A Beginner's Guide to Complex Dynamics

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For pure mathematicians a dynamical system is usually a map, f, from some space X to itself and we study how this evolves under iteration.

Where we begin

Complex dynamics studies the iteration of some function f over the complex numbers, $f : \mathbb{C} \to \mathbb{C}$. In this talk we'll focus mostly on polynomials. We'll be interested in studying the orbit of a complex number z, that is set $\{z, f(z), f(f(z)), \cdots, f^{\circ n}(z), \cdots\}$.

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Definition

A polynomial of degree d is a function $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$.

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Examples: $f(z) = 2$, $g(z) = 8z$, $h(z) = 20z^{1630} + z^{430} + 503032$.

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- $d = 2$, very rich and surprisingly difficult!

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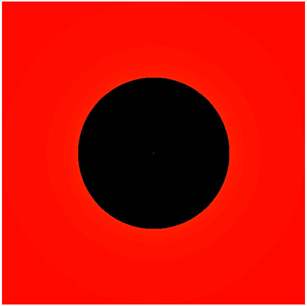
This means we only have to keep track of one parameter. This will be useful later on. We can now begin to look at examples.

First example

Consider $f_0(z) = z^2$



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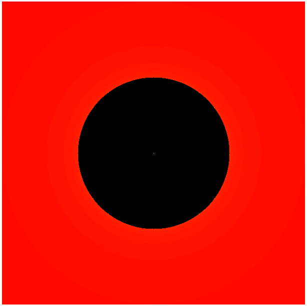


Figure: The decomposition of $\mathbb{C}$ by behaviour of orbits under z^2. The black region is called the filled Julia set.

$0 \mapsto 0,\ 1 \mapsto 1,\ -1 \mapsto 1,\ 0.99 \to \cdots \to 0,\ 1.01 \to \cdots \to \infty$

f_{-1}

Now consider $f_{-1}(z) = z^2 - 1$

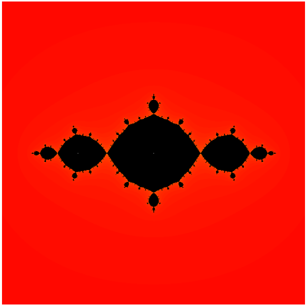
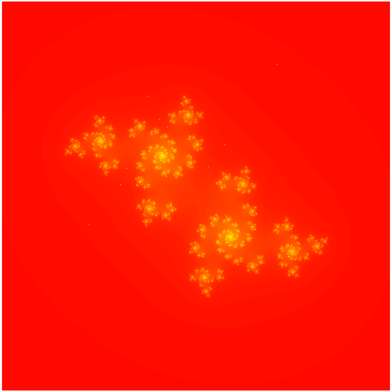
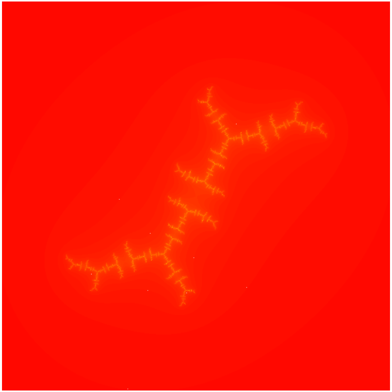


Figure: The filled Julia set for $z^2 - 1$

Now this decomposition is a complicated fractal.

Two fundamentally different examples

Here almost all orbits move to infinity, a Cantor set.

The Mandelbrot Set

The Mandelbrot set $\mathcal{M}$ is the set of points c for which the filled Julia set K_{f_c} of f_c is connected. Equivalently, it is the set of c values for which the orbit of 0 remain bounded i.e.

$$\mathcal{M} = \{c \in \mathbb{C} | \ f_c^{\circ n}(0) \not\to \infty \ \text{ as } n \to \infty\}.$$

The orbit of zero, the critical point, determines the connectivity of K_{f_c}.

A theorem of Tan Lei tells us that near certain points of $\mathcal{M}$, the filled Julia set for the corresponding parameter will look "similar".

The Mandelbrot Set

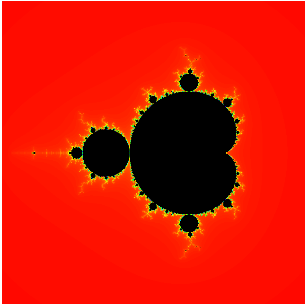


Figure: The Mandelbrot set $\mathcal{M}$.

This video shows the correspondence between $\mathcal{M}$ and J.

My work

I am interested in (roughly) the shape of J for the map $H : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$H = H_{c,a} : (x, y) \mapsto (p_c(x) + ay, x)$$

where $p_c(x) = x^2 + c$, $c \in \mathbb{C}$, and how this varies as we change our parameter values. Computer pictures have been very useful in the one dimensional study so we'd like a similar situation in $\mathbb{C}^2$. We cannot draw such pictures directly, but we may attempt to produce a guiding picture.

See my webpage for some examples.

Thanks for your attention!