The Quadratic Family I

The aim of this initial talk is to motivate the study of the quadratic family of polynomials. I will mention some theorems without proofs with the intention of returning when we have the tools to prove these.

As dynamicists we are interested in understanding quadratic polynomials as dynamical systems, in particular we are interested in questions such as:

"Given a starting point z_0, what happens to orbits? What starting points lead to bounded orbits?" is a natural question.

We can also ask that given a parametrised family of maps, what happens to such starting points as the parameter varies?

We will consider $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \alpha z^2 + \beta z + \gamma$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha \neq 0$. We are interested in simplifying our study. We can do so as follows.

Proposition Every complex quadratic map is topologically conjugate to a map of the form $z^2 + c$, $c \in \mathbb{C}$, by some affine map. This c is unique.

Proof Let f be as above. Let $g = z^2 + c$, $h = az + b$.

$$h \circ f = g \circ h \iff a(\alpha z^2 + \beta z + \gamma) = (az+b)^2 + c$$

$$\iff \alpha a z^2 + a\beta z + a\gamma = a^2 z^2 + 2abz + b^2 + c.$$

Equating coefficients of: z^2 we see $a = \alpha$

z we see $b = \beta/2$

constant terms we see $a\gamma + b = b^2 + c$ i.e. $c = \alpha\gamma + \frac{\beta}{2} - \left(\frac{\beta}{2}\right)^2$

This c is unique.

We have therefore reduced from studying three parameters to one parameter. The parameter plane is $\mathbb{C}$. This means we can draw sets on or subsets of this parameter plane. This is not possible in general.

More generally a polynomial of degree $d \geq 2$ is affinely conjugate to one of the form $F(z) = z^d + c_{d-2} z^{d-2} + \dots + c_0$

Thus for a cubic, the parameter space is $\mathbb{C}^2$ so we can not directly draw a picture. This means the quadratic family is a natural place to study and test our intuition. Pictures will be utilised and subsets of this space explored. We will eventually see some such results.

N.B. $q_c(z) = z^2 + c$ has critical point $z = 0$ and critical value $f(0) = c$.

The fundamental objects we have been studying are the Julia set and its complement the Fatou set. To begin studying these it is natural to ask where in the dynamical plane these live, that is, when c is fixed, where is the Julia set of $z^2 + c$.

<u>Proposition</u> For every polynomial P_c, every filled Julia set is bounded, in particular it is contained completely in a disc centred at the origin of Radius $R = \max\{2, |c|\}$

Remark: This uses an alternative definition of the Julia set of a polynomial as the boundary of $K_P = \{z \in \mathbb{C} \mid P_c^{\circ n}(z) \not\to \infty$ as $n \to \infty\}$.

We will see this later.

We will also see that for a polynomial P_c with connected Julia set, $J_P \subset \overline{\mathbb{D}_2}$.

Before we proceed it will be useful to prove this equivalent definition. Define the filled Julia set $K = K(f) = \{z \in \mathbb{C} \mid f^{\circ n}(z) \not\to \infty \text{ as } n \to \infty\}$

<u>Lemma</u> (9.4 in Milnor) Let $f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$, $d \geq 2$.

For any polynomial f of degree $d \geq 2$, the filled Julia set $K(f)$ is compact, with connected complement. The topological boundary $\partial K = J(f)$ and with interior equal to the union of all bounded components U of $F = \mathbb{C} \setminus J$. Thus, K is equal to the union of all such U, together with J itself. Any such bounded component U is simply connected.

<u>Proof</u>

$f(z)/z^d$ converges to a_d as $|z| \to \infty$. We can assume $a_d = 1$. Since $a_d \neq 0$, $d \geq 2$ we can choose c with $c^{d-1} = a_d$ and note the linearly conjugate polynomial $c f(z/c)$ is monic. Chose a constant $r_0 \geq 2$ so that $\left| f(z)/z^d - 1 \right| < 1/2$ for $|z| > r_0$. It follows $|f(z)| > \frac{|z^d|}{2} > 2|z|$ for $|z| > r_0$

Thus any point with $|z| > r_0$ belongs to attracting basin $\mathcal{A} = \mathcal{A}(\infty)$ of infinity. K can be identified with $\hat{\mathbb{C}} \setminus \mathcal{A}$ so K is compact.

Corollary 4.12 in Milnor tells us $\partial K = \partial \mathcal{A} = J_f$.

Corollary 4.12 (of Transitivity Theorem)

If $\mathcal{A} \subset \hat{\mathbb{C}}$ is the basin of attraction for some periodic orbit, $|\lambda| < 1$, then the topological boundary $\partial\mathcal{A} = \bar{\mathcal{A}} \setminus \mathcal{A}$ is equal to the entire Julia set.

Every connected component of $F = \hat{\mathbb{C}} \setminus J$ either coincides with with some connected component of $\mathcal{A}$ or else is disjoint from $\mathcal{A}$.

Proof

If N a neighbourhood of any point in J, transitivity theorem then tells us that some $f^{\circ n}(N)$ intersects $\mathcal{A}$ so N intersects $\mathcal{A}$. So $J \subset \bar{\mathcal{A}}$ but J is disjoint from $\mathcal{A}$ so $J \subset \partial\mathcal{A}$.

On the other hand, if N is a neighbourhood of a point of $\partial\mathcal{A}$, then any limit of iterates $f^{\circ n}|_N$ must have a jump discontinuity between $\mathcal{A}$ and $\partial\mathcal{A}$, so $\partial\mathcal{A} \subset J$. Finally, any connected Fatou component intersecting $\mathcal{A}$ cannot intersect $\partial\mathcal{A}$ and must coincide with some component of $\mathcal{A}$. ■

The remainder of the proof of Lemma 9.4 can be found in Milnor.