sim_c t'$ iff t and t' have the same itinerary. In the strictly preperiodic case, $t \sim_c t'$ iff

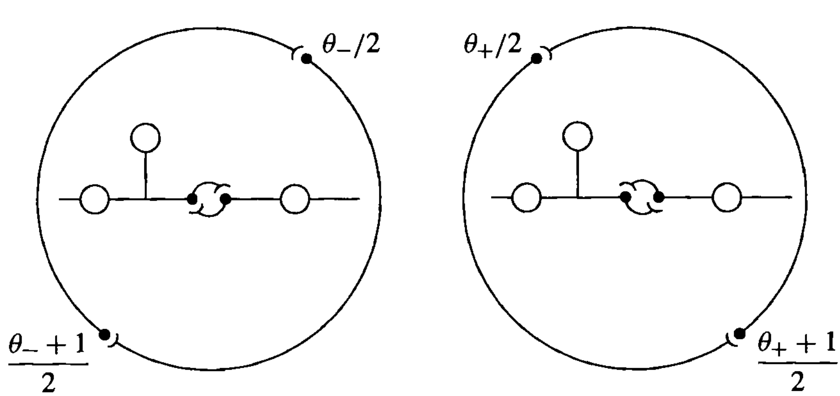


Figure III.4.

either they have the same itinerary, or their itineraries differ at one place n_0, and $s_{n_0+i}(t) = s_{n_0+i}(t') = s_{i-1}(\theta)$ for every $i \geq 1$.

The equivalence relation $\sim_c$ has the following properties:

(1) The classes are finite, and their cardinality is bounded. Any class with more than 2 elements is contained in $\mathbf{Q}/\mathbf{R}$. Indeed, if a point x in K_c has ≥ 3 external arguments, it is an iterated pre-image of a branch point of H_c, and it is preperiodic.

(2) The orbit of any point $x \in K_c$ having at least 2 external arguments contains a point on the spine $[\beta, \beta']_{K_c}$. Indeed, if t and t' are two arguments of x, choose n such that the n^{th} digit in the binary expansion of t and t' differ. Then $f^{n-1}(x)$ has an argument in $[0, 1/2]$ and one in $[1/2, 1]$.

IV. A model for the Mandelbrot set

IV.1 External rays of points of M. Recall that the Mandelbrot set M satisfies (D0), (D1), (D2), and that it is not known whether it satisfies (D3), i.e., whether it is locally connected. In this chapter, we shall however construct a pinched disk model for M, together with a map χ from M to its model which will be a homeomorphism iff M is locally connected. We must first review a certain number of facts concerning M which can be found in [DH].

The capacity radius r_M of M is equal to 1. We say that the ray $\mathfrak{R}(M, t)$ *lands* at c iff $\psi_M(r \cdot \mathbf{e}(t))$ tends to c when r tends to 1. In that case we also say that t is an external argument of c with respect to M, and we write $c = \gamma_M(t)$ (it is not known whether the map γ_M is defined on all of $\mathbf{T}$).

Propositions 1 and 2 below relate external arguments with respect to M, i.e., in the parameter plane, to external arguments with respect to some filled Julia set, i.e., in a dynamical plane. They are the key to all the combinatoric information we can get on M.

Let c_0 be a point such that 0 is periodic of some period k under f_{c_0}. The point c_0 belongs to $\overset{\circ}{M}$; denote by W the connected component of $\overset{\circ}{M}$ containing c_0. Then ∂W is a connected component of an $\mathbf{R}$-algebraic curve, it is smooth or has one cusp. For each $c \in W$, the map f_c has a unique attractive cycle; let us denote by $\varphi_W(c)$ its multiplier. Then the map $\varphi_W\colon W \to \mathbf{D}$ is a conformal homeomorphism, and it extends to a homeomorphism $\overline{W} \to \overline{\mathbf{D}}$ still denoted by φ_W. In this situation we say that W is a *hyperbolic component* of $\overset{\circ}{M}$ and that c_0 is its center. The point $\varphi_W^{-1}(\mathbf{e}(t))$ is called the point of ∂W of inner argument t and denoted by $\gamma_W(t)$. In particular $\gamma_W(0) = \varphi_W^{-1}(1)$ is the *root* of W.

It is conjectured that all connected components of $\overset{\circ}{M}$ are hyperbolic: this is the generic hyperbolicity conjecture for quadratic polynomials.

A point c in M such that 0 is strictly preperiodic under f_c is usually called a *Misiurewicz point* (even though it would be more appropriate to give this name to all points c such that 0 is non-recurrent).

Set $M^{\mathbf{Q}} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$, where

- $\mathcal{D}_0$ is the set of centers of hyperbolic components, i.e., of values of c such that 0 is periodic under f_c;
- $\mathcal{D}_1$ is the set of roots of hyperbolic components, i.e., of values of c such that f_c has a rational indifferent cycle;
- $\mathcal{D}_2$ is the set of Misiurewicz points, i.e., of values of c such that 0 is strictly preperiodic under f_c.

There is a bijection $\mathcal{D}_0 \to \mathcal{D}_1$ which associates the root of a hyperbolic component to its center.

PROPOSITION 1. Let $c \in \mathcal{D}_2$ be a Misiurewicz point. Then, in K_c, the point $x_1 = c$ has a finite number $\nu > 0$ of external arguments $\theta_1, \ldots, \theta_\nu$. The external rays $\mathcal{R}(M, \theta_1), \ldots, \mathcal{R}(M, \theta_\nu)$ of M land at c. No other external ray of M lands at c.

PROPOSITION 2. Let $c_0 \in \mathcal{D}_0$ and $c_1 \in \mathcal{D}_1$ be the center and the root of a hyperbolic component W of $\overset{\circ}{M}$. Let U_1 be the connected component of $\overset{\circ}{K}_{c_0}$ containing the critical value c_0. The root point of U_1 has two external arguments θ_- and θ_+ with respect to K_{c_0}. They are periodic of period k under $t \mapsto 2 \cdot t$, i.e., they are rational with denominator dividing $2^k - 1$. The rays $\mathcal{R}(M, \theta_-)$ and $\mathcal{R}(M, \theta_+)$ land at c_1. No other external ray of M lands at c_1.

If $c_0 \neq 0$, the curve $L_{c_1} = \mathcal{R}(M, \theta_-) \cap \{c_1\} \cap \mathcal{R}(M, \theta_+)$ separates 0 from c_0.

PROPOSITION 3. For $\theta \in \mathbf{Q}/\mathbf{Z}$, the ray $\mathcal{R}(M, \theta)$ lands at a point $c = \gamma_M(\theta) \in \partial M$, which is the root of a hyperbolic component or a Misiurewicz point.

More precisely, for $\theta = p/2^l \cdot (2^k - 1) \in [0, 1]$ with p odd and k minimal, we have the following: If $l = 0$, i.e., if θ is rational with odd denominator, c is the root of a hyperbolic component W of period k (i.e., for $c' \in W$, $f_{c'}$ has an attractive cycle of order k). If $l > 0$, the point c is a Misiurewicz point: $f_c^{l+1}(0)$ is periodic but $f_c^l(0)$ is not; k can be written as $k' \cdot k''$, where k' is the period of $f_c^{l+1}(0)$ and k'' the number of branches of the Hubbard tree (extended or not) at that point.

The map $\theta \mapsto \gamma_M(\theta)$ maps $\mathbf{Q}/\mathbf{Z}$ onto $\mathfrak{D}_1 \cup \mathfrak{D}_2$. For $c \in \mathfrak{D}_1$, the set $\gamma_M^{-1}(c)$ contains two points, both with odd denominator (except for $c = 1/4 = \gamma_M(0)$). The involution σ on the set of rationals with odd denominator which interchanges them has been described by Lavaurs in [L] .

IV.2 Tuning. The following can be found in [D3], with a sketch of a proof.

Let c_0 and c_1 be the center and the root of a hyperbolic component W, and let θ_- and θ_+ be the two arguments of c_1 with respect to M. There is a map $M \to M$ called the *tuning map* $x \mapsto c_0 \perp x$, which is a homeomorphism of M onto its *tuning copy* $c_0 \perp M$. Its effect on external arguments is the following: Let x be either a root of a hyperbolic component or a Misiurewicz point, and let t be an external argument of x in M. Then to t there corresponds an external argument t' of $c_0 \perp x$ given by the following algorithm:

Expand θ_-, θ_+ and t in base 2 (the bar means that the sequence under it is repeated indefinitely):

$$\theta_- = .\overline{u_1^0 \dots u_k^0},$$
$$\theta_+ = .\overline{u_1^1 \dots u_k^1},$$
$$t = .s_1 \dots s_n \dots$$

Then $t' = .u_1^{s_1} \dots u_k^{s_1} u_1^{s_2} \dots u_k^{s_2} u_1^{s_3} \dots .$

(Note that if t is a dyadic number, it has two dyadic expansions, and we get two values of t': at such a point, the set M extends beyond the tuning copy.)

For W a hyperbolic component and $t \in \mathbf{Q}/\mathbf{Z}$, there is a unique hyperbolic component $W(t)$ with root $\gamma_W(t)$. We call it the *satellite* of W of inner argument t. Denote by W_0 the main component of $\mathring{M}$, i.e., the one which contains 0. For $t \in]0, 1[\cap \mathbf{Q}$, the root arguments of $W_0(t)$, i.e., the external arguments of $\gamma_{W_0}(t)$ are ([DH0]):

$$\theta_- = \sum_{0<s<t} \frac{1}{2^{q(s)} - 1}$$

$$\theta_+ = \sum_{0<s\leq t} \frac{1}{2^{q(s)} - 1}$$

where the sum is over rational values of s, denoting by $q(s)$ the denominator of s written as an irreducible fraction. For an arbitrary hyperbolic component W, the root arguments of $W(t)$ are then given by the tuning algorithm.

IV.3 The Q-veins of M. If c_0 is the center of a hyperbolic component W of $\mathring{M}$, we define the *arguments* **Q**-*associated* to c_0 (or to W) to be the external arguments of points $\gamma_W(t)$, $t \in \mathbf{Q}/\mathbf{Z}$, i.e., the root arguments of the satellites of W. For $c \in \mathcal{D}_1 \cap \mathcal{D}_2$, the arguments associated to c are simply the external arguments of c with respect to M.

For τ a dyadic point in $\mathbf{T}$ (or in $[0, 1]$), we define the **Q**-*vein* $N_\tau^{\mathbf{Q}}$ as the set of points $c \in M^{\mathbf{Q}}$ for which there are two **Q**-associated arguments t, t' such that τ is the leading point of $[t, t']$.

This definition is coined in such a way that, if M is locally connected, then $N_\tau^{\mathbf{Q}} = N_\tau \cap M^{\mathbf{Q}}$. (We have to make an exception in the case τ is one of the root arguments of a component W with center c, but then the denominator must be an odd power of 2 and this occurs only for $\tau = 0$; we set $\gamma_M(0) = \{1/4\}$).

We say that c is the origin of N_τ if $c \in N_\tau$, and there is a dyadic angle τ' of smaller order such that $c \in N_{\tau'}$.

An important result is the following ([DH] II, Th $1_{\mathcal{D}}$, p. 128):

PROPOSITION 4. For every $\tau = p/2^k \in [0, 1]$, $k > 0$, the **Q**-vein $N_\tau^{\mathbf{Q}}$ has an origin in $\mathcal{D}_0 \cup \mathcal{D}_2$.

Let s_- and s_+ be the smallest and greatest arguments associated with the roots of N_τ, let $x \in M^{\mathbf{Q}} - N_\tau^{\mathbf{Q}}$ be a point with an associated argument $u \in]s_-, s_+[$ and $c \in N_\tau^{\mathbf{Q}}$. We say that c is the projection of x on $N_\tau^{\mathbf{Q}}$ if c has two associated arguments t, t' such that $u \in]t, t'[$ and $\tau \notin]t, t'[$ (this condition is independent of the choice of u).

An adaptation of the proof of Proposition 4 gives the following variant:

PROPOSITION 5. With τ and x as above, the point x has a projection on $N_\tau^{\mathbf{Q}}$.

IV.4 A disked-tree model for M. The proof of Proposition 4 involves a way of detecting whether the origin of $N_\tau^{\mathbf{Q}}$ is in $\mathcal{D}_0$ or in $\mathcal{D}_2$, and of computing its associated arguments. In view of this, one can construct by induction on n a tree $T_n(M)$—just an abstract tree provided with an imbedding class in $\mathbf{C}$: the tree T_{n+1} is obtained by adding to T_n arcs supposed to be the veins of order $n+1$ at the appropriate places.

If W is a hyperbolic component of order $k > 1$, its center belongs to a vein $N_\tau^{\mathbf{Q}}$ of order $< k$, i.e., with $\tau = p/2^{k'}$, $k' < k$. Indeed, let $\theta_- = .\overline{u_1^0 \dots u_k^0}$ and $\theta_+ = .\overline{u_1^1 \dots u_k^1}$ be the root arguments of W, with $0 \le \theta_- < \theta_+ \le 1$. For some $k' \le k$, one has $u_{k'}^0 = 0$, $u_{k'}^1 = 1$, $u_i^0 = u_i^1$ for $i < k'$; then the leading point of $]\theta_-, \theta_+[$ is $\tau = .u_1^0 \dots u_{k'-1}^0 1$. As a consequence, if for $\tau = p/2^k$ the origin of $N_\tau^{\mathbf{Q}}$ is the center of a component W, then the period of W is $\le k$. We can then define an n^{th} approximating disked tree M_n of M by adding to the tree T_n the topological disc $\overline{W}$ for each hyperbolic component W of period $\le n$. Indeed conditions (I.1) and (I.2) of section II.3 will be fulfilled.

Up to now M_n is just an abstract disked tree provided with an embedding class in $\mathbf{C}$ (whether some disks should touch is decided by computing the root arguments). We shall discuss at the end of this paper the question of realizing M_n as a subset of M.

On the other hand, as we have seen in section II.4, the knowledge of all the disked trees M_n does not provide a complete knowledge of M, even assuming the (MLC) conjecture.

If K is a compact set satisfying (D0) to (D3) or more generally a pinched disc, and $x \in K$, the legal arc $[\gamma_K(0), x]_K$ is of the form $J_0 \cup \dots \cup J_k$ or $\bigcup_{i \in \mathbf{N}} J_i \cup \{x\}$, where for each i, the set J_i is an arc in a vein $N_{\tau_i}(K)$. We call the finite or infinite sequence (τ_i) the *address* of x in K. One can transfer this definition in the setting of $M^{\mathbf{Q}}$.

The following result by Lavaurs summarizes our knowledge of the combinatorics of $M^{\mathbf{Q}}$:

PROPOSITION 6 ([L]). For $c \in M^{\mathbf{Q}}$, the address of c in $M^{\mathbf{Q}}$ is the same as its address in K_c.

period	1	2	3	4	5	6
denominator of root-argument	1	3	7	15	31	63

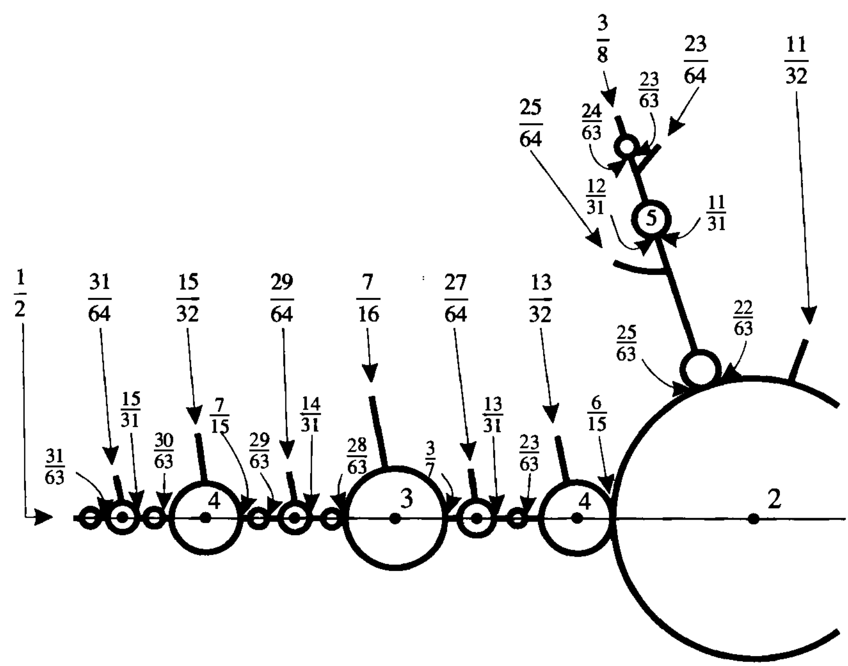
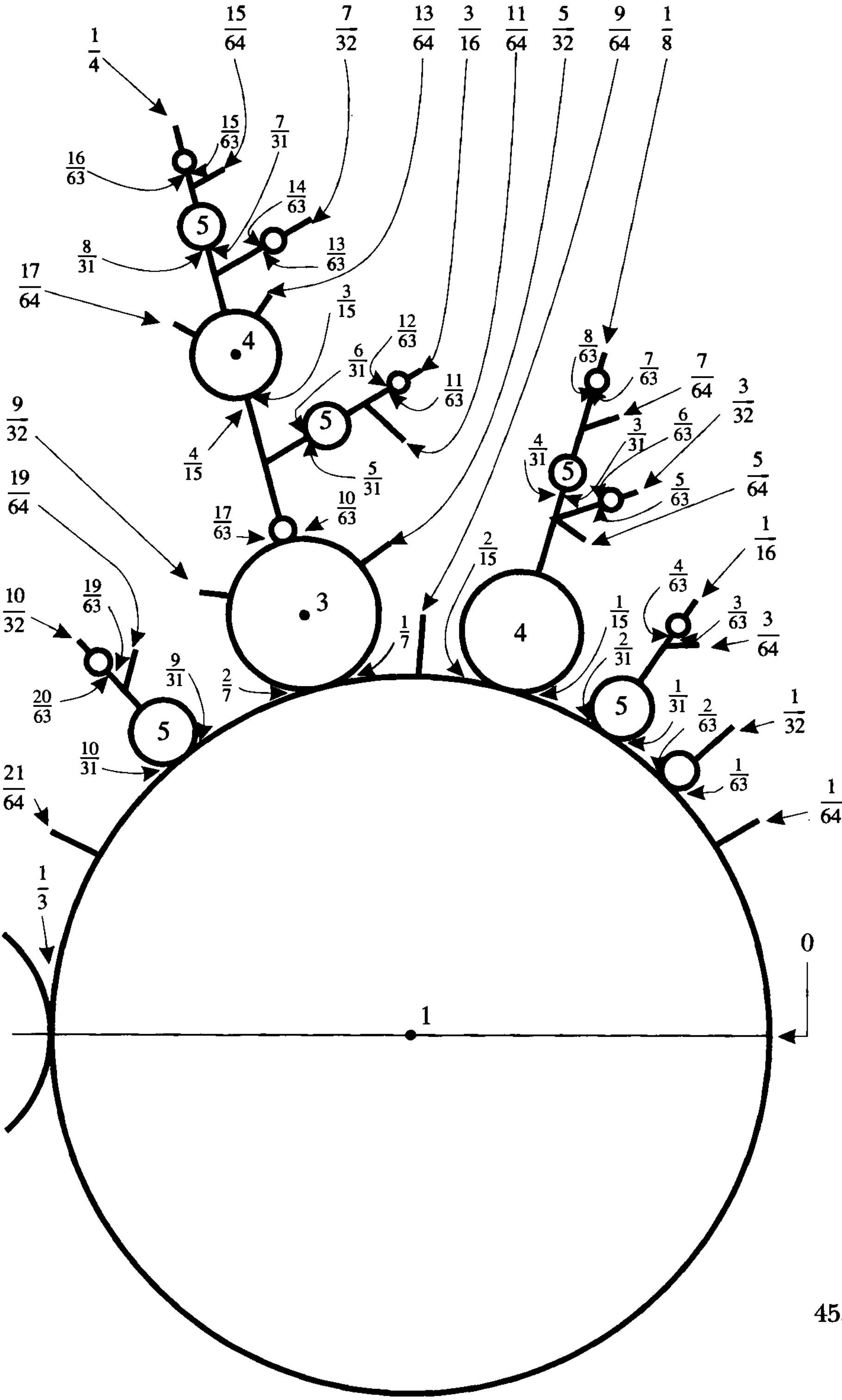


Figure IV.1. The 6^{th} approximating disked tree of the Mandelbrot set.



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IV.5 The equivalence relations $\sim_M^{\mathbf{Q}}$ and $\sim_M$. We define the equivalence relation $\sim_M^{\mathbf{Q}}$ on $\mathbf{Q}/\mathbf{Z}$ by $t \sim_M^{\mathbf{Q}} t' \iff \gamma_M(t) = \gamma_M(t')$. It is unlinked (condition (E2) of section I.2), with finite classes.

For each hyperbolic component W of $\overset{\circ}{M}$, there is an open set $V = V_W$ of D such that ∂V is the union of the geodesics joining the two arguments of $\gamma_W(u)$ for $u \in \mathbf{Q}/\mathbf{Z}$, together with a Cantor set in ∂D. The open set V_W is disjoint from the convex hull of any class of $\sim_M^{\mathbf{Q}}$.

We consider the relation $\sim_M$ on $\mathbf{T}$ whose graph is the closure of the graph of $\sim_M^{\mathbf{Q}}$.

THEOREM 3. (a) The relation $\sim_M$ is an equivalence relation.

(b) For $t \in \mathbf{Q}/\mathbf{Z}$, the class of t in $\sim_M$ is just its class in $\sim_M^{\mathbf{Q}}$.

(c) Every class of $\sim_M$ with ≥ 3 elements is a class of $\sim_M^{\mathbf{Q}}$.

(d) The relation $\sim_M$ is the only closed unlinked equivalence relation on $\mathbf{T}$ inducing $\sim_M^{\mathbf{Q}}$ on $\mathbf{Q}/\mathbf{Z}$, and whose classes have a convex hull disjoint from the open sets V_W.

It follows from Theorem 3 (d) that, if M is locally connected, then $\sim_M$ is precisely the equivalence relation defined by the Caratheodory loop of M.

SKETCH OF PROOF OF THEOREM 3. Let us abstract from the situation the hypotheses which are used in the proof of Theorem 3, and after that we shall indicate the steps of the proof.

We start with an equivalence relation $\sim^{\mathbf{Q}}$ on $\mathbf{Q}/\mathbf{Z}$ and a family $(V_i)_{i \in I}$ of open sets in $\mathbf{D}$. We denote by $L^{\mathbf{Q}}$ the union of convex hulls of classes of $\sim^{\mathbf{Q}}$. We define an *argument* associated to V_i as a $t \in \mathbf{Q}/\mathbf{Z}$ such that $\mathbf{e}(t)$ is one extremity of a geodesic contained in ∂V_i. If x is a class of $\sim^{\mathbf{Q}}$, an argument associated to x is simply an element of x.

We assume the following properties:

(i) The equivalence $\sim^{\mathbf{Q}}$ is unlinked with finite classes.

(ii) The open sets (V_i) are connected and disjoint.

(iii) For each i, $V_i \cap L^{\mathbf{Q}} = \emptyset$ and $\partial V_i \subset L^{\mathbf{Q}} \cup S^1$.

(iv) For $\tau = p/2^k$, $k > 0$, there is an α, which is either one of the open sets V_i or a class of $\sim^{\mathbf{Q}}$, having two associated arguments t, t' such that τ is the leading point of $[t, t']$, and two associated arguments t_1, t_1' such that the leading point τ_1 of $[t_1, t_1']$ is of smaller order.

(v) Given s, s' with $0 \leq s < s' \leq 1$ and $s \sim^{\mathbf{Q}} s'$, let τ be the leading point of $]s, s'[$. Suppose x is a class of $\sim^{\mathbf{Q}}$ contained in $[s, \tau[$ or $]\tau, s']$. Then there is a c, which is either one of the open sets V_i or a class of $\sim^{\mathbf{Q}}$, such that c has two associated arguments u, u' with $x \subset]u, u'[$ and $\tau \notin [u, u']$, and two associated arguments v, v' with $\tau \in [v, v']$.

We now consider the relation $\sim$ on $\mathbf{T}$ whose graph is the closure of the graph of $\sim^{\mathbf{Q}}$ (it is not obvious at this point that it is an equivalence relation).
We get the following lemmas:

LEMMA 1. Suppose that $t_1 \sim t_2$ and $t_3 \sim t_4$, and that the geodesics $[t_1, t_2]$ and $[t_3, t_4]$ cross in $\mathbf{D}$. Then $t_i \sim t_j$ for i, j in $\{1, 2, 3, 4\}$.

LEMMA 2. Let t, t' and t'' be three distinct points in $\mathbf{T}$ with $t \sim t'$ and $t \sim t''$. Then $t \in \mathbf{Q}/\mathbf{Z}$, and the class of t in $\sim^{\mathbf{Q}}$ has ≥ 3 elements.

Hint: Let τ_1, τ_2, τ_3 be the leading points of the three components of $\mathbf{T} - \{t, t', t''\}$. Let τ be the one of them with highest order. Apply property (iv) to τ.

LEMMA 3. Suppose $t \in \mathbf{Q}/\mathbf{Z}$ and $t \sim t'$ with $t' \neq t$. Then t is not alone in its class for $\sim^{\mathbf{Q}}$.

Hint: Supposing $0 \leq t < t' \leq 1$, let τ be the leading point of $]t, t'[$. Apply property (iv) to τ to get s, s', and then property (v) to τ and t.

LEMMA 4. Suppose that $t \sim t'$ and that $t \in \mathbf{Q}/\mathbf{Z}$. Then $t' \in \mathbf{Q}/\mathbf{Z}$ and $t \sim^{\mathbf{Q}} t'$.

Hint: Suppose $t' \notin \mathbf{Q}/\mathbf{Z}$. Using Lemma 3, let t'' be a point distinct from t with $t' \sim^{\mathbf{Q}} t$. Assume t or t' is one of the point of the class of t in $\sim^{\mathbf{Q}}$ adjacent to t'. Then proceed as for Lemma 2 (consider several cases). If $t' \in \mathbf{Q}/\mathbf{Z}$, proceed as for Lemma 3 .

With Lemmas 1 to 4 , parts (a), (b) and (c) of Theorem 3 are immediate. In order to prove part (d), we need the following:

LEMMA 5. Suppose $t \not\sim t'$. Then either t and t' belong to the boundary of a common V_i, or there is a class of $\sim^{\mathbf{Q}}$ whose convex hull separates t and t'.

Hint: Suppose $0 \leq t < t' \leq 1$. Let $[t_1, t_1']$ be an interval contained in $[t, t']$, satisfying $t_1 \sim t_1'$, and maximal for these properties. Let τ_1, τ_2, τ_3 be the leading points of $]t, t_1[$, $]t_1, t_1'[$ and $]t_1', t'[$, and let τ be one of them with greatest order as a dyadic point. Then apply property (iv) to τ.

IV.6 The map $\chi: M \to M_{\text{abs}}$. Consider the pinched disc $X_M = \overline{\mathbf{D}}/\simeq_M$ defined by $\sim_M$. The connected components of $\mathring{X}_M$ correspond to the hyperbolic components of $\mathring{M}$: this follows from Proposition 4 and the Remark in section II.3.

For W a hyperbolic component of $\mathring{M}$, let us denote by Y_W the corresponding component of $\mathring{X}_M$. There is a natural homeomorphism $\eta_W: \partial Y_W \to \partial W$; this homeomorphism can be extended to a homeomorphism $h_W: \overline{Y}_W \to \overline{W}$, but there is no very natural way to choose h_W. On the other hand, there is a natural homeomorphism $\varphi_W: \overline{W} \to \overline{\mathbf{D}}$ (section IV.1, (2)).

As we have seen in section I, if M is locally connected, then it is homeomorphic to X_M, but in order to get a homeomorphism $h: X_M \to M$ we have to choose for each hyperbolic component W an extension h_W of η_W.

Without assuming that M is locally connected, we can define the space M_{abs} as the pinched disc X_M provided with a homeomorphism $h_W: \overline{Y}_W \to \overline{W}$ extending η_W for each hyperbolic component W.

We can also get a definition which is more sophisticated but natural (i.e., not depending on any choice) by taking X_M and replacing each component Y_W of the interior by a copy of $\mathbf{D}$ attached using the homeomorphism $\varphi_W \circ \eta_W: \partial Y_W \to S^1$. There is then a unique way of defining the topology so as to obtain a compact space which is metrizable and locally connected.

The space M_{abs} comes as a subspace of a space E, obtained in the same way from $\mathbf{C}/\simeq_M$.

We can define a correspondence between $\mathbf{C}$ and E in the following way:

We define an allowed graph in $\mathbf{C}$ to be a finite union of topological arcs which may be

- an arc of the closure of an external ray of M with rational argument;
- an arc of an equipotential of M;
- an arc of the form $\varphi_W^{-1}(J)$, where W is a hyperbolic component of $\mathring{M}$, and J is a segment in a ray $J[0, \mathrm{e}(t)]$ with $t \in \mathbf{Q}/\mathbf{Z}$ or an arc of a circle of radius < 1 centered in 0.

If Γ is an allowed graph, the closures of the connected components of $\overline{\mathbf{C}} - \Gamma$ form an *allowed tessellation* $\mathcal{P} = (P_i)_{i \in I}$ of $\overline{\mathbf{C}}$, and there is a corresponding tessellation $\mathcal{P}^E = (P_i^E)$ of $\overline{E} = E \cup \{\infty\}$.

We declare that a point $x \in \overline{E}$ corresponds to a point $c \in \overline{\mathbf{C}}$ iff, for any allowed tessellation $\mathcal{P} = (P_i)_{i \in I}$ of $\mathbf{C}$, there is an $i \in I$ such that $c \in P_i$ and $x \in P_i^E$.

PROPOSITION 7. This correspondence is a continuous map $\overline{\mathbf{C}} \to \overline{E}$, which induces a map $\chi : MJ \to M_{\text{abs}}$.

PROOF. (1) The graph of the correspondence is closed, since it is

$$\bigcap_{\mathcal{P}} \bigcup_{i \in I_{\mathcal{P}}} P_i \times P_i^E.$$

(2) For $c \in \mathbf{C}$, there is a unique $x \in E$ corresponding to c. This is immediate if $c \in \overline{\mathbf{C}} - M$, or if c is in a hyperbolic component of $\mathring{M}$; in the other cases the uniqueness follows from Lemma 5 above. The existence is immediate for $c \in M^{\mathbf{Q}}$; for $c \in \partial M - M^{\mathbf{Q}}$ it is obtained by the following compacity argument: For each allowed tessellation $\mathcal{P}$, denote by $P(\mathcal{P}, c)$ the unique piece of $\mathcal{P}$ containing x, and by $P^E(\mathcal{P}, c)$ the corresponding piece in E. This piece is compact and non-empty. Given two allowed tessellations, there is an allowed tessellation finer than both. So the set of points x corresponding to c, which is

$$\bigcap_{\mathcal{P}} P^E(\mathcal{P}, c)$$

is not empty.

(3) The map $\overline{\mathbf{C}} \to \overline{E}$ is continuous because its graph is closed and $\overline{E}$ is compact. A compacity argument analogous to the one above shows that it is surjective. It induces a map $\chi : M \to M_{\text{abs}}$, which is surjective since M is the inverse image of M_{abs}.

IV.7 Reformulation of (MLC) and the generic hyperbolicity conjecture. A polynomial f is said to be *hyperbolic* if every critical point of f is attracted to an attracting cycle or to ∞. In the space $\mathbf{C}^{d-1}$ of monic centered polynomials of degree d, the hyperbolic ones form an open set $\mathcal{H}_d$. It is conjectured that this open set is dense.

For quadratic polynomials, the set $\mathcal{H}_2$ is the union of $\mathbf{C} - M$ and the hyperbolic components of $\mathring{M}$. In degree 2, the above conjecture is equivalent to the statement that every connected component of $\mathring{M}$ is hyperbolic.

THEOREM 4. (a) The conjecture (MLC) is equivalent to the injectivity of $\chi_M : M \to M_{\text{abs}}$.

(b) The density of $\mathcal{H}_2$ in $\mathbf{C}$ is equivalent to the statement that, for every $x \in M_{\text{abs}}$, the set $\chi^{-1}(x)$ has an empty interior.

PROOF. (a) The space M_{abs} is homeomorphic to X_M, which is locally connected as a quotient of $\overline{\mathbf{D}}$. If χ_M is injective, then it is a homeomorphism since M is compact, and M is locally connected too.

If M is locally connected, then it follows from Theorem 1 that χ is a homeomorphism.

(b) If $\chi^{-1}(x)$ is not reduced to a point, we call it a *queer set*, and if its interior is not empty we call a component of it a *queer component.*

If $\mathring{M}$ has a connected component W which is not hyperbolic, then W cannot intersect an allowed graph, so for each allowed tessellation W is contained in one piece. Therefore $\chi(W)$ is reduced to a point and W is a queer component.

Conversely if $\chi^{-1}(x)$ has a nonempty interior W then W is in $\mathring{M}$ and is disjoint from all hyperbolic components, so there is a non-hyperbolic component of $\mathring{M}$.

V. Work in progress

In the previous chapter we have shown how irritating it is not to have a proof of the (MLC) conjecture. In this chapter we explain how close we are to a proof with a theorem of Jean-Christophe Yoccoz (unpublished) which will be described in detail in H. Hubbard's paper in these Proceedings. We also review some related results of B. Branner, J. H. Hubbard, J. Kahn and the author.

V.1 A result on cubic polynomials. The starting point of Yoccoz' theorem mentioned above is a result on cubic polynomials. In 1986, Bodil Branner and John Hamal Hubbard studied complex cubic polynomials with non-connected Julia sets, i.e., those for which at least one critical point escapes to infinity. In the space $\mathbf{C}^2$ of monic centered cubic polynomials with labeled critical points $f_{a,b}: z \mapsto z^3 - 3 \cdot a^2 \cdot z + b$ they form an open set Ω, which can be written as $\Omega_+ \cup \Omega_-$, where Ω_+ (resp., Ω_-) corresponds to the case where the critical point a (resp., $-a$) escapes at least as fast as the other.

The set Ω_+ is fibered onto $\mathbf{C} - \overline{\mathbf{D}}$ by $H: (a, b) \mapsto \lim\left(f_{a,b}^{n+1}(a)\right)^{1/3^n}$, the fiber being a closed trefoil, i.e., the union of three closed disc with one point in common. In each of these discs L, there is a closed set E_L which is the set of points (a, b) such that the critical point $-a$ does not escape. B. Branner and H. Hubbard proved that E_L has a non-countable infinity of connected components, among which a countable infinity are copies of M, and the others are points ([BH]; cf. also Branner's paper in these Proceedings, in which she describes the way these points waltz around each other when the assigned value of $H(a, b)$ turns thrice around $\overline{\mathbf{D}}$, and the solenoid they may generate).

The delicate point in the proof of [BH] is the fact that components which are expected to be points are actually points.

They first prove a result in the dynamic plane, namely that, for (a, b) in such a component X, the set $K(f_{a,b})$ is a Cantor set. In order to show that the components of $K(f_{a,b})$ are points, they look at the annuli between two critical equipotentials. Whenever such an annulus A surrounds the critical point $-a$, then $f_{a,b}\colon A \to f_{a,b}(A)$ is a covering map of degree 2, and $\mathrm{mod}(A) = \frac{1}{2}\,\mathrm{mod}(f_{a,b}(A))$.

They gather the relevant combinatorial information concerning those annuli in a "tableau", and eventually prove that, under the given hypotheses, the sum of the moduli of the annuli surrounding a component of $K(f_{a,b})$ is infinite (which implies that this component is reduced to a point).

After that, they transfer this result to the parameter plane: they observe that the annuli which surround the critical value $f_{a,b}(-a)$, for $(a, b) \in X$ are reproduced holomorphically in the parameter plane as annuli surrounding X. The sum of the moduli is still infinite, and this forces X to be a point.

Since the result of Branner and Hubbard shows that sets which are expected to be points are actually points, it was reasonable to imagine that their method could be adapted to get results in the direction of (MLC), which can be stated as an injectivity result. Actually, J.-C. Yoccoz had already noticed some correspondence between the behavior of cubic polynomials and quadratic ones, a similarity made very concrete in a specific case by B. Branner in [BD]. However here there was a big difficulty which seemed unsurmountable: the transfer of results to the parameter plane.

V.2 Local connectivity of M at untuned points. J.-C. Yoccoz proves that, for a point $c \in M$ which is not tuned and not in the closure of the main component of $\overset{\circ}{M}$,

$$(*) \qquad \chi^{-1}(\chi(c)) = \{Jc\},$$

where $\chi\colon M \to M_{\mathrm{abs}}$ is the natural map.

He first proves that, for c untuned, the space K_c is locally connected. One can define allowed graphs in the dynamical plane of f_c as in section IV.6, replacing M by K_c (it is known that an external ray of K_c with rational argument always lands ([DH] I, VIII Prop. 2, p. 70)).

Yoccoz constructs for each $z \in K_c$ a sequence of nested annuli $A_n = \overset{\circ}{P}_n - P'_n$ where P_n and P'_n are closed topological discs bounded by allowed graphs, with P'_n a neighborhood of x and $P'_n \cap K_c$ connected. He then proves that the sum of their moduli is infinite, so $\bigcap P'_n = \{x\}$, and the P'_n form a fundamental system of neighborhoods of x. (I am cheating a bit, because there are some special cases which have to be treated separately).

Yoccoz formulates the combinatorial analysis which leads to the divergence of the series of moduli in a language which is more general and possibly more powerful than that of [BH]. But H. Hubbard checked (see his paper in this Proceedings) that it can be formulated in the language of tableaus, and then it is very similar to the analysis made in [BH] . The tuned case for quadratic polynomials corresponds to the case which gives copies of M for cubic polynomials.

The real difficulty starts when we try to transfer the results from the dynamical plane to the parameter plane. To each annulus A_n in the dynamical plane of f_c corresponds an annulus A_n^M in the parameter plane, but A_n^M is not just a copy of A_n. We only have a holomorphic bijection between $A_n^M - M$ and $A_n - K_c$; the sets $M \cap A_n^M$ and $K_c \cap A_n$ are not even homeomorphic (the first one has interior points in $\mathbf{C}$ and the other doesn't).

By an extremely fine analysis of the behavior of A_n and A_n^M at the neighborhood of K_c and M respectively, Yoccoz was able to get a bound independent of n for the ratio of their moduli. It follows that the sum of the moduli of the annuli A_n^M is infinite and this implies the theorem.

V.3 Local connectivity at other points. For c in the main component W_0 of $\overset{\circ}{M}$, the relation $(*)$ is immediate. For $c \in \partial W_0$, the reader will find a sketch of a proof in Hubbard's paper. There are two cases: c with rational (resp., irrational) inner argument.

For $c = \gamma(t)$, $t \in \mathbf{Q}/\mathbf{Z}$, there are two external rays $\mathfrak{R}(M, \theta_-)$ and $\mathfrak{R}(M, \theta_+)$ of M landing at c. Denote by L_c the curve $\mathfrak{R}(M, \theta_-) \cup \mathfrak{R}(M, \theta_+) \cup \{c\}$. The component V_c of $\mathbf{C} - L_c$ which does not contain 0 is called the *wake* of c, and $M^*(t) = V_c \cap M$ is the *strict limb* of M with inner argument t (the *limb* $M(t)$ is $M^*(t) \cup \{c\}$).

The relation $(*)$ for all points in ∂W_0 with irrational inner argument is equivalent to the fact

$$M - \overline{W}_0 = \bigcup_{t \in \mathbf{Q}/\mathbf{Z}} M^*(t).$$

This fact can be deduced from an inequality of Yoccoz which gives a bound for the diameter of the limbs of M (manuscript Orsay, see also Pommerenke [P]). But actually we only need the following result, a preliminary result for the mentioned inequality.

LEMMA. Let f be a polynomial of degree $d \geq 2$ with $K(f)$ connected, and x a repelling periodic point of period k for f. Then there is a finite number $\nu > 0$ of external rays of $K(f)$ which land at x. Their arguments are periodic of period dividing $k \cdot \nu$ for $t \mapsto d \cdot t$, i.e., they are rational with denominator dividing $d^{k \cdot \nu} - 1$

This lemma should also be enough for the case $c \in \partial W_0$ with rational argument.

Once we know (*) for all untuned points, we can practically say that we have it for all the points which are only finitely tuned. We can adapt the proof—and it requires only slight modifications (see sketch in Hubbard's paper). We can also deduce the finitely tuned case from the untuned case: For c_0 the center of a hyperbolic component, the tuning map $x \mapsto c_0 \perp x$ can be defined both as a map $M \to M$ and as a map $M_{\text{abs}} \to M_{\text{abs}}$. We then have to show that $\chi(c_0 \perp x) = c_0 \perp \chi(x)$ (which is easy), and that $\chi^{-1}(c_0 \perp M_{\text{abs}}) = c_0 \perp M$ (which is essentially one of Hubbard's lemmas).

I do not think that a complete proof of this fact has ever been written down in detail, up to this day.

V.4 Veins of M. For each n we have described in section IV.3 the n^{th} approximating disked tree M_n of M. The question arises naturally of whether M_n can be realized as a subset of M. More precisely, whether the veins can be realized as subsets of M, since there is no problem with the hyperbolic components.

In [BD] we proved it for the vein $N_{1/4}$, using *holomorphic surgery*. The proof extends easily to the vein $N_{1/2^n}$, more generally to the main vein of each limb (the vein N_τ, where τ is the leading point of $[\theta_-, \theta_+]$). Actually we could extend it to any individual vein we have tried, but we have not found an algorithm telling us how to do the surgery for any vein N_τ.

Jeremy Kahn has proposed another approach: We can define the main vein $N_{1/2}(M)$ to be the segment $[-2, 1/4]$ of $\mathbf{R}$. Given a dyadic angle τ, we can consider the vein $N_\tau(M_{\mathrm{abs}})$ of M_{abs}, and the subset $N'_\tau = \chi^{-1}(N_\tau(M_{\mathrm{abs}}))$ of M. If $x \in N_\tau(M_{\mathrm{abs}})$ is a finitely tuned point, then $\chi^{-1}(x)$ is just a point by Yoccoz' theorem.

LEMMA. If the vein $N_\tau(M_{\mathrm{abs}})$ enters a tuned copy $c_0 \perp M_{\mathrm{abs}}$ at its root point c_1, then it contains $c_0 \perp N_{1/2}(M_{\mathrm{abs}})$.

PROOF. Let $\theta_- = .\overline{u_1^0 \dots u_k^0}$ and $\theta_+ = .\overline{u_1^1 \dots u_k^1}$ be the external arguments of c_1. Then τ is the leading point of $[\theta_-, \theta_+]$, so

$$\begin{aligned} \tau &= .u_1^0 \dots u_{k'}^0 1, \\ \theta_- &= .u_1^0 \dots u_{k'}^0 0 \dots, \\ \theta_+ &= .u_1^0 \dots u_{k'}^0 1 \dots, \end{aligned}$$

with $k' < k$. The two arguments θ'_- and θ'_+ of $c_0 \perp (-2)$ are given by the tuning algorithm applied to $t = \frac{1}{2} = .0\overline{1} = .1\overline{0}$, i.e.,

$$\begin{aligned} \theta'_- &= .u_1^0 \dots u_k^0 \overline{u_1^1 \dots u_k^1}, \\ \theta'_+ &= .u_1^1 \dots u_k^1 \overline{u_1^0 \dots u_k^0}, \end{aligned}$$

and the leading point of $[\theta'_-, \theta'_+]$ is again τ.

Now the topological arc $N_\tau(M_{\mathrm{abs}})$ can be described as the union of a Cantor set C made of finitely tuned points and a countable family of open arcs $c_i \perp N^*$, where $N^* = \chi(]-2, 1/4[) \subset M_{\mathrm{abs}}$. Then N'_τ contains the set $N_\tau(M) = \chi^{-1}(C) \cup \bigcup(c_i \perp]-2, 1/4[)$, which is the union of the same Cantor set and a countable family of open arcs attached in the same way. So $N_\tau(M)$ is again a topological arc.

V.5 The limits of the method. The method of diverging series of moduli of annuli seems to be powerful, but it has its limits. The computation which leads from the combinatorial study to the divergence makes use of specific properties of the situation. The proof of the Branner-Hubbard theorem does not extend to polynomials of degree 4 with one double critical point, even though this case seems very analogous. Similarly the proof of Yoccoz does not extend to cubic polynomials of the form $z \mapsto z^3 + c$, which otherwise behave very much like quadratic polynomials. In both cases there are examples where the series converges, and for such examples one cannot decide.

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