Natural flat observer fields in spherically-symmetric space-times

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An *observer field* in a space-time is a time-like unit vector field. It is *natural* if the integral curves (field lines) are geodesic and the perpendicular 3–plane field is integrable (giving normal space slices). It then follows that the field determines a coherent notion of time: a time coordinate that is constant on the perpendicular space slices and whose difference between two space slices is the proper time along any field line.

A natural observer field is *flat* if the normal space slices are metrically flat. We find a dual pair of spherically-symmetric natural flat observer fields for a large family of spherically-symmetric space-times including Schwarzschild and Schwarzschild-de-Sitter space-time. In these cases one of these observer fields is expanding and the other contracting and it is natural to describe the expanding field as the "escape" field and the dual contracting field as the "capture" field. Observer fields are a useful setting for understanding redshift and the fields described here are used in a possible explanation of redshift explored in [5].

83C20; 83C15, 83C40, 83C57, 83F05

Introduction

A pseudo-Riemannian manifold L is a manifold equipped with non-degenerate quadratic form g on its tangent bundle called the *metric*. A *space-time* is a pseudo-Riemannian 4–manifold equipped with a metric of signature $(-,+,+,+)$. The metric is often written as ds^2, a symmetric quadratic expression in differential 1–forms. A tangent vector v is *time-like* if $g(v) < 0$, *space-like* if $g(v) > 0$ and *null* if $g(v) = 0$. The set of null vectors at a point form the *light-cone* at that point and this is a cone on two copies of S^2. A choice of one of these determines the *future* at that point and we assume *time orientability*, ie a global choice of future pointing light cones. An *observer field* on a space-time L is a smooth future-oriented time-like unit vector field on L.

We consider space-times which admit metrics of the form:

$$ds^2 = -Q\,dt^2 + P\,dr^2 + r^2\,d\Omega^2 \tag{1}$$

where P and Q are positive functions of r and t on a suitable domain. Here t is thought of as time, r as radius and $d\Omega^2$, the standard metric on the 2–sphere, is an abbreviation for $d\theta^2 + \sin^2\theta\,d\phi^2$ (or more symmetrically for $\sum_{j=1}^3 dz_j^2$ restricted to $\sum_{j=1}^3 z_j^2 = 1$). These metrics are all spherically-symmetric and represent locally the general spherically-symmetric space-time.

A significant subclass is when P and Q are functions of t alone. In this case the metric is "static" in the sense that time generates a Killing vector field.[1] An important special case is when in addition $P = 1/Q$. Included in this special case are the Schwarzschild metric defined (where positive) by $Q = 1-2M/r$ and the de Sitter metric $Q = 1-(r/a)^2$, and also the combined Schwarzschild de Sitter metric defined by $Q = 1-2M/r-(r/a)^2$ (for $M/a < 1/\sqrt{27}$). Here M is mass (half the Schwarzschild radius) and a is the "radius of the visible universe". The de Sitter metric is one of the standard metrics on (part of) de Sitter space, see for example Moschella [4]. The Schwarzschild de Sitter metric is appropriate for the exterior of a massive body in de Sitter space. It satisfies Einstein's equation for a vacuum with cosmological constant $3/a^2$ and it can be naturally extended past the cosmological horizon where $Q = 0$ at approximately $r = a$ (see [2, 5]).

We will show that all metrics of the form (1) admit precisely two spherically-symmetric foliations by flat space slices on the open set U defined by $Q > 0$ and $P > 1$. In the case $P = 1/Q$ this open set is defined by $0 < Q < 1$ (which in the special case of the Schwarzschild de Sitter metric, for M/a small, is the region between the event horizon near $r = 2M$ and the cosmological horizon near $r = a$).

We are interested in the normal vector field to these foliations and, in particular, when the integral curves are geodesic and therefore form a natural observer field. For the static case (when P and Q are both functions of t only) we shall find the precise condition for this to happen, namely iff PQ is constant. If this is the case we can make a simple change of coordinates (multiply t by a constant) to get $P = 1/Q$, the special case mentioned earlier.

In the Schwarzschild case one of these fields points outwards and reaches ∞ with outward velocity zero. In the Schwarzschild de Sitter metric, the outward pointing field can be extended across the coordinate singularity near $r = a$ and tends asymptotically

[1] This should not be confused with coherence as defined in the abstract: the rate of proper time along t–lines varies according to the function $Q(t)$.

to fit with the time lines in the standard expansive coordinates on de Sitter space. Thus in these cases it is natural to think of this outward pointing field as the "escape field". The other field is the dual "capture field". The escape field is expansive in the sense that, flowing along the escape field, the volume on space slices is expanded and dually the capture field is contractive. The expansive field is a candidate for part of the observer field in a redshifted universe. For more detail here see [5].

Notes The existence of flat space slices for the Schwarzschild and de Sitter metrics is well known and this is also known for more general metrics (see eg [3] and its bibliography). However, to our knowledge, the normal geodesic field and coherent time have not been observed before.

1 Flat space slices

We look for spherically-symmetric flat space slices for the metric (1). Consider a connected Riemannian 3–manifold V foliated by scaled copies of the 2–sphere S^2 such that SO(3) acts by isometries preserving this foliation. Let v be the vector field defined by a choice of non-zero vector normal to the leaves of the foliation and let dv be the corresponding line element of unit length (ie dv is the 1–form such that $dv(v) = ||v||$ and $dv(u) = 0$ for vectors u lying in leaves of the foliation). Let $x = \int dv$ be length measured along v, then x can be regarded (locally) as a parameter for the family. Then S^2_x is isometric to a Euclidean 2–sphere of some well-defined radius which we denote $r(x)$. The metric on V can now be written $dv^2 + r(x)^2 d\Omega^2$. We claim that this metric is flat iff $dr/dx = \pm 1$. This can be checked by calculating curvature (not hard because the relevant tensors have many zero entries) but is much more easily seen by thinking geometrically. A flat 3–manifold is locally isometric to $\mathbb{R}^3$. Futher 2–spheres are rigid; they only embed smoothly and isometrically in $\mathbb{R}^3$ as round spheres and the same is true for any open subset (this is a classical result due to Liebman, see Alexandrov [1] for a proof). So the local isometry carries each leaf of the foliation to part of a round 2–sphere. These pack together in only one way – like concentric spheres – and it follows that the radii vary exactly as they do for concentric spheres in $\mathbb{R}^3$ which is what we want.

Now suppose that we have a spherically-symmetric space slice of the metric (1). This is determined by a curve γ in the (r, t)–plane. Let ds be the metric restricted to γ then from (1) we have

$$ds^2 = -Q\, dt^2 + P\, dr^2 \tag{2}$$

along γ. At each point of γ we have a 2–sphere of radius r and by the result just proved, the slice is flat iff $ds^2/dr^2 = 1$. Substituting in (2) we find that

$$1 = -Q\,\frac{dt^2}{dr^2} + P$$

which gives

$$\frac{dt}{dr} = \varepsilon\sqrt{\frac{P-1}{Q}} \quad \text{where} \quad \varepsilon = \pm 1\,. \tag{3}$$

This has real roots on the open set U defined by $P > 1$. (Note that we have already assumed that Q is positive.) Thus on U there are precisely two flat space slices through each point. The positive square root, $\varepsilon = +1$, gives a positive slope and then the corresponding normal vectors also have positive slope, ie moving outwards. In the Schwarzschild and Schwarzschild de Sitter cases we shall see that this is the "escape" field. The negative square root gives the dual "capture" field.

In general we refer to the flat slices with outward normals as the *outward* slices and by continuity the outward slices fit together to foliate U. Dually there is another foliation given by inward slices.

2 Radial geodesic vector fields in the static case

By symmetry the normal vectors to the foliations described above all lie in the (r,t)–plane, in other words they are radial. We shall determine the radial geodesic vector fields in the metric (1). We shall do this in the static case (where P and Q are independent of t) using the conservation law method given in [6]. The static case is sufficiently simple that we shall be able to find the exact condition for the geodesic field to be normal. Furthermore, in this case, translation by t is an isometry and each of the two families of flat slices we found above can be described simply as comprising all t–translates of one particular slice.

Suppose that we have a weightless test particle moving in the metric (1) with zero angular velocity. It moves along a radial geodesic. The radial geodesics can be considered as parametrised curves $(t(\,), r(\,))$ satisfying the Euler–Lagrange equations for the Lagrangian

$$\mathcal{L}\left(t, r, \dot t, \dot r\right) = \frac{1}{2}\left(-Q(r)\,\dot t^2 + P(r)\,\dot r^2\right)$$

where $\dot{}$ denotes differentiation wrt the parameter.

t–translational symmetry implies

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -Q\,\dot{t}$$

is constant. This is negative for a positively oriented time-like geodesic so we have

$$-Q\,\dot{t} = -E \qquad \text{ie}$$

(4) $$\dot{t} = \frac{E}{Q} \qquad \text{where } E > 0 \text{ is constant.}$$

"Energy" conservation implies

$$\mathcal{H}(p,\dot{q}) = \langle\, p,\dot{q}\,\rangle - \mathcal{L} = \frac{1}{2}(-\frac{p_t^2}{Q} + \frac{p_r^2}{P})$$

is conserved and wlog we can take its value to be respectively $-\frac{1}{2}$, 0, $\frac{1}{2}$ in the time-like, null, space-like cases respectively, because all other values of $\mathcal{H}$ are related to one of these by an affine reparametrisation of the geodesic; the choice $-\frac{1}{2}$ in the timelike case makes the parameter into proper time. Here q is the pair (t,r), $\dot{q}$ the pair $(\dot{t},\dot{r})$ and p the pair

$$\big(p_t, p_r\big) = \left(\frac{\partial \mathcal{L}}{\partial \dot{t}}, \frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \big(-Q\,\dot{t}, P\,\dot{r}\big)\ .$$

Hence

$$\frac{1}{2}\left(-\frac{E^2}{Q} + P\,\dot{r}^2\right) = -\frac{1}{2}$$

since we are in the time-like case. This implies

$$\dot{r}^2 = \frac{1}{P}\left(\frac{E^2}{Q} - 1\right) = \frac{E^2}{PQ} - \frac{1}{P}$$

and then from (4) we have the general radial geodesic vector field given by

(5) $$\big(\dot{t},\dot{r}\big) = \left(\frac{E}{Q}, \varepsilon\sqrt{\frac{E^2}{PQ} - \frac{1}{P}}\right)$$

where E is a positive constant and $\varepsilon = \pm 1$.

3 Normality

The radial tangents to the flat slices given by (3) are up to scale

$$(\delta t, \delta r) = (\varepsilon\sqrt{P-1}, \sqrt{Q})$$

and this is normal to $(\dot t, \dot r)$ in the metric (1) iff

$$\langle(\dot t, \dot r), (\delta t, \delta r)\rangle = -Q\,\dot t\,\delta t + P\,\dot r\,\delta r = 0\,.$$

Substituting from the last equation and (5) we have:

$$\begin{aligned} 0 &= -Q\left(\frac{E}{Q}\right)\left(\varepsilon\sqrt{P-1}\right) + P\left(\varepsilon\sqrt{\frac{E^2}{PQ} - \frac{1}{P}}\right)\sqrt{Q} \\ &= -\varepsilon E\sqrt{P-1} + \varepsilon\sqrt{E^2P - PQ} \\ &= -\varepsilon\sqrt{E^2P - E^2} + \varepsilon\sqrt{E^2P - PQ} \end{aligned}$$

But this is zero iff $PQ = E^2$. As remarked earlier, if this is the case we can make a change of coordinates (multiply t by a constant) to get $P = 1/Q$, the special case mentioned earlier.

If $P = 1/Q$ then the case $E = 1$ of the geodesic vector field (5) takes the simpler form:

$$(\dot t, \dot r) = \left(\frac{1}{Q}, \varepsilon\sqrt{1-Q}\right) \tag{6}$$

4 Coherent time

Now assume we are in the static case and that $P = 1/Q$. Since the geodesic vector field found above is natural, we know that it defines a coherent time. This fact is proved in general in the next section. Here we give a direct proof which gives more information.

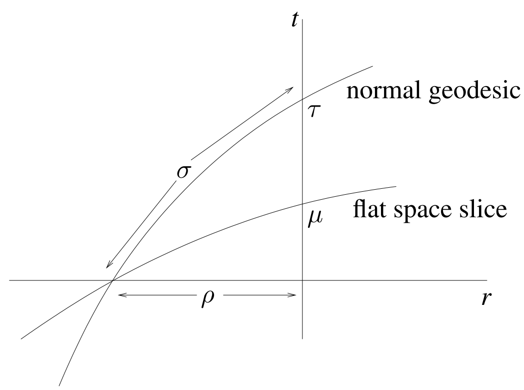
We want to compare proper time along the geodesics with t. To avoid confusion, we will use coordinates (τ, ρ) on the geodesics and (t, r) on the space slices. We use σ as distance parameter (proper time) along geodesics. For definiteness assume that $\varepsilon = +1$. The calculation in the case $\varepsilon = -1$ is similar and the result is the same.

In terms of (τ, ρ) (6) becomes:

$$\left(\frac{d\tau}{d\sigma}, \frac{d\rho}{d\sigma}\right) = \left(\frac{1}{Q}, \sqrt{1-Q}\right) \tag{7}$$

Suppose that a particle on a geodesic line moves a small distance σ increasing its r coordinate by ρ say and its t coordinate by τ. It is now on a new space slice. The old space slice contains a point (μ, ρ) and the t difference between the two slices is:

$$\tau - \mu = \rho\frac{d\tau}{d\rho} - \rho\frac{dt}{dr}$$



But
$$\frac{d\tau}{d\rho} = \frac{1}{Q\sqrt{1-Q}} \quad \text{and} \quad \frac{dt}{dr} = \frac{\sqrt{1-Q}}{Q}$$
from (7) and (3). Hence
$$\tau - \mu = \frac{\rho}{Q}\left(\frac{1}{\sqrt{1-Q}} - \sqrt{1-Q}\right)$$
which simplifies to
$$\frac{\rho}{\sqrt{1-Q}}\,.$$
But
$$\sigma = \rho\frac{d\sigma}{d\rho} = \rho\frac{1}{\sqrt{1-Q}}$$
using (7) which is the same.

Integrating, we deduce that proper time measured along any geodesic in the natural observer field gives the same parametrisation of the set of flat space slices as the Killing coordinate t. Further we can now define a coherent time in U by taking the flat space slices to be slices of constant time and measuring time between them by using proper time along the geodesics.

5 Natural coordinates

In this section we prove that a natural observer field gives a coherent time in general. This reproves the result of the last section but without providing the connection with the Killing cordinate t.

We consider metrics with natural observer fields which are not necessarily spherically-symmetric. Suppose that we have a space-time with an observer field which has normal space slices (ie such that the normal plane field is integrable). Then we can use this data to construct coordinates by measuring "time" along the observer field and "space" in the normal slices, which we call *natural coordinates*. In these coodinates the metric has the form:

$$ds^2 = -Q\,dt^2 + P(d\mathbf{z}) \tag{8}$$

where Q is a positive function and $P = P_{ij}$ is a positive definite quadratic form on $d\mathbf{z} = (dz_1, dz_2, dz_3)$ both being functions of t, z_i. The condition that t–lines (ie curves with $\mathbf{z} = \text{const}$) are normal to $\mathbf{z}$–slices (ie submanifolds with $t = \text{const}$) is precisely that there are no terms involving products of dt with dz_i.

Proposition *A vector field parallel to t–lines in the metric (8) is geodesic iff t provides a coherent time with the $\mathbf{z}$–slices being the slices of constant time and in this case, the metric has the same form as (8) but with Q a function of t alone.*

We have have used the cumbersome terminology "a vector field parallel to t–lines" to stress that we are not assuming that the parametrisation given by t is the correct one.

The proposition gives another proof of the result of the last section (though of course it does not give the connection with the Killing coordinate t that we found there). This is an appropriate point to make the warning that we are using t for a different coordinate in this section than in previous ones (where we used σ for the coherent natural time coordinate).

Proof We use Christoffel symbols. Let $\alpha\partial_0$ be a vector field parallel to the t–lines where we are indexing coordinates by 0 for t and i for z_i. This is geodesic iff

$$\begin{aligned} 0 &= \nabla_0(\alpha\partial_0) \\ &= \dot\alpha\partial_0 + \alpha\nabla_0\partial_0 \\ &= \dot\alpha\partial_0 + \alpha\left(\sum_{i=0}^{3}\Gamma^i_{0,0}\partial_i\right) \end{aligned} \tag{9}$$

where $\dot{}$ is differentiation wrt t. But using the fact that $g_{ij} = g^{ij} = 0$ if one (but not both) of i, j is zero we have

$$\Gamma^0_{0,0} = g^{00}\partial_0 g_{00} = \dot Q/Q$$

$$\Gamma^i_{0,0} = -\frac{1}{2}\sum_{j=1}^{3} g^{ji}\partial_j g_{00} = -\frac{1}{2}\sum R_{ji}Q'_j$$

where R is the inverse matrix to P and $Q'_j = \partial_j Q$ Then from the ∂_0 component of (9) we have

$$\dot{\alpha} + \alpha\dot{Q}/Q = 0 \tag{10}$$

and from the other components (and noting that R is symmetric) we deduce

$$R\mathbf{Q}' = \mathbf{0}$$

where $\mathbf{Q}'$ is the column vector $\{Q'_i\}$. But P is non-singular everywhere and hence so is R and we deduce that all the partial derivatives of Q wrt the z_i are zero. Thus Q is a function of t alone and from (10) we deduce that α is also a function of t alone and that αQ is constant. It is now clear that t gives a coherent time and indeed we can reparametrise t to make $Q = 1$ if we wish.

The converse of this last part is obvious and we have the result. □

6 Expansion and contraction

We now go back to spherically-symmetric metrics in the static case with $PQ = 1$ and for definiteness assume $\varepsilon = +1$, ie consider the outward field and flat slices. The notation reverts to the notation used in earlier sections. There is a natural choice for coordinates in the slices. Since $dr = ds$ we can use r for radial distance and then Ω completes the coordinate system. Flowing along the normal geodesic field gives a diffeomorphism between different space slices which is determined entirely by the function of r used. This diffeomorphism expands or contracts uniformly in the two Ω directions and by a possibly different scale factor in the r direction.

Geodesic field lines are naturally parametrised by σ (proper time) and then from (7) we have

$$\frac{dr}{d\sigma} = \sqrt{1-Q}$$

where we have replaced ρ by r since there is no confusion here. Then the rate of expansion in the Ω coordinates can be read off as

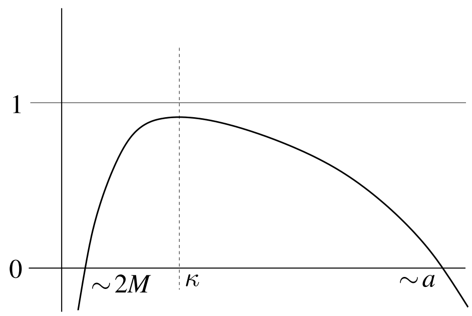
$$\frac{1}{r}\frac{dr}{d\sigma} = \frac{\sqrt{1-Q}}{r} \tag{11}$$

(r becomes $r + \delta r$ so the expansion is $\delta r/r$ per unit length) and in the r direction as

$$\frac{d}{dr}\frac{dr}{d\sigma} = \frac{d}{dr}\left(\sqrt{1-Q}\right) = \frac{-Q'}{2\sqrt{1-Q}} \tag{12}$$

where $'$ is differentiation wrt r. Notice that the expansion rate in the Ω coordinates is always positive but in the r coordinate may be negative (ie contraction rather than expansion). To be precise we have contraction in the r coordinate when $Q' > 0$ and expansion when $Q' < 0$.

To get some idea of what this means for particular metrics let's look in detail at the special case of the Schwarzschild de Sitter metric where Q is $1 - 2M/r - r^2/a^2$. The graph of Q is sketched below. Q is zero when r is roughly $2M$ (the event horizon) and again when r is roughly a (the cosmological horizon). Q' is positive up to the *critical radius* $\kappa = a^{\frac{2}{3}} M^{\frac{1}{3}}$ and negative from there onwards.



Looking first at the region near the central mass (where $r^2/a^2 \ll 2M/r$) then we have $1 - Q \sim 2M/r$ and expansion rate in the Ω coordinates of approximately T where

$$T = \sqrt{2M} r^{-3/2}$$

and in the r direction of approximately

$$\frac{1}{2}\left(\frac{-2M}{r^2}\right)\frac{1}{\sqrt{2M/r}} = -\frac{1}{2}T$$

which is contraction but at half the rate of the expansion in the other two directions.

We now have to decide what "expansion" should mean when we have both expansion and contraction. Thinking of a box expanding, if the three coords expand by e_1, e_2, e_3 (ie so that 1 unit becomes $1 + qe_1$ etc after a small time q) then the volume of the box is multiplied by $(1 + qe_1)(1 + qe_2)(1 + qe_3)$ and we get a rate of volume expansion of $e_1 + e_2 + e_3$. Applying this to the flat space slices we have rate of expansion (measured in σ coordinates, ie distance along the geodesics) $(3/2)T$.

Dividing by the number of space dimensions we can interpret this as an average linear expansion rate of $T/2$.

Looking next at the region near the cosmological horizon when the r^2/a^2 term dominates we get expansion in all three directions of approximately $1/a$ which is the average expansion and corresponds to an FLRW metric with warping function $\exp(t/a)$. In between these two, there is general interpolating volume expansion with the contraction direction becoming inoperative at the critical radius.

As explained in detail in [5] following [2], the coordinate singularities where $Q = 0$ are both removable and in particular the metric extends past the cosmological horizon and tends asymptotically to the standard expansive metric on a small modification of de Sitter space. Using the analysis of Section 1 the flat slices must merge into the unique pair of spherically-symmetric flat slices for the expansive metric on de Sitter space with core geodesic corresponding to the path of the central mass. Thus the normal geodesics merge into the corresponding time-lines and this justifies calling this the "escape" field.

Dually the case $\varepsilon = -1$ gives the "capture" field.

In the Schwarzschild case, the expansion (and contraction) both tend to zero as $r \to \infty$ and the normal field tends to a field parallel to the t–axis, ie with outward velocity zero. So again this is the "escape" field (and the dual field is the "capture" field).

7 Final remarks

As remarked in the introduction it is important not to confuse a static time coordinate with one providing a coherent time. The Killing coord t in metric (1) does not provide a coherent time. It provides a coherent coordinate but this IS NOT TIME for an observer following it. The two are related by Q. An small interval δt in the Killing coord is experienced as a proper time $Q(\delta t)$ by an observer on a t–line. This is underlined by the proposition. The t–coordinate provides a coherent time iff t–lines are geodesic. This never happens in for example the Schwarzschild-de-Sitter metric.

You can have an observer field with coherent time without space slices being flat, again by the proposition. All that is needed is that the field is geodesic. Further you can have flat slices with normal observer field not geodesic and hence not providing a coherent time. For example any of the metrics (1) with P and Q functions of t but with PQ not constant. Thus "irrotational" ie having integral normal slices is NOT the same as providing coherent time. For this you also need geodesic normal field again by the proposition. This point is often confused in the literature.

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