Cerbelli and Giona's map is pseudo-Anosov

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Abstract

It is shown that a piecewise affine area-preserving homeomorphism of the 2-torus studied by Cerbelli and Giona is pseudo-Anosov. This enables one to prove various of their conjectures and compute its topological entropy.

1 Introduction

In [CG], an interesting example of an area-preserving homeomorphism of a 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is introduced:

$$\begin{aligned} x' &= x + f(y) \\ y' &= y + x' \end{aligned} \tag{1}$$

with $f(y) = 2y$ if $y \in [0, \frac{1}{2}]$ mod $\mathbb{Z}$, $2 - 2y$ if $y \in [\frac{1}{2}, 1]$ mod $\mathbb{Z}$. They calculate explicitly many of its dynamical features. In particular, they construct explicit invariant foliations $\mathcal{F}^{\pm}$ whose leaves are contracted exponentially in forwards and backwards time respectively. They prove it to be mixing with respect to area, and to have positive Lyapunov exponent. They prove it has a sign-alternation property, namely the leaves pass in opposite directions arbitrarily close to any point, so it is not topologically conjugate to Anosov. The authors propose it as an archetype of continuous area-preserving maps exhibiting "non-uniform chaos".

The purpose of this note is to show that (1) belongs to a well established class of maps, known as *pseudo-Anosov*. This opens up a body of theory to apply to it.

2 Pseudo-Anosov maps

A *pseudo-Anosov map* is a homeomorphism f of a compact surface for which there exists $\lambda > 1$ and a transverse pair of invariant continuous foliations $\mathcal{F}^{\pm}$ carrying transverse measures $\mu^{\pm}$ which are expanded by precisely λ each iteration of $f^{\pm 1}$, respectively, but which possess a finite non-zero number of singularities at each of which $\mathcal{F}^{\pm}$ are modelled up to homeomorphism on the curves of constant real and imaginary parts (modulo sign if k is odd) of $z^{k/2}$ near $z = 0$ in $\mathbb{C}$ for some $k \in \mathbb{N}\backslash\{2\}$ (called a *k-prong* singularity).

Some authors allow the case of no singularities, but such maps are essentially Anosov and although some results apply to both, the Anosov case has to be excluded for others. Some authors require f to be C^∞ except at the singularities, but this seems unnecessary to me. Some authors call a pseudo-Anosov map with non-empty set S of 1-prongs "pseudo-Anosov relative to S". Most authors also allow an extension to compact surfaces with boundary, but this possibility is not required for the purposes of this note.

Simple examples of pseudo-Anosov homeomorphisms are provided by choosing any hyperbolic automorphism A of the 2-torus and identifying points of T^2 under reflection through $(0,0)$. This yields a pseudo-Anosov homeomorphism of the 2-sphere S^2. The dilatation λ is the modulus of the eigenvalue of A larger than one, $\mathcal{F}^\pm$ are the images of the lines parallel to the backwards and forwards contracting eigenspaces of A under the identification, $\mu^\pm$ are given by length on $\mathbb{T}^2$ of a leaf of $\mathcal{F}^\mp$ respectively, and there are precisely 4 singularities, all 1-prongs, namely the points $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, which are their own reflections through $(0,0)$.

Pseudo-Anosov homeomorphisms have many interesting dynamical properties. In particular, they have a Markov partition, meaning a decomposition up to overlap along the edges, into sets bounded by segments of leaves from $\mathcal{F}^\pm$ such that any $\mathcal{F}^+$ boundary segment is mapped by f inside another one and any $\mathcal{F}^-$ boundary segment is mapped by f^{-1} inside another one. This leads to a correspondence of the dynamics, up to ambiguities for boundary points, with a subshift of finite type (topological Markov chain). They are topologically mixing and have positive topological entropy, in fact $h_{\text{top}} = \log\lambda$. If f has a singularity with odd k, then the leaves of each foliation fold back across themselves in opposite directions. The product of $\mu^\pm$ is an invariant measure μ; it is mixing and is the measure of maximal entropy. Any continuous perturbation g of a pseudo-Anosov map f preserving the set of 1-prongs has at least as much dynamics (more precisely, it has an invariant subset Y with a semi-conjugacy of $g|_Y$ to f).

For introductions to pseudo-Anosov maps, see [B, M].

3 Proof that (1) is pseudo-Anosov

From figure 4 of [CG] showing the invariant foliations, it can be seen that there are precisely four singularities: $(0,0)$ and $(0,\frac{1}{2})$ are 1-prongs, $(\frac{1}{2},0)$ and $(\frac{1}{2},\frac{1}{2})$ are 3-prongs (the homeomorphism to the standard models is piecewise smooth).

To identify the dilatation factor λ and determine the transverse measures $\mu^\pm$, note that it is possible to construct a Markov partition by extending leaves of $\mathcal{F}^\pm$ from the singularities judiciously. Figure 1 shows the one to be used here, though it might be possible to simplify it (e.g. reduce the number of partition elements).

Let Γ be the directed graph indicating which partition elements are traversed by the images of each under f (shown in Figure 2). If the image of one region were to cross another one more than once, the crossings would count individually and without sign, but for this example all traversals are simple.

Let M be the matrix indicating the numbers M_{ij} of traversals of the image of region i over region j:

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{2}$$

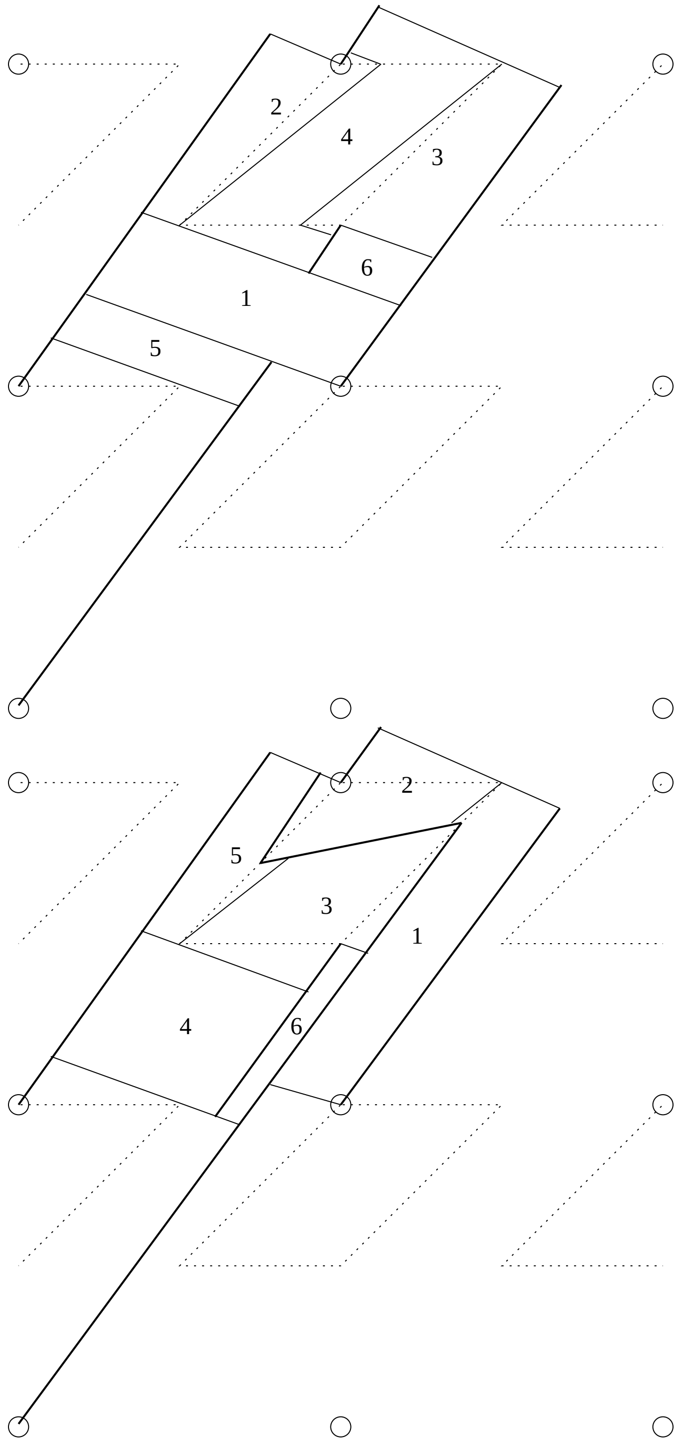

Figure 1: A Markov partition for (1) and its image. Thick lines belong to $\mathcal{F}^-$ and thin lines to $\mathcal{F}^+$. The leaves are straight and parallel except for changes in slope at each intersection with the dotted lines.

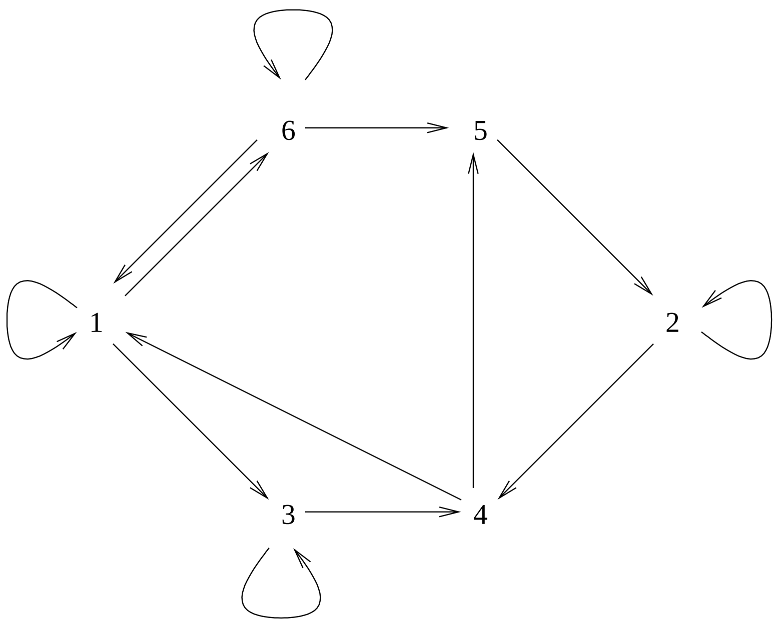

Figure 2: The graph Γ of allowed transitions for the Markov partition of Figure 1.

Since M is non-negative, irreducible and aperiodic, it has a unique eigenvalue λ of maximum modulus and λ is positive (in fact greater than 1 because $\det M \in \mathbb{Z}\backslash\{0\}$), $\lambda \approx 2.296630262$. This is the dilatation factor for the map. To specify it exactly, λ is the largest root of the factor $x^4 - 2x^3 - 2x + 1$ of the characteristic polynomial of M.

Furthermore the eigenvalue λ is simple and has an eigenvector with all components positive. The components of the eigenvector give the weights μ_j^+ for the bundle of leaves of $\mathcal{F}^+$ crossing the region j (up to an arbitrary normalisation). To three significant figures, the weights are $(0.832, 0.327, 0.327, 0.424, 0.143, 0.752)$. To determine the weight of an arbitrary interval I of leaves of $\mathcal{F}^+$, apply f^n for $n > 0$ to the subset of some region representing I and let $\mu_n^+(I)$ be the product of λ^{-n} and the sum of the weights of the regions fully traversed by $f^n(I)$; then $\mu^+(I) = \lim_{n\to\infty} \mu_n^+(I)$ exists and is independent of the choice of initial region.

The measure μ^- is constructed similarly, using the transition matrix for f^{-1} (which is the transpose of M, so the same λ is obtained).

Thus, the defining properties of a pseudo-Anosov homeomorphism are verified for the Cerbelli and Giona map.

4 Some Consequences

One consequence of the above proof is that it gives the exact value for the topological entropy, namely $h_{\text{top}} = \log \lambda \approx 0.831443$, for which [CG] obtained a numerical approximation (which agrees well with this) and a lower bound.

Another consequence is that it constructs the measures $\mu^\pm$, which are the "w-measures" of [CG], hence allowing analysis of their multifractal nature. In particular, it proves the conjecture of [CG] that they are singular. More precisely in any curve transverse to $\mathcal{F}^+$ there is a subset of full μ^+-measure and zero Lebesgue measure, because there are periodic points with different positive Lyapunov exponent (for the idea of a proof of this implication, see Corollary 1 in section 4 of [S2], which although for Anosov diffeomorphisms, depends really only on the existence of a Markov partition and piecewise smoothness). To see the existence of periodic orbits with different exponents, the analysis of [CG] shows that the positive Lyapunov exponent of any orbit is $\alpha \log(2+\sqrt{3})$, where α is the fraction of time spent in $x \in [0, \frac{1}{2}]$; there are periodic orbits with α ranging from $\frac{1}{3}$ to 1 (for the extremes, the orbits are on discontinuities of the derivative of the map, but they still have well defined one-sided exponents). The same holds for μ^- using negative Lyapunov exponents. The multifractal nature of $\mu^\pm$ could be quantified by (for μ^+) letting S_i be a transversal to $\mathcal{F}^+$ in region i (e.g. a leaf of $\mathcal{F}^-$) and considering the expanding map on $S = \cup_i S_i$ induced by applying f and then sliding along the $\mathcal{F}^+$ foliation (similarly for μ^- using f^{-1}).

A third consequence is that the product of $\mu^\pm$ is the measure of maximal entropy. This result for a general pseudo-Anosov map is quoted in [B]; a proof can be constructed along the lines of [S1], which although for Anosov diffeomorphisms, depends really only on the existence of a Markov partition and piecewise smoothness.

Another consequence is that it allows an understanding of continuous perturbations fixing the 1-prongs. Firstly, they possess at least all the dynamics of f (the construction of the invariant subset Y and semi-conjugacy from $g|_Y$ to f is relatively explicit [Ha]). Secondly, smooth perturbations can be made which remain pseudo-Anosov [GK], or which fatten chosen leaves ending on singularities into strips containing additional dynamics ("derived from pseudo-Anosov" maps).

5 Final Remarks

It would be interesting to investigate whether some other examples, like those of Wojtkowski [W1, W2], might be pseudo-Anosov for selected parameter values. Note that Burton and Easton's [BE] are not (they are essentially Anosov with a fixed point blown up into a square of fixed points).

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