Topics   Authors

Eqm Stat Mech   Tom   Miasu

Markov processes   Aditya   W. J. Stewart

Shortest routes
Traffic flow
} Chris   Aleksandar?

Multiagent games   Dayal, Dominic

Oscillator networks   Dhruvraj Kosuk, Michael, Defne, Alistair

Others e.g. financial networks, electricity distn

29 Feb

Oscillator Networks

Uniform Hyperbolicity with a view to normal hyperbolicity

Need to study uniform hyperbolicity for "cocycles" over a dynamical system

Do in discrete-time because easier.

Suppose $f : M \circlearrowleft$ C^1 diffeomorphism (can do non-autonomous case $(f_t)_{t \in \mathbb{Z}}$ sequence of such)

Define $F : M^{\mathbb{Z}} \circlearrowleft$ by

$$F(x)_t = f_{t-1}(x_{t-1}) \quad x \in M^{\mathbb{Z}}$$

$$x = (x_t)_{t \in \mathbb{Z}},\ x_t \in M$$

Orbits/trajectories of $f \leftrightarrow$ fixed points of F

Say an orbit x (or set of orbits) is u.hyp. for f if x is a uniformly non-degenerate fixed point of F : $I - DF_x$ invertible &

$\exists K > 0$ s.t. $\|(I - DF_x)^{-1}\| \leq K^{-1}$

wrt sup norm $\|\xi\| = \sup_t |\xi_t|$ $\quad |\xi_t|$ is a norm on TM

This says forced linearised dynamics $\xi_{t+1} = f'_{t,x_t}\xi_t + \eta_t$ <u>has</u> <u>unique</u> bounded soln ξ for bdd forcing η & $\|\xi\| \leq K^{-1}\|\eta\|$.

Generalise to u.hyp of a "co-cycle". Cocycles are the matrix solns of linear dynamics

$$\xi_{t+1} = A_t(x_t)\,\xi_t \qquad \text{for}$$

ξ_t now in an arbitrary vector bundle V over M and $A : M \to L(V,V)$ an arbitrary (cont.) matrix-valued fn on M. (generalising the case of $V = TM$ and $A = f'$)

We say the co-cycle is u.hyp if defining $B_x : V^{\mathbb{Z}} \to V^{\mathbb{Z}}$ by $B[\xi]_t = A(x_{t-1})\,\xi_{t-1}$

then $\exists K > 0$ s.t. $\|(I - B_x)^{-1}\| \leq K^{-1}$

Let $L_x = I - B_x$, so

$$L[\xi]_t = \xi_t - A_{t-1}\,\xi_{t-1}. \qquad A_{t-1} = A(x_{t-1})$$

By analogue for discrete-time of previous theory, if $\|L_x^{-1}\| \leq K^{-1}$ then

$$L^{-1}[\eta]_t = \sum_{s \in \mathbb{Z}} G_{t,s}\,\eta_s \quad \text{for a Green function } G \text{ \&}$$

$$|G_{t,s}| \leq C\mu^{|t-s|} \quad \text{some } \mu < 1$$

Say x is a pseudo-orbit for $f: M \circlearrowleft$

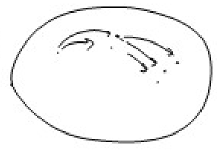
if $x_{t+1} = f(x_t) + \delta_t$

some bdd seq $\delta = (\delta_t)_{t \in \mathbb{Z}}$ $\|\delta\| \leq \delta$

More precisely, $d(x, F(x)) \leq \delta$

for d a metric on M. e.g. change f a bit to some nearby $\tilde{f}$ & take orbits of $\tilde{f}$.

Want to show that for δ small enough the cocycle remains u.hyp for all pseudo-orbits & with $\|L^{-1}\| \leq \tilde{K}^{-1}$ with $\tilde{K}$ only slightly smaller than K, assuming cocycle is u.hyp for every orbit of f and $\|L^{-1}\| \leq K^{-1}$.



Strategy of proof is to make operators R, Q which are bounded & approx. right (left) inverses for L on the pseudo-orbit i.e. $\|I - LR\| \leq \varepsilon_R < 1$

$\|I - QL\| \leq \varepsilon_Q < 1$. cf. Palmer

Then (LR) is invertible & $\|(LR)^{-1}\| \leq \frac{1}{1-\varepsilon_R}$

(QL) is invertible & $\|(QL)^{-1}\| \leq \frac{1}{1-\varepsilon_Q}$

So $R(LR)^{-1}$ is a true right inverse to L:

$$L(R(LR)^{-1}) = I$$

And $(QL)^{-1}Q$ is a true left inverse to L

So L is invertible & $L^{-1} = R(LR)^{-1} = (QL)^{-1}Q$

& $\|L^{-1}\| \leq \frac{\|R\|}{1-\varepsilon_R}$ & $\leq \frac{\|Q\|}{1-\varepsilon_Q}$

My choice is $R_{ts} = G^s_{ts}$ for $t-T \leq s < t+T$, 0 otherwise

where G^s is the Green fn for true orbit of x_s

for a T large (if $\delta \ll 1$) to be determined

& $Q_{ts} = G^t_{ts}$ for $t-T \leq s < t+T$, 0 o/w

Then $$LR[\eta]_t = \sum_{t-T \leq s < t+T} G^s_{ts} \eta_s - A_{t-1} \sum_{t-T-1 \leq s < t+T-1} G^s_{t-1,s} \eta_s$$

Get $(I - LR)[\eta]_{t+1} =$

$$\Big(\sum_{t+1-T \leq s < t+T} (A_t - A_t^s)\, G^s_{ts}\, \eta_s \Big) + A_t\, G^{t-T}_{t,t-T}\, \eta_{t-T} - A_t^{t+T}\, G^{t+T}_{t,t+T}\, \eta_{t+T}$$

Bound $A_t - A_t^s$ using $x_t - x_t^s = f(x_{t-1}) + \delta_{t-1} - f(x^s_{t-1})$

(see notes) & $|f'| \leq \lambda$,

to get $|A_t - A_t^s| \leq \ell \delta \dfrac{\lambda^{t-s}}{\lambda - 1}$ for $t > s$

Choose $T \approx \dfrac{\log 1/\delta}{\log \lambda}$ (if $\lambda, \mu > 1$)

& get $\|I - LR\| \leqslant O\left(\delta^{\frac{\log 1/\mu}{\log \lambda}}\right)$

Ending proving $\|L^{-1}\| \leqslant \dfrac{1}{K - O\left(\delta^{\frac{\log 1/\mu}{\log \lambda}}\right)}$

Next week: no lecture!

Meet again on 17 Nov