Uniform hyperbolicity of cocycles for pseudo-orbits in discrete time

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1 Discrete-time

Suppose $f : M \to M$ is a C^1 diffeomorphism of a manifold M with a norm on its tangent bundle. We can allow f to be time-dependent with small notational change, but the extension will be clear once the autonomous case is understood.

Define $F : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ by $F(x)_t = f(x_{t-1})$ and use supremum norm for sequences of tangent vectors. Orbits of f correspond to fixed points of F. Uniformly hyperbolic orbits of f correspond to non-degenerate fixed points of F ($I - DF$ invertible with bounded inverse).

More generally, suppose $A : M \to L(V, V)$ is a continuous matrix function on M, acting on a normed vector space V (really we should make it a vector bundle over M). We suppose A is Lipschitz, with Lipschitz constant ℓ (though any module of continuity would suffice). An example is the derivative f', with V being the tangent bundle TM.

We study the linear dynamics on V generated by

$$\xi_{t+1} = A(x_t)\xi_t,$$

for orbits or pseudo-orbits $x = (x_t)_{t\in\mathbb{Z}}$ of f. Its matrix solutions (i.e. taking $\xi_t \in L(V, V)$ instead of just V are called a *cocycle* for f.

We say that the cocycle is *uniformly hyperbolic* for a sequence $x = (x_T)_{t\in\mathbb{Z}}$ on M (or a set of such sequences and matrix functions A) if there exists $K > 0$ such that for the operator B_x on sequences $\xi = (\xi_t)_{t\in\mathbb{Z}}$ in V defined by

$$B_x[\xi]_t = A(x_{t-1})\xi_{t-1}$$

then $I - B_x$ is invertible with $\|(I - B_x)^{-1}\| \leq K^{-1}$.

For $x \in M^{\mathbb{Z}}$, let $L_x = I - B_x$. So

$$L_x[\xi]_t = \xi_t - A_{t-1}\xi_{t-1}$$

where A_{t-1} is short for $A(x_{t-1})$.

By preceding theory, if $\|L^{-1}\| \leq K^{-1}$ then $L^{-1}[\eta]_t = \sum_{s\in\mathbb{Z}} G_{ts}\eta_s$ for some matrix function G_{ts} called the Green function, satisfying

$$|G_{ts}| \leq C\mu^{|t-s|} \tag{1}$$

for some $C > 0$ and $\mu < 1$, related to K.

From $LL^{-1} = I$ we obtain

$$G_{ts} = A_{t-1} G_{t-1,s} + \delta_{ts}, \tag{2}$$

where δ_{ts} is the identity matrix for $t = s$, zero otherwise. From $L^{-1}L = I$ we obtain

$$G_{ts} = G_{t,s+1} A_s + \delta_{ts}. \tag{3}$$

Suppose x is a *pseudo-orbit*, i.e. $d(F(x), x) \leq \delta$ small, with each $x_t \in \Lambda$, some uniformly hyperbolic set for A invariant under f (could allow x_t close to Λ later). We wish to prove that x is uniformly hyperbolic for A with only slightly smaller K.

The strategy (cf. [Pa]) is to make an approximate right inverse R for L, in the sense that $\|I - LR\| \leq \varepsilon_R < 1$, and an approximate left inverse Q, $\|I - QL\| \leq \varepsilon_Q < 1$. Then LR is invertible with $\|(LR)^{-1}\| \leq (1-\varepsilon_R)^{-1}$ and QL is invertible with $\|(QL)^{-1}\| \leq (1-\varepsilon_Q)^{-1}$. So $R(LR)^{-1}$ is a true right inverse to L and $(QL)^{-1}Q$ is a true left inverse to L. Then L is invertible and $\|L^{-1}\| \leq \|R\|/(1-\varepsilon_R)$ and $\|Q\|/(1-\varepsilon_Q)$ (assuming R or Q is bounded).

We take

$$R_{ts} = G^s_{ts}, \; Q_{ts} = G^t_{ts} \text{ for } t - T \leq s < t + T, \tag{4}$$

for some T to be determined (could allow different $T_\pm$), zero otherwise, where for any $u \in \mathbb{Z}$, G^u is the Green function for the true orbit of x_u.

Then

$$LR[\eta]_t = \sum_{t-T \leq s < t+T} G^s_{ts} \eta_s - A_{t-1} \sum_{t-1-T \leq s < t-1+T} G^s_{t-1,s} \eta_s.$$

Substitute (2) in the first sum, but using the label A^s to indicate that A is evaluated along the orbit of x_s. Then, shifting t to $t+1$ to simplify the expression,

$$(I - LR)[\eta]_{t+1} = \left(\sum_{t+1-T \leq s < t+T} (A_t - A^s_t) G^s_{t,s} \eta_s \right) + A_t G^{t-T}_{t,t-T} \eta_{t-T} - A^{t+T}_t G^{t+T}_{t,t+T} \eta_{t+T}. \tag{5}$$

To bound $A_t - A^s_t$ we use that A is Lipschitz with Lipschitz constant ℓ and that

$$x_t - x^s_t = f(x_{t-1}) + \delta_{t-1} - f(x^s_{t-1}),$$

where x^s denotes the orbit of x_s and $|\delta_{t-1}| \leq \delta$. So

$$|x_t - x^s_t| \leq \lambda |x_{t-1} - x^s_{t-1}| + \delta,$$

where λ is an upper bound for $|f'|$. For $t > s$ this implies $|x_t - x^s_t| \leq \delta \frac{\lambda^{t-s}-1}{\lambda - 1}$. For simplicity we will choose $\lambda > 1$ and use the bound $|x_t - x^s_t| \leq \delta \frac{\lambda^{t-s}}{\lambda-1}$. So for $t > s$ we have

$$|A_t - A^s_t| \leq \ell \delta \frac{\lambda^{t-s}}{\lambda - 1}.$$

For $t < s$ we use instead

$$x_t - x_t^s = f^{-1}(x_{t+1} - \delta_t) - f^{-1}(x_{t+1}^s)$$

to obtain

$$|x_t - x_t^s| \leq \lambda(|x_{t+1} - x_{t+1}^s| + \delta),$$

where we have chosen λ to also be an upper bound on $|f^{-1\prime}|$ (could use a separate constant). Thus for $t < s$ we obtain $|x_t - x_t^s| \leq \delta \frac{\lambda^{s-t+1} - \lambda}{\lambda - 1}$. Again we will bound this by just $\delta \frac{\lambda^{s-t+1}}{\lambda - 1}$. So for $t < s$ we have

$$|A_t - A_t^s| \leq \ell\delta \frac{\lambda^{s-t+1}}{\lambda - 1}.$$

Thus the part of the sum in (5) with $s < t$ is bounded by

$$\frac{\ell\delta C|\eta|}{\lambda - 1} \frac{(\lambda\mu)^T - \lambda\mu}{\lambda\mu - 1},$$

where the second ratio is interpreted as $T - 1$ if $\lambda\mu = 1$. Similarly we obtain λ times this as a bound on the forward part of the sum. The term with $s = t$ is zero. So we obtain

$$|\sum_{t+1-T \leq s < t+T} (A_t - A_t^s) G_{t,s}^s \eta_s| \leq \frac{(\lambda + 1)\ell C\delta}{\lambda - 1} \frac{(\lambda\mu)^T - \lambda\mu}{\lambda\mu - 1} |\eta|.$$

The boundary terms in (5) are each bounded by $\lambda C \mu^T |\eta|$. Thus

$$\|I - LR\| \leq \frac{\lambda + 1}{\lambda - 1} \ell C\delta \frac{(\lambda\mu)^T - \lambda\mu}{\lambda\mu - 1} + 2\lambda C\mu^T.$$

We choose T roughly to minimise the RHS by making the two terms comparable. The solution depends on the size of $\lambda\mu$ relative to 1. We could always take a big over-estimate of λ to achieve $\lambda\mu > 1$ but for the purposes of not throwing away too much in our estimates, we'll consider the three cases.

If $\lambda\mu$ is significantly larger than 1 then we take $\lambda^T \delta \approx 1$, so $T \approx \frac{\log 1/\delta}{\log \lambda}$. This is also the largest we can take T to be sure that x_t remains in a local chart around x_t^s. Then $\|I - LR\|$ has size of order $\mu^T \approx \delta^{\frac{\log 1/\mu}{\log \lambda}}$, which goes to zero as $\delta \to 0$.

If $\lambda\mu$ is significantly less than 1 then we take $\delta \approx \mu^T$, so $T \approx \frac{\log \delta}{\log \mu}$. Then $\|I - LR\|$ has size of order δ, which goes to zero faster.

If $\lambda\mu$ is near 1 then we solve $T\delta \approx \mu^T$ which makes T a little larger than $\frac{\log \delta}{\log \mu}$ and $\|I - LR\|$ of order $\delta \frac{\log \delta}{\log \mu}$ which still goes to zero with δ.

Next we bound R. The easiest is just to use the bounds (1) on the Green function. So $\|R\| \leq \sum_{t-T \leq s < t+T} C\mu^{t-s} \leq \frac{1+\mu}{1-\mu} C$. One could do much better by comparing G^s with G^t and using $|\sum_{s \in \mathbb{Z}} G_{ts}^t \eta_s| \leq K^{-1}|\eta|$ and the Green function estimates to bound the tails, but the comparison of G^s with G^t requires some more work, and we can instead use Q to obtain a tight bound on the final result.

For the left inverse, first we bound Q. Recall $Q[\eta]_t = \sum_{t-T\leq s<t+T} G^t_{ts}\eta_s$. If we took the sum over all $s \in \mathbb{Z}$ then we would have $(L^t)^{-1}[\eta]_t$, which is bounded by $K^{-1}|\eta|$, where L^t is the operator corresponding to the orbit of x_t. The sum of the added tails is bounded by $C\frac{1+\mu}{1-\mu}\mu^T|\eta|$. Thus

$$\|Q\| \leq K^{-1} + C\frac{1+\mu}{1-\mu}\mu^T.$$

Next we bound $I - QL$.

$$QL[\xi]_t = \sum_{t-T\leq s<t+T} G^t_{ts}(\xi_s - A_{s-1}\xi_{s-1}).$$

Substituting (3) for the $G^t_{ts}\xi_s$ term (but using A^t), we obtain $(I - QL)[\xi]_t =$

$$\sum_{t-T\leq s\leq t+T-2} G^t_{t,s+1}(A_s - A^t_s)\xi_s + G^t_{t,t-T}A_{t-T-1}\xi_{t-T-1} - G^t_{t,t+T}A^t_{t+T-1}\xi_{t+T-1}.$$

Reversing the roles of s and t in the analysis for $A_t - A^s_t$ above, we bound $|x_s - x^t_s| \leq \frac{\delta}{\lambda-1}\lambda^{s-t}$ for $s > t$ and by $\frac{\delta}{\lambda-1}\lambda^{t-s+1}$ for $s < t$. So

$$\|I - QL\| \leq \frac{1+1/\lambda}{\lambda-1}C\ell\delta\frac{(\lambda\mu)^T - \lambda\mu}{\lambda\mu - 1} + 2C\lambda\mu^T.$$

The same choice of T as before makes these two terms comparable and so $\|I - QL\|$ is of order $\delta^{\frac{\log 1/\mu}{\log\lambda}}$ again if $\lambda\mu > 1$ (and the corresponding expressions for the other two cases).

Thus L is invertible for δ small enough, and

$$\|L^{-1}\| \leq \frac{\|Q\|}{1-\varepsilon_Q} \leq \frac{K^{-1} + O(\delta^{\frac{\log 1/\mu}{\log\lambda}})}{1 - O(\delta^{\frac{\log 1/\mu}{\log\lambda}})} = \frac{1}{K - O(\delta^{\frac{\log 1/\mu}{\log\lambda}})}$$

as desired, where we've taken the case $\lambda\mu > 1$ (else change the error term to $O(\delta)$ or $O(\delta\frac{\log\delta}{\log\mu})$).

There are other possible choices of R and Q, e.g. using the projections for the true orbits multiplied by the product of derivatives for the pseudo-orbit. This simplifies some of the analysis, but produces worse approximations.

References

[Pa] Palmer K, Shadowing in dynamical systems (Kluwer, 2000)