Note that Frenkel-Kontorova chains have an economic interpretation too, as dynamic optimisation

Suppose you own a field and can harvest crop at times t_n, $n \in \mathbb{Z}$ of your choosing and you get a profit $h(t_{n-1}, t_n)$ from harvesting at time t_n given previous harvest was at t_{n-1}

Suppose $h(t+1, t'+1) = h(t, t')$

1 = one year and a concavity condition $\partial_1 \partial_2 h > 0$. How to choose $(t_n)_{n \in \mathbb{Z}}$ to maximise "total profit" $\sum_{n \in \mathbb{Z}} h(t_{n-1}, t_n)$?

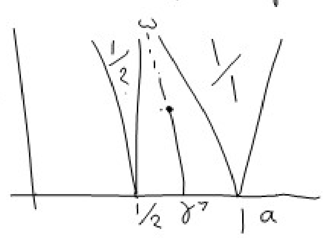
Can interpret this as find $(t_n)_{n \in \mathbb{Z}}$ which maximise $\sum_{n=a}^{b} h(t_{n-1}, t_n)$ over $(t_n)_{a \le n < b}$

$\forall\, a < b \in \mathbb{Z}$

allowing also for insertion or deletion of one or more harvests

The optimal schedules have a mean spacing $\lim_{n-m \to \infty} \frac{t_n - t_m}{n-m}$ and as a function of parameters of h, get phase diagram like

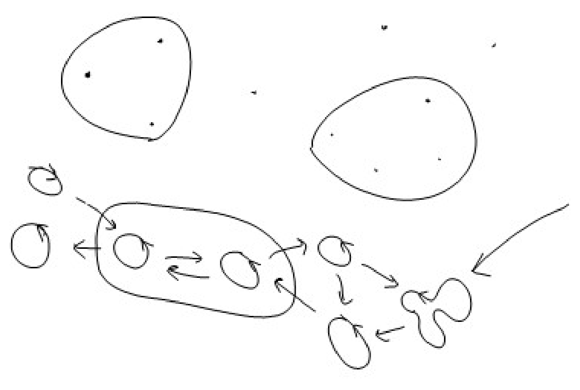
$t_n = \tau(n\omega + \theta)$



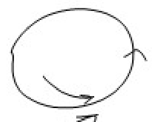
Back to the pdf §4 Interacting agents.

pre-order $x \le y$ & $y \le x$ $\not\Rightarrow$ $x = y$ Write $x \sim y$

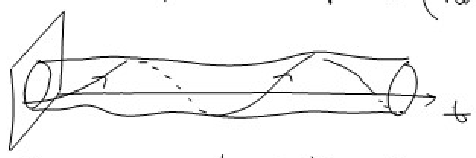
order: $x \le y$ & $y \le x \Rightarrow x = y$.



Autonomous oscillator = limit cycle



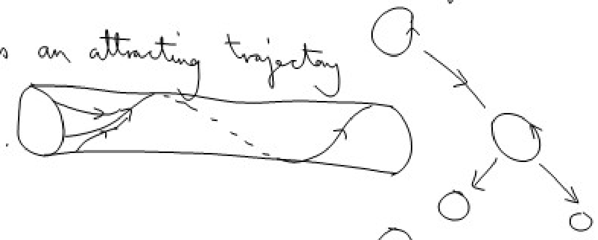
Non-autonomous oscillator =
attracting cylinder in state space × time
for each input function of time (not too strong)



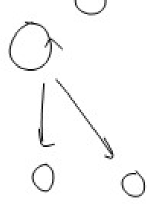
Use normal hyperbolicity theory.

Say 2 oscillators *phase-lock* if this attracting $\mathbb{T}^2 \times \mathbb{R}$ contains an attracting $\mathbb{T}^1 \times \mathbb{R}$. So can replace the pair of oscillators by one effective oscillator.

Also can eliminate some oscillators:
if dynamics on invariant cylinder has an attracting trajectory



Next I'll go into normal hyperbolicity theory



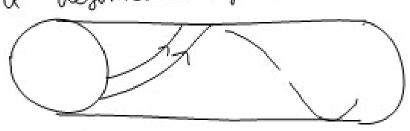
Consider
$$\begin{cases} \frac{d\theta}{dt} = \dot{\theta} = \Theta(\theta, r, t) & \theta \in \mathbb{T}^m \\ \frac{dr}{dt} = \dot{r} = R(\theta, r, t) & r \in (\mathbb{R}^n, 0) \end{cases}$$
$t \in \mathbb{R}$

Unforced case (autonomous), $m = 1$
choose coordinates so that $\theta(t) = \omega t + \theta(0)$, $r(t) = 0$ on the limit cycle $\theta = t/T \ast 2\pi$

period T

Limit cycle is *hyperbolic* if time-T map ϕ for linearised dynamics
$$\dot{\xi} = R_r(\omega t, 0, t)\,\xi$$
in r-directions has no eigenvalue on unit circle
$$R_r = \left(\frac{\partial R_i}{\partial r_j}\right)$$

If add forcing then hope to prove there is an invariant cylinder near $r=0$, a deformation of :



$\theta(t) = \omega t + \theta(0)$ $\longrightarrow t$
$r(t) = 0$

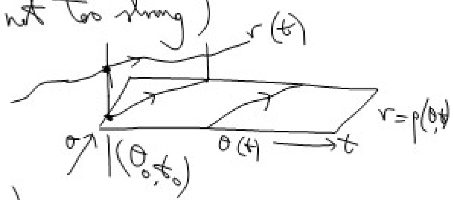
The strategy is a contraction map T on a space of candidates : graphs $r = \rho(\theta, t)$ of suitable functions ρ . $\tilde{\rho} = T\rho$

For any (θ_0, t_0) take trajectory $\theta(t)$ of $\dot{\theta} = \Theta(\theta, \rho(\theta, t), t)$ from (θ_0, t_0)

Then solve $\dot{r} = R(\theta(t), r, t)$ for the locally unique trajectory $r(t)$

($\exists$ & ! "existence and uniqueness" comes from assuming hyperbolicity of the limit cycle if forcing is not too strong)

Define $\tilde{\rho}(\theta_0, t_0) = r(t_0)$



To obtain the locally unique $r(t)$ use hyperbolicity theory

Write in general setting and do linear case : $\dot{x} = A(t)x \quad x \in V$

Write the matrix solution X of $\dot{X} = A(t)X$ from initial condition $X(s,s) = I$ as $X(t,s)$: $\partial_1 X(t,s) = A(t)X(t,s)$

Then solution $x(t)$ from initial condition $x(s)$ is

$$x(t) = X(t,s)\,x(s).$$

Note that $X(t,s)X(s,t) = I$

so differentiate wrt s & get $\partial_2 X(t,s) = -X(t,s)A(s)$

e.g. trajectory $y(t)$ of vector field $\dot{y} = v(y,t)$

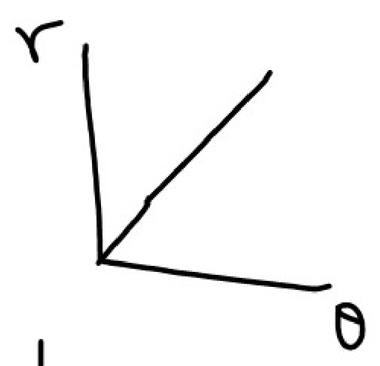
take $A(t) = v_y(y(t),t)$

Here we will need 3 other ways that $\dot{x} = A(t)x$ arises: ① linearised normal dynamics

$$A(t) = R_r(\theta(t), r(t), t)$$

② slope dynamics linearised about 0 :

a <u>slope</u> is a linear map S from θ to r



Slope evolves by a Ricatti equation

$$\dot{S} = R_\theta + R_r S - S\,\Theta_\theta - S\,\Theta_r S$$

Linearise about 0 gives $\dot{\sigma} = A(t)\sigma = R_r\sigma - \sigma\Theta_\theta$

③ slope dynamics modified: given graph

$r = \rho(\theta, t)$ consider

$$\dot{\sigma} = A(t)\sigma = R_r\sigma - \sigma\left(\Theta_\theta + \Theta_r\,\rho_\theta\right)$$