Towards the quadratic family

Last week we motivated our study of the quadratic family and showed that this is a natural place to begin our study. There is very rich structure and behaviour and importantly we can draw a parameter space for the family z^2+c, $c \in \mathbb{C}$.

To prove results about the quadratics it is worth taking a step back to prove some more general results.

Some of these results are contained in the previous notes and we will give an alternative proof of the theorem that every attracting basin contains a critical point for a rational map. (c.f. problem 8-g Milnor or Carleson & Gamelin)

Theorem

If f is a rational map of degree $d \geq 2$ then the immediate basin of every attracting periodic orbit contains at least one critical point. Hence, the number of attracting periodic orbits is finite, less than or equal to the number of critical points.

(Immediate attracting basin is connected component of attracting basin containing z_0)

Proof

Suppose z_0 is an attracting fixed point, $0 < |\lambda| < 1$.

Take U_0 a small neighbourhood of z_0, invariant under f, on which the analytic branch of $f^{-1} = g_1$ satisfying $g_1(z_0) = z_0$ is defined. (g_1 will be repelling at z_0) g_1 will map U_0 into $\mathcal{A}(z_0)$. g_1 will be injective. Let $U_1 = g_1(U_0)$, which is simply connected, with $U_0 \subset U_1$.

We can construct a sequence $U_{n+1} = g_1(U_n) \supset U_n$. If this does not terminate, $g_n : U_0 \to U_n$, the sequence of analytic functions on U_0 omits J and thus is normal on U_0. This is impossible. z_0 is a repelling fixed point of g, so $z_0 \in J_g$. We eventually reach a U_n such that we cannot extend g. There is then a critical point $p \in \mathcal{A}^*(z_0)$ such that $f(p) \in U_n$

If z_0 periodic of period $n > 1$, and $|(f^{\circ n}(z_0))'| < 1$, the argument above tells us every component of $\mathcal{A}(z_0)$ contains a critical point of f^n. By the chain rule, $\mathcal{A}^*(z_0)$ must also contain a critical point. ∎

The proof in Milnor use a global linearising map.

We also have a result about the topology of immediate attracting basins. (c.f polynomials - motivation for $\mathcal{M}$) we will not discuss this today.

<u>Theorem</u>

Let $\mathcal{A}_0$ be the immediate attracting basin of an attracting fixed point. Then $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ is either connected or consists of uncountably many points.

Recall the transitivity theorem told us that forward iterates of an arbitrary neighborhood of an arbitrary point in the Julia set cover the entire Julia set and all but at most two points of $\hat{\mathbb{C}}$.

A corollary of this is the following.

<u>Corollary</u>

If $\mathcal{A} \subset \hat{\mathbb{C}}$ is the basin of attraction for some attracting periodic orbit, then the topological boundary $\partial\mathcal{A} = \bar{\mathcal{A}} \setminus \mathcal{A}$ is equal to the entire Julia set. Every connected components of $\hat{\mathbb{C}} \setminus J$ either coincides with some connected components of $\mathcal{A}$ or else is disjoint from $\mathcal{A}$.

Proof

If N is a neighbourhood of a point of J then transitivity implies some $f^{\circ n}(N)$ intersects $\mathcal{A}$ (since $\mathcal{A}$ is attracting basin) hence N itself intersects $\mathcal{A}$ so J. But J is disjoint from $\mathcal{A}$ so $J \subset \partial\mathcal{A}$.

On the other hand, N is a neighbourhood of a point in $\partial\mathcal{A}$ then any limit of iterates $f^{\circ n}|_N$ must have a jump discontinuity between $\mathcal{A}$ and $\partial\mathcal{A}$. Hence $\partial\mathcal{A} \subset J$. Finally any Fatou component which is connected and intersects $\mathcal{A}$, since it cannot intersect $\partial\mathcal{A}$, must coincide with some component of $\mathcal{A}$.

$\blacksquare$

We define $K_p := \{z \in \mathbb{C} \mid p^{\circ n}(z) \not\to \infty$ as $n \to \infty\}$ and it is easy to see $\partial K_p = J_p$ for polynomials. Therefore, for polynomials it is useful to study K_p to understand J_p. A theorem of Fatou and Julia tells us about the structure of K_p.

Theorem (Fatou and Julia 1919)

Let Ω_p denote the set of critical points for a polynomial p. Then

$$\Omega_p \subset K_p \iff J_p \text{ connected}$$

$$\Omega_p \cap K_p = \emptyset \Rightarrow J_p \text{ is a Cantor set.}$$

For quadratics, there is only one critical point so J is either connected or totally disconnected. The dichotomy follows from the next two theorems.

Theorem

The Julia set J_P is connected if and only if there is no finite critical point of P in $\mathcal{A}(\infty)$, that is, if and only if the orbit of every finite critical point is bounded.

and

Theorem

If $P^{\circ n}(q) \to \infty$ for each critical point q, then J is totally disconnected.

The first of these uses the Green's function in the proof. The Green's function is an important tool in the study of the dynamics of polynomials. We will look at this in depth soon. In particular, this is closely related to the Böttcher isomorphism and we will utilise this to study, for example, external rays.