Complex Dynamics (Week 0)

Dynamics in one complex variable (John Milnor) pages 1-55.

Fatou / Julia sets + Examples + Properties

Def^n: A collection $\mathcal{F}$ of maps from $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (or compact Riemann surfaces $S \to T$) is called <u>normal</u> on $\Omega \subseteq \hat{\mathbb{C}}$ if

$$\forall (f_n)_n \subseteq \mathcal{F} \quad \exists (n_k)_k \; n_k \to \infty \text{ s.t. } (f_{n_k})_k \text{ converges locally uniformly } \forall K \subseteq \Omega \text{ compact.}$$

normal at $z \in \hat{\mathbb{C}}$ if $\exists$ open $z \in U \subseteq \hat{\mathbb{C}}$ s.t. $\mathcal{F}$ is normal on U.

Remark: In case T is non-compact, also normal if

$$\forall K \subset S,\ K' \subset T \text{ compact we have } f_n(K) \cap K' = \emptyset \text{ for suff. large } n.$$

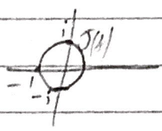
Def^n: (Fatou / Julia sets) Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ non-constant holomorphic map (i.e. rational function) $f^n: \hat{\mathbb{C}} \circlearrowleft$ be the n-fold product. Set $\mathcal{F} = \{f^n : n \in \mathbb{N}\}$ collection of iterates.

The <u>Fatou set</u> of f is the domain normality of $\mathcal{F}$. The complement of the Fatou set is called the <u>Julia set</u>, denoted $J(f)$.

Remark: Fatou set is open / Julia set is closed.

For $p \in \hat{\mathbb{C}}$ if $\exists U \subseteq \hat{\mathbb{C}}$ open containing p so that $\mathcal{F} = \{f^n : n \in \mathbb{N}\}$ restricted to U is normal then p belongs to the Fatou set $F(f)$. (U subset of the Fatou set). Otherwise $p \in J(f)$.

Examples: $f(x) = z^2$



Also p.42

Invariance lemma: Let $f: \hat{\mathbb{C}} \circlearrowleft$ holomorphic. Then $J(f)$ is fully invariant i.e. $z \in J(f) \iff f(z) \in J(f)$.

Equivalently, $F(f) = f(F(f)) = f^{-1}(F(f))$

Proof. <u>$f^{-1}(F(f)) \subseteq F(f)$:</u> Let $z \in f^{-1}(F(f))$ then $f(z) \in F(f)$. Let $(f^{n_j})_j \subseteq \mathcal{F}$. Since $f(z) \in F(f)$

$\exists U \subseteq \hat{\mathbb{C}}$ open containing $f(z)$ s.t. $(f^{n_{j_k}+1})_k$ converges locally uniformly $g: U \to \hat{\mathbb{C}}$. Then since f is <u>continuous</u>

$z \in f^{-1}(U) \subseteq \hat{\mathbb{C}}$ <u>open</u> s.t. $(f^{n_{j_k}})_k$ converges uniformly to $g \circ f : f^{-1}(U) \to \hat{\mathbb{C}}$.

<u>$f(F(f)) \subseteq F(f)$:</u> Let $y \in f(F(f))$. Then $\exists z \in F(f): y = f(z)$. Let $(f^{n_j})_j \subseteq \mathcal{F}$. Since $z \in F(f)$

$\exists U \subseteq \hat{\mathbb{C}}$ open containing z s.t. $f^{n_{j_k}+1}: U \to \hat{\mathbb{C}}$ converges locally uniformly as $k \to \infty$. Then since f is <u>open</u> (since rational)

$y \in f(U) \subseteq \hat{\mathbb{C}}$ open s.t. $f^{n_{j_k}}: f(U) \to \hat{\mathbb{C}}$ converges locally uniformly as $k \to \infty$. $\square$

$\boxed{\text{Iteration lemma:}}$ $J(f) = J(f^k)$ $k \geq 1$.

Proof: $J(f) = \{z \in \hat{\mathbb{C}} : \mathcal{F} = \{f^n : n \in \mathbb{N}\}$ is not normal at $z\}$ so $J(f^k) \subseteq J(f)$.

$\boxed{J(f) \subseteq J(f^k)}$ Equivalently $F(f^k) \subseteq F(f)$

① $\boxed{\text{Arzelà-Ascoli Th}^m:}$ $\mathcal{F}$ is normal $\iff$ $\mathcal{F}$ equicontinuous on every compact $K \subseteq \hat{\mathbb{C}}$.

Fix $\varepsilon > 0$

Let $z_0 \in F(f^k)$. $\mathcal{F}_k = \{f^{kn} : n \in \mathbb{N}\}$ is normal at z_0, ① $\Rightarrow$ equicontinuous at z_0. Then

$\exists \delta > 0$: $z \in B(z_0, \delta)$ then $d(f^{kn} z, f^{kn} z_0) < \varepsilon$ for all $n \in \mathbb{N}$.

② Every holomorphic map $\hat{\mathbb{C}} \circlearrowleft$ satisfies a Lipschitz condition.

In particular $f, f^2, \ldots, f^{k-1}$ satisfy a Lipschitz condition with constants $L_1, \ldots, L_{k-1} \leq L$

Then for $z \in B(z_0, \delta)$ we have $d(f^m z, f^m z_0) = d(f^{kn+r} z, f^{kn+r} z_0) \overset{②}{\leq} L\, d(f^{kn} z, f^{kn} z_0) \leq$ ($0 \leq r \leq k-1$)

$\leq L\varepsilon$.

Hence $z_0 \in F(f)$. ∎

$\boxed{\text{Transitivity lemma:}}$ $z \in J(f) \subseteq \hat{\mathbb{C}}$ and $z \in N \subseteq \hat{\mathbb{C}}$ open. Then $U = \bigcup_{n=0}^{\infty} f^n(N)$ contains:

(i) $J(f)$

(ii) $\hat{\mathbb{C}}$ except at most two points (called exceptional Milnor p. 47)

Proof: $\boxed{\text{Montel's Th}^m \text{ (Fundamental Normality test):}}$ $\mathcal{F}$ family of holomorphic functions from $\Omega \to \hat{\mathbb{C}}$ (⊆ $\hat{\mathbb{C}}$)

omitting $a, b \in \hat{\mathbb{C}}$ from their range is normal on Ω.

Assume $|\hat{\mathbb{C}} \setminus U| \geq 3$. Then $|\hat{\mathbb{C}} \setminus U| \geq 2$ and so $\mathcal{F} = \{f^n : n \in \mathbb{N}\}$ is normal on N and so $z \in J(f) \cap F(f)$ ↯

Since $f(U) \subseteq U$ any pre-image of $z \in \hat{\mathbb{C}} \setminus U$ must lie in $\hat{\mathbb{C}} \setminus U$ which is finite hence z is periodic and grand orbit finite hence in the Fatou set. Hence $J \subset U$. ∎

$\boxed{\text{No isolated points lemma:}}$ If f has degree ≥ 2 then $J(f)$ is a perfect set (p. 47)

Proof: $J(f) \neq \emptyset$ see lemma 4.8. $J(f)$ is not finite since it would be a grand orbit hence in $F(f)$, i.e. $|J(f)| = \infty$

Hence $J(f)$ contains a limit point z_0. $\{z \in \hat{\mathbb{C}} \mid f^n(z) = z_0,\ n \in \mathbb{N}\}$ is dense in $J(f)$ (closed of compact hence compact)

It suffices $J(f) = \overline{\bigcup_{k=0}^{\infty} f^{-k}(z_0)}$. Since $z_0 \in J(f)$ then by f-invariance $f^{-k}(z_0) \subset J(f)$ for all $k \in \mathbb{N}$.

Since $J(f)$ is closed $\overline{\bigcup_{k=0}^{\infty} f^{-k}(z_0)} \subseteq J(f)$. Now let $z_1 \in J(f)$ and $z_1 \in N$ open neighborhood. By

transitivity $\exists m \in \mathbb{N}$: $f^m(N) \ni z_1$. Hence $f^{-m}(z_1) \in U$. ∎

$f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

Example of Cantor Julia set: $f(z) = z^2 - 6$

Claim: $J(f) \subseteq \overline{B(0,3)}$

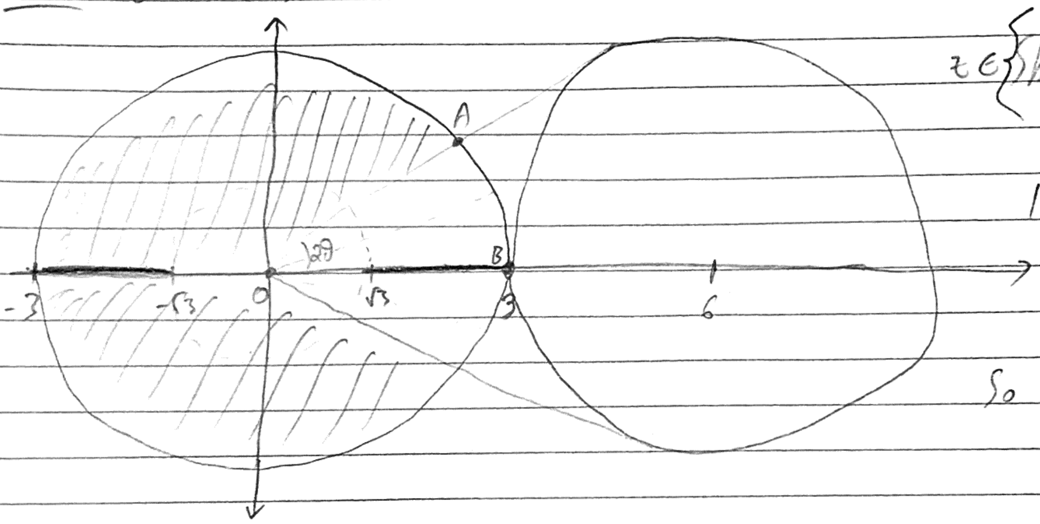
Proof: Let $z \in \hat{\mathbb{C}} \setminus \overline{B(0,3)}$ $\quad |z| > 3$

Then $|f(z)| - |z| = |z^2 - 6| - |z| \geq z^2 - 6 - |z| = |z|(|z| - 1) - 6 \geq 3 \cdot 2 - 6 = 0$

So $|f^n(z)| \to \infty$ so $z \in F(f)$

Claim: $J(f) \cap (-\sqrt{3}, \sqrt{3}) = \emptyset$

Proof: Let $z \in (-\sqrt{3}, \sqrt{3})$ then $f(z) < -3$ so $z \notin J(f)$



$z \in \{\text{shaded region}\} \setminus \mathbb{R} \Rightarrow |f(z)| > 3$

If $z_0 = re^{i\varphi}$ is in OAB then iterating increases the angle φ until some iterate $f^k(z_0) \in \{\text{shaded region}\}$

So $J(f) \subseteq [-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$

Now $I_0 = [-3, -\sqrt{3}]$

$I_1 = (-\sqrt{3}, \sqrt{3})$ $\quad$ then $f(I_0) = I_0 \cup I_1 \cup I_2$

$I_2 = [\sqrt{3}, 3]$ $\quad$ $f(I_2) = I_0 \cup I_1 \cup I_2$

but $J(f) \subseteq I_0 \cup I_2$ and $f(J(f)) = J(f) = f^{-1}(J(f))$

Hence $\bigcup_{k=0}^{\infty} f^{-k}(I_1) \cap J(f) = \emptyset$ i.e. $J(f)$ is a Cantor set.