Attracting/Repelling Fixed Points

Definition (Multiplier) f holomorphic on a neighbourhood of a fixed point $z \neq \infty$. The multiplier of z_0 is defined by $f'(z_0)$. In general, if z_0 is periodic for f, the multiplier is $(f^m)'(z_0)$, where m is the least period of z_0.

If ∞ is the periodic point, it's multiplier is $\lim_{z \to \infty} \frac{1}{(f^m)'(z)}$.

There are four names for periodic orbits, based on their multipliers.
(the multiplier is the same for all points $z_1, \dots, z_m$ in an orbit, as $(f^m)'(z_i) = f'(z_1) \cdots f'(z_m)$)

Orbit multiplier λ	Type of orbit
$\lambda = 0$	Superattracting
$0 \leq \lvert\lambda\rvert < 1$	Attracting (geometrically)
$\lvert\lambda\rvert = 1$	Indifferent
$\lvert\lambda\rvert > 1$	Repelling

Example: $f(z) = z + \frac{1}{z}$ has ~~one superattracting~~ indifferent fixed point at ∞. $f(z) = z^2 - 6$ has two repelling fixed points at $\frac{1 \pm \sqrt{5}}{2}$.

Definition (Basin of Attraction) The basin of attraction of an attracting periodic orbit τ is the open set $A \subset \hat{\mathbb{C}}$ given by $\{z \in \hat{\mathbb{C}} \mid \lim_{n \to \infty} f^{nm}(z) \in \tau\}$, where m is the period of τ.

Lemma: The basin of attraction of any attracting periodic orbit is contained in the Fatou set. Every repelling periodic orbit is in the Julia set.

Proof: Let z_0 be a periodic point period m with multiplier λ. If $|\lambda| > 1$, then $\forall n \in \mathbb{N}$ $|(f^{nm})'(z_0)| = |\lambda^n| \xrightarrow[n \to \infty]{} \infty$ so by Weierstrass Unif. Conv. Thm. no subseq. of $(f^{nm})_{n \in \mathbb{N}}$ may uniformly converge on a nbhd of z_0. By 'Fatou set of f = Fatou set of f^n' the orbit of z_0 is in $J(f)$. We may assume z_0 is a fixed point by
Now suppose $|\lambda| < 1$. By Taylor's theorem, $\exists C, r_0 > 0$ such that $\forall z$ with $|z - z_0| < r_0$, $|f(z) - z_0 - \lambda(z - z_0)| \leq C|z - z_0|^2$. Let $0 < r < r_0$ s.t $|\lambda| + Cr < 1$.

then for z with $|z - z_0| < r$,

$|f(z) - z_0| \leq (|\lambda| + Cr)|z - z_0| < |z - z_0|$

so $|z - z_0| < r \Rightarrow |f(z) - z_0| < r$

and so $|f^n(z) - z_0| \leq (|\lambda| + Cr)^n |z - z_0|$

$\forall n \in \mathbb{N}$, so $f^n \xrightarrow[n\to\infty]{\text{unif}} [z \mapsto z_0]$.

Now, for any $z \in A$, $f^n(z) \xrightarrow[n\to\infty]{} z_0$,

so $\exists N \in \mathbb{N}$ with $|f^N(z) - z_0| < \frac{r}{2}$.

Let $\delta > 0$ s.t $|z - u| < \delta \Rightarrow |f^N(u) - f^N(z)| < \frac{r}{2}$,

so $\forall u \in B(z, \delta)$ $|f^N(u) - z_0| < r$;

so $|f^n(u) - z_0| \leq (|\lambda| + Cr)^{n-N} r$ $\forall n \geq N$,

thus $f^n \xrightarrow[n\to\infty]{\text{unif}} [z \mapsto z_0]$ on $B(z, \delta)$,

so z is normal for f. $\square$

Don't Do

Definition (Parabolic Points) A periodic point $z_0 = f^m(z_0)$ is parabolic if it's multiplier λ is a root of unity, but no iterate of f is the identity map.

Examples: $f(z) = z + \frac{1}{z}$ has a parabolic fixed point at ∞.

$f(z) = \frac{1}{z}$ has no parabolic periodic points as $f^2 = \mathrm{Id}$.

Lemma: Parabolic periodic points are contained in the Julia set.

Definition: A fixed point z_0 of f is topologically attracting if there is a neighbourhood $U \ni z_0$ so that $f^n|_U$ converges uniformly to the constant map $z \longmapsto z_0$.

Lemma: Top. attracting $\Longleftrightarrow$ $|\lambda| < 1$.

Proof: "$\Leftarrow$" - this is just the proof that attracting periodic points are in the Fatou set.

"$\Rightarrow$" As $f^n|_U \xrightarrow{\text{unif}} [z \mapsto z_0]$, any sufficiently small disk $\mathbb{D}(z_0, \varepsilon)$, must satisfy that $\exists n \in \mathbb{N}$ with $f^n\, \mathbb{D}(z_0, \varepsilon) \subsetneq \mathbb{D}(z_0, \varepsilon)$, so f cannot be a rotation, thus by Schwarz Lemma, $|\lambda^n| < 1$, ~~so~~ $|\lambda| < 1$.

$\square$

Theorem 8.2 (Koenigs Linearization)
If the multiplier λ satisfies $|\lambda| \neq 0, 1$, then $\exists$ a local holomorphic conjugacy ϕ (thought of as a change of co-ordinates $\phi(z) = w$) with $\phi(z_0) = 0$, so that $\phi \circ f \circ \phi^{-1} = [w \longmapsto \lambda w]$ for all w in some neighbourhood of the origin. Furthermore, ϕ is unique up to multiplication by a constant.

Proof: It suffices to prove KL for $z_0 = 0$, as this is always the case up to a change of basis. Suppose $|\lambda| < 1$.

By earlier arguments we know $\exists r > 0$ s.t $\forall z \in \mathbb{D}_r$, $|f(z)| \leq c|z|$ for $|\lambda| < c < 1$.
We also have that $\exists C > 0$ s.t $\forall z \in \mathbb{D}_r$, $|f(z) - \lambda z| \leq C|z^2|$. Choose $c < 1$ so that $c^2 < |\lambda| < c$. Then by the above, $n \in \mathbb{N}, z \in \mathbb{D}_r$ gives

$$|f^{n+1}(z) - \lambda f^n(z)| \leq C|f^n(z)|^2$$
$$\leq Cr^2 c^{2n}$$

Let $k = Cr^2/|\lambda|$. Then

$$\left| \frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \right| = \frac{k}{|\lambda^n|} c^{2n} = k\left(\frac{c^2}{|\lambda|}\right)^n$$

Thus
$$(g_n): \mathbb{D}_r \longrightarrow \mathbb{D}_r$$
$$z \longmapsto \frac{f^n(z)}{\lambda^n}$$
is uniforly Cauchy.

so uniformly convergent to some $\phi: \mathbb{D}_r \longrightarrow \mathbb{D}_r$ holomorphic (By Thm 1.4). $\phi(0) = \lim_{n\to\infty} \frac{f^n(0)}{\lambda^n} = 0$.
For $z \in \mathbb{D}_r$

$$\phi(f(z)) = \lim_{n\to\infty} \frac{f^n(f(z))}{\lambda^n} = \lambda \lim_{n\to\infty} \frac{f^{n+1}(z)}{\lambda^{n+1}} = \lambda\phi(z)$$

Under the change of coordinates $\phi(z) = w$, we have $\phi f \phi^{-1}(w) = \lambda w \quad \forall w \in \mathbb{D}_r$.

For $|\lambda| > 1$, there is a neighbourhood of 0 on which f^{-1} is defined* and holomorphic. We apply the previous argument to this map, with multiplier $0 < |\lambda^{-1}| < 1$.

<u>Uniqueness</u>: Suppose ϕ, ψ satisfy the properties, so $\forall w \in \mathbb{D}_r$,

$$\phi \circ f \circ \phi^{-1}(w) = \psi \circ f \circ \psi^{-1}(w) = \lambda w,$$

and $\phi(0) = \psi(0) = 0$.

$$(\phi \circ \psi^{-1}) \circ \psi \circ f \circ \psi^{-1} \circ (\psi \circ \phi^{-1}) = \phi \circ f \circ \phi^{-1}$$
$$= \psi \circ f \circ \psi^{-1}$$

so $\phi \circ \psi^{-1}$ commutes with $w \mapsto \lambda w$.

~~Then~~ $\forall w \in \mathbb{D}_r$, using Taylor expansion

$$\phi \circ \psi^{-1}(w) = b_1 w + b_2 w^2 + b_3 w^3 + \dots$$

By the above, $\lambda b_n = \lambda^n b_n \ \forall n \in \mathbb{N}$,

so as $|\lambda| \neq 0, 1$, $b_n = 0 \ \forall n \geq 2$.

So $\phi \circ \psi^{-1}(w) = b_1 w \implies \phi(w) = b_1 \psi(w)$.

$\square$