Velocity symmetries
Approximate symmetries

Mathematics for Fusion Power part 4

N. Kallininikos

February 14, 2024
Outline

1. Velocity (Hamiltonian) symmetries of FGCM

2. Approximate (Hamiltonian) symmetries of FGCM
First-order guiding-center motion (FGCM) - Exact treatment

For a given magnetic field $B$ on a 3-dimensional manifold $M$, the Hamiltonian structure of FGCM is

$$\omega = -\beta - d(p\|b^b)$$  \hspace{1cm} (1)

$$H = \frac{p^2\|}{2} + \mu|B|$$  \hspace{1cm} (2)

on the 4-dimensional GC bundle $N$ over $M$, where

- $\beta = i_B\Omega =$ magnetic flux form,
- $b = B/|B|$ and $b^b = i_b g$,
- $g =$ metric tensor on $M$
- $\Omega =$ associated volume form on $M$
- $\mu =$ magnetic moment

and normalised units $m = e = 1$. 

In terms of the *modified* magnetic field,

$$\tilde{B} = B + p_\parallel c$$  \hspace{1cm} (3)$$

where $c = \text{curl} b$, the symplectic form can be written as

$$\omega = - \beta - d(p_\parallel b^b) = - i_B \Omega - dp_\parallel \wedge b^b - p_\parallel db^b$$

$$= - i_B \Omega - p_\parallel ic \Omega - dp_\parallel \wedge b^b$$

$$= - \tilde{\beta} - dp_\parallel \wedge b^b$$  \hspace{1cm} (4)$$

where $\tilde{\beta} = i_{\tilde{B}} \Omega$ is the modified flux.

Note that $\text{div} \, \tilde{B} = 0$. This means that $\tilde{\beta}$ is closed on $M$. But it’s not closed on $N$. To see these, write $\tilde{\beta} = \beta + p_\parallel db^b$ to derive $d\tilde{\beta} = dp_\parallel \wedge db^b$ and therefrom $d\tilde{\beta} \wedge dp_\parallel = 0$. 
The GC 2-form $\omega$ is nondegenerate if and only if $\tilde{B}_\parallel \neq 0$, where $\tilde{B}_\parallel = b \cdot \tilde{B}$.

**Proof.** $\omega$ is nondegenerate if and only if $i_U \omega = 0 \iff U = 0$, where $U = (u, w)$ is a vector field on $N$. Now, using (4)

$$i_U \omega = - i_u i_B \Omega - wb^b + (u \cdot b) dp_\parallel = - (\tilde{B} \times u)^b - wb^b + (u \cdot b) dp_\parallel$$

So $i_U \omega = 0$ splits to $u \cdot b = 0$ and $\tilde{B} \times u + wb = 0$, which in turn splits to

$$\begin{cases} (\tilde{B} \times u) \times b = 0 \\ wb \cdot \tilde{B} = 0 \end{cases} \Rightarrow \begin{cases} (\tilde{B} \cdot b) u - (u \cdot b) \tilde{B} = 0 \\ \tilde{B}_\parallel w = 0 \end{cases} \Rightarrow \begin{cases} \tilde{B}_\parallel u = 0 \\ \tilde{B}_\parallel w = 0 \end{cases}$$

i.e. $U = 0$ is the only solution if-f $\tilde{B}_\parallel \neq 0$.  \[\square\]
Velocity symmetries of FGCM - Exact treatment

Consider a symmetry generated by a vector field

\[ U = (u, w) \]

on the guiding-centre phase space \( N \), where

- \( u \) is the 3D part on the physical space \( M \)
- \( w \) is 1D in the \( p_\parallel \)-direction
- \( u, w \) depend on both \((Q, p_\parallel)\)
- \( u, w \) are considered independent of \( \mu \) (at least for now)

Recall that the conditions for a Hamiltonian symmetry are

\[ L_U \omega = 0 \]
\[ L_U H = 0 \]
Velocity symmetries of FGCM - Exact treatment (continued)

1. From \( L_U H = 0 \), we have \( wp_\parallel + \mu L_u |B| = 0 \). For all values of \( \mu \), this splits to

\[
\begin{align*}
  w &= 0 \quad (5) \\
  L_u |B| &= 0 \quad (6)
\end{align*}
\]

2. For \( w = 0 \), \( L_U \omega \) reduces to \( L_u \omega \) and

\[
L_u \omega = - L_u \beta - L_u d(p_\parallel b^b) = - i_u d\beta - d(i_u \beta) - dL_u (p_\parallel b^b) \\
= - d(i_u i_B \Omega) - d(p_\parallel L_u b^b) = - d(i_u i_B \Omega + p_\parallel L_u b^b)
\]

since \( d\beta = 0 \). Thus, \( L_U \omega = 0 \) if and only if

\[
\boxed{i_u i_B \Omega + p_\parallel L_u b^b = d\psi}
\]

for some function \( \psi \) (defined at least locally) on \( N \).
Velocity symmetries of FGCM - Exact treatment (continued)

3. This condition is in turn equivalent to:

\[ i_u i_B \Omega + p_i i_u d b^\parallel + p_i d(i_u b^\parallel) = d\psi \]
\[ i_u i_B \Omega + p_i i_u i_c \Omega + d(p_i i_u b^\parallel) - (i_u b^\parallel) dp_i = d\psi \]

and can be written as

\[ i_u i_\tilde{B} \Omega - (u \cdot b) dp_i = - dK \] (8)

where

\[ K = - \psi + (u \cdot b)p_i \] (9)

Equation (8) splits to

\[ u \times \tilde{B} = \nabla K \] (10)
\[ u \cdot b = \partial_{p_i} K \] (11)
Velocity symmetries of FGCM - Exact treatment (continued)

4. $K$ is the invariant associated to the symmetry generator $U$:

$$i_U \omega = i_u \omega = i_u (-\tilde{\beta} - dp_\parallel \land b^b) = -i_u i_{\tilde{B}} \Omega + (i_u b^b) dp_\parallel = dK$$

5. The compatibility condition between (10)-(11) yields

$$\nabla (u \cdot b) = \partial_{p_\parallel} (u \times \tilde{B}) \quad (12)$$

while the compatibility condition of (10) is $\text{curl} (u \times \tilde{B}) = 0$ and

$$\text{curl} (u \times \tilde{B}) = (\text{div} \tilde{B}) u - (\text{div} u) \tilde{B} + (\tilde{B} \cdot \nabla) u - (u \cdot \nabla) \tilde{B}$$

$$= - (\text{div} u) \tilde{B} + [\tilde{B}, u],$$

hence reads

$$[u, \tilde{B}] + (\text{div} u) \tilde{B} = 0 \quad (13)$$
Velocity symmetries of FGCM - Exact treatment (continued)

6a. Writing $L_u \omega = L_u (-\tilde{\beta} - dp_\parallel \wedge b^b) = -L_u \tilde{\beta} - dp_\parallel \wedge L_u b^b$, note first that yet another way of expressing $L_u \omega = 0$ is

$$L_u \tilde{\beta} + dp_\parallel \wedge L_u b^b = 0 \quad (14)$$

This implies, in particular,

$$L_u \tilde{\beta} \wedge dp_\parallel = 0 \quad (15)$$

6b. Secondly, $i_{\tilde{B}} L_u \tilde{\beta} = (L_u i_{\tilde{B}} - i_{[u,\tilde{B}]})\tilde{\beta} = L_u i_{\tilde{B}} \tilde{\beta} + \div u i_{\tilde{B}} \tilde{\beta}$, using (13), and since $i_{\tilde{B}} \tilde{\beta} = i_{\tilde{B}} i_{\tilde{B}} \Omega = 0$, we have

$$i_{\tilde{B}} L_u \tilde{\beta} = 0 \quad (16)$$

6c. Thirdly, applying $i_{\tilde{B}}$ to (14) and using (16), we also deduce

$$i_{\tilde{B}} L_u b^b = 0 \quad (17)$$
7. Using this,

\[ i_{[u, \tilde{B}]} \tilde{b}^b = (L_u i_{\tilde{B}} - i_{\tilde{B}} L_u) \tilde{b}^b = L_u \tilde{B}_|| \]

hence the \( b \)-component of the compatibility (13) reads

\[ L_u \tilde{B}_|| + (\text{div } u) \tilde{B}_|| = 0 \]  \hspace{1cm} (18)

8. Finally, for \( \bar{b} = \tilde{B} / \tilde{B}_|| \)

\[ [u, \bar{b}] = L_u (\tilde{B}_||^{-1}) \tilde{B} + \tilde{B}_||^{-1} [u, \tilde{B}] = -\tilde{B}_||^{-2} L_u (\tilde{B}_||) \tilde{B} + \tilde{B}_||^{-1} [u, \tilde{B}] \]

and so we deduce from (13),(18) that

\[ [u, \bar{b}] = 0 \]  \hspace{1cm} (19)
Velocity symmetries of FGCM - Exact treatment (continued)

In summary, what we can say so far are

**Theorem 1**

Given a magnetic field $B$, a vector field $U = (u, w)$ on $N$ generates a Hamiltonian symmetry of FGCM if-f $L_u \tilde{\beta} + dp\| \wedge L_u b^b = 0$, $L_u |B| = 0$, $w = 0$.

**Theorem 2**

If a $p\|$-dependent vector field $u$ on $M$ generates a Hamiltonian symmetry of FGCM, then

- $[u, \tilde{B}] + (\text{div } u)\tilde{B} = 0$
- $[u, \bar{b}] = 0$
- $\nabla (u \cdot b) = \partial_{p\|} (u \times \tilde{B})$
- $i_{\tilde{B}} L_u b^b = 0$
- $i_{\tilde{B}} L_u \tilde{\beta} = 0$, $L_u \tilde{\beta} \wedge dp\| = 0$
Velocity symmetries of FGCM - Exact treatment (continued)

In summary, what we can say so far are

**Theorem 1**

Given a magnetic field $B$, a vector field $U = (u, w)$ on $N$ generates a Ham. symmetry of FGCM if-f

\[ u \times \tilde{B} = \nabla K, \quad u \cdot b = \partial_{p\parallel} K, \]
\[ u \cdot \nabla |B| = 0, \quad w = 0, \]

where $K$ is the associated invariant.

**Theorem 2**

If a $p_{\parallel}$-dependent vector field $u$ on $M$ generates a Hamiltonian symmetry of FGCM, then

1. $[u, \tilde{B}] + (\text{div } u) \tilde{B} = 0$
2. $[u, \bar{b}] = 0$
3. $\nabla (u \cdot b) = \partial_{p\parallel} (u \times \tilde{B})$
4. $i_{\tilde{B}} L_u b^\flat = 0$
5. $i_{\tilde{B}} L_u \tilde{\beta} = 0, \quad L_u \tilde{\beta} \wedge dp_{\parallel} = 0$
FGCM & Symmetries - Approximate treatment

- FGCM is the 1st-order approximation of GCM wrt $\varepsilon = m/e \ll 1$

  $$\omega = -\beta - \varepsilon d(p_\parallel b^b)$$
  $$H = \varepsilon (p_\parallel^2/2 + \mu|B|)$$

So, natural to consider:

- Approximate vector fields of 1st-order
  $$U = U_0 + \varepsilon U_1$$

- Approximate symmetries of 1st-order
  $$L_U \omega = O(\varepsilon^2)$$
  $$L_U H = O(\varepsilon^2)$$
FGCM & Symmetries - Approximate treatment

- FGCM is the 1st-order approximation of GCM wrt 
  \( \varepsilon = m/e \ll 1 \)

  \[
  \omega = -\beta - \varepsilon d(p\parallel b^b) \\
  H = \varepsilon (p\parallel^2/2 + \mu|B|)
  \]

So, natural to consider:

- Approximate vector fields of 1st-order
  \( U = U_0 + \varepsilon U_1 \)

- Approximate symmetries of 1st-order
  \( L_U \omega \approx 0 \) \\
  \( L_U H \approx 0 \)

From now on, we write \( A = B + O(\varepsilon^n) \) as \( A \approx B \) for any two tensors of the same type. For FGCM, we take \( n = 2 \).
Approximate version of Noether’s theorem

A vector field $U$ generates an approximate symmetry of an approximate Hamiltonian system $(\omega, H)$ if-f there exists an approximate constant of motion $K$ such that $i_U \omega \approx dK$.

**Proof.** For any $K = K_0 + \varepsilon K_1 + \cdots$, a vector field $U$ s.t. $i_U \omega \approx dK$ is well-defined for $\omega = \omega_0 + \varepsilon \omega_1 + \cdots$, since $\omega_0$ is nondegenerate,

$$i_{U_0} \omega_0 = dK_0$$

$$i_{U_1} \omega_0 + i_{U_0} \omega_1 = dK_1$$

$$i_{U_2} \omega_0 + i_{U_1} \omega_1 + i_{U_0} \omega_2 = dK_2$$

(20)

Thus, $L_U \omega \approx 0$ and, if $L_X K \approx 0$, $L_U H \approx 0$ too, because

$$L_U \omega = d i_U \omega$$

(21)

$$L_U H = i_U dH = i_U i_X \omega = -i_X dK = -L_X K.$$  

(22)

In the other direction, if $U$ generates an approximate Hamiltonian symmetry, then (21) gives $i_U \omega \approx dK$ for some (suppose global) function $K$, and (22) gives $L_X K \approx 0$. 

$\square$
FGCM & Symmetries - Approximate treatment (continued)

Complication:

For $\varepsilon = 0$ the GC 2-form, $\omega_0 = -\beta$, is degenerate of rank 2 (i.e., *presymplectic* of constant rank) for $B \neq 0$ because $i_U \beta = i_u i_B \Omega = (B \times u)^\flat$ for any vector field $U = (u, w)$ on $N$, and therefore setting $i_U \beta = 0$, we see that

The kernel of $\beta$ (naturally pullbacked) on $N$ consists of all the vector fields $(fb, g)$ for arbitrary functions $f, g$

hence is two-dimensional.

This produces

**Trivial symmetries**

A trivial approximate symmetry is generated by any vector field $S$ s.t. $i_S \omega \approx 0$. For the GC 2-form $\omega$, $S = \varepsilon S_1$ with $S_1 \in \ker \beta$. 
Approximate Symmetries of FGCM (Burby, K, MacKay)

Theorem 3

Given a magnetic field \( B \), a v.f. \( U = (u, w) = (u_0 + \varepsilon u_1, w_0 + \varepsilon w_1) \) on \( N \) generates an approximate Ham. symmetry of FGCM if-f \( L_{u_0} \beta = 0 \), \( p\|L_{u_0} b^p + i_{u_1} i_B \Omega = d\psi_1 \), \( L_{u_0} |B| = 0 \), \( w_0 = 0 \) for a function \( \psi_1 \) on \( N \).

Proof. Take \( L_U H \approx 0 \), \( L_U \omega \approx 0 \) and split up by different powers of \( \varepsilon \), dropping any 2nd-order terms. The first condition gives

\[
p\| w_0 + \mu L_{u_0} |B| = 0
\]

thus \( w_0 = 0 \), \( L_{u_0} |B| = 0 \) for all \( \mu \). For \( w_0 = 0 \), \( L_U \omega \approx 0 \) reduces to \( L_u \omega \approx 0 \), so from the second condition, we have

\[
L_{u_0} \beta = 0
\]

\[
L_{u_0} d(p\| b^p) + L_{u_1} \beta = 0
\]

from the 0th- and 1st-order terms, respectively. Same as in the exact treatment (see eq. (7)), the latter gives \( p\|L_{u_0} b^p + i_{u_1} i_B \Omega = d\psi_1 \) for some function \( \psi_1 \) on \( N \). Straightforwardly, the converse is also true. \( \Box \)
Approximate Symmetries of FGCM (continued)

**Flux surfaces.** From $L_{u_0} \beta = 0$, we have $i_{u_0} \beta = d\psi_0$ for some function $\psi_0$ on $N$, because $\beta$ is closed. The $p_\parallel$-component gives $\partial_{p_\parallel} \psi_0 = 0$, and since $i_{u_0} \beta = i_{u_0} i_B \Omega = (B \times u_0)^b$, we deduce then

$$B \times u_0 = \nabla \psi_0$$

(23)

**Theorem 4**

If a vector field $U = (u_0 + \varepsilon u_1, \varepsilon w_1)$ on $N$ generates an approximate Hamiltonian symmetry of FGCM, then:

- $\text{div} u_0 = 0$, $[u_0, B] = 0$, $b \cdot V_0 = 0$;
- $B \cdot \nabla \psi_1 = 0$;
- $B \cdot \nabla (b \cdot u_0) = c \cdot \nabla \psi_0$;
- $p_\parallel u_0 \cdot \nabla (b \cdot u_0) = u_0 \cdot \nabla \psi_1 + u_1 \cdot \nabla \psi_0$;
- $p_\parallel [u_0, c] + [u_1, B] + (\text{div} u_1) B = 0$

where $c = \text{curl} b$ and $V_0 = c \times u_0 + \nabla (b \cdot u_0)$. 
Approximate Symmetries of FGCM (continued)

**Theorem 5**

Given a magnetic field \( B \), a v.f. \( U = (u, w) = (u_0 + \varepsilon u_1, w_0 + \varepsilon w_1) \) on \( N \) generates an approximate Ham. symmetry of FGCM up to trivial symmetries if-f \( L_{u_0} \beta = 0, L_{u_0} |B| = 0, w = 0 \), and

\[
\begin{align*}
  u_1 &= b \times (p \parallel V_0 - \nabla \psi_1)/|B| \\
  b \cdot \nabla \psi_1 &= p \parallel b \cdot V_0 \\
  \partial_{p \parallel} \psi_1 &= p \parallel b \cdot \partial_{p \parallel} u_0
\end{align*}
\]  

(24)

(25)

(26)

**Proof.** From \( L_{u_0} b = i_{u_0} db + di_{u_0} b = i_{u_0} i_c \Omega + d(b \cdot u_0) \), note that

\[
L_{u_0} b = V_0 + (b \cdot \partial_{p \parallel} u_0) dp \parallel
\]  

(27)

Thus, the condition \( p \parallel L_{u_0} b + i_{u_1} i_B \Omega = d\psi_1 \) of Thm 3 splits to

\[
B \times u_1 + p \parallel V_0 = \nabla \psi_1
\]  

(28)

and (26). Dotting (28) with \( b \) gives (25), while crossing with \( b \) we find

\[
u_1 = b \times (p \parallel V_0 - \nabla \psi_1)/|B| + (b \cdot u_1) b
\]

Dropping the trivial symmetry \( \varepsilon((b \cdot u_1)b, w_1) \) completes the proof. \( \square \)
Approximate Symmetries of FGCM (continued)

Approximate invariant

The corresponding approximate constant of motion is now given by

\[ K = -\psi_0 - \varepsilon(\psi_1 - p_\parallel b \cdot u_0) \]  

(29)

Proposition 1

Assume the \( p_\parallel \)-dependent vector field \( u_0 + \varepsilon u_1 \) on \( M \) generates an approximate Ham. symmetry of FGCM.

1. \( u_0 \) is spatial if and only if \( \psi_1 \) is.
2. If \( u_0 \) is spatial, then \( V_0 = \partial_{p_\parallel} u_1 \times B \).

Proof. From \( B \times u_0 = \nabla \psi_0 \) (23) we have \( B \times \partial_{p_\parallel} u_0 = 0 \) and together with \( p_\parallel b \cdot \partial_{p_\parallel} u_0 = \partial_{p_\parallel} \psi_1 \) (26) we deduce \( \partial_{p_\parallel} \psi_1 = 0 \) if-f \( \partial_{p_\parallel} u_0 = 0 \).

The second one follows from \( B \times u_1 + p_\parallel V_0 = \nabla \psi_1 \) (28), since if \( u_0 \) is spatial then so are \( \psi_1, V_0 \).
Approximate Symmetries of FGCM (continued)

Corollary

Given a magnetic field $B$, a vector field $u = u_0 + \varepsilon u_1$ on $M$ generates an approximate quasisymmetry if $u_0$ is a quasisymmetry and $L_{u_1} \beta = 0$.

Proof. From (27) we have $L_{u_0} b^b = V^b$, and from Prop 1 we have in turn $L_{u_0} b^b = 0$ and $\partial_{p \parallel} \psi_1 = 0$. Therefore the symmetry condition $p_{\parallel} L_{u_0} b^b + i_{u_1} i_B \Omega = d\psi_1$ of Thm 3 reduces to $i_{u_1} i_B \Omega = d\psi_1$, which says $L_{u_1} \beta = 0$. The rest of the symmetry conditions, $L_{u_0} \beta = 0$, $L_{u_0} |B| = 0$, together with $L_{u_0} b^b = 0$ prove that $u_0$ is a quasisymmetry.

Weak quasisymmetry (Rodríguez, Helander & Bhattacharjee)

is an approximate Hamiltonian symmetry of FGCM on $M$ which is spatial to leading order and nontrivially linear in $p_{\parallel}$ to first order.
Theorem 6

Let \( u_0 \) be a vector field on \( M \) with \( V_0 \neq 0 \). The vector field \( u = u_0 + \varepsilon u_1 \) generates a weak quasisymmetry up to trivial symmetries if and only if \( L_{u_0} \beta = 0 \), \( \text{div} \, u_0 = 0 \), \( L_{u_0} |B| = 0 \) and \( u_1 = b \times \left( p_{\parallel} V_0 - \nabla \psi_1 \right)/|B| \) with \( \psi_1 \) a flux function on \( M \).

Proof. If \( u \) generates a weak quasisymmetry then from Thms 4-5 we see that the conditions hold.

In the opposite direction, note first from (27) that \( L_{u_0} b^b = V_0^b \) since \( u_0 \) is spatial. Now, \( \text{div} \, u_0 = 0 \) is equivalent to \( b \cdot V_0 = 0 \), when \( L_{u_0} \beta = 0 \) and \( L_{u_0} |B| = 0 \). To see this, apply \( L_{u_0} \) to the relation \( b^b \wedge \beta = |B| \Omega \) to find \( L_{u_0} b^b \wedge \beta = |B| L_{u_0} \Omega \), where \( L_{u_0} \Omega = di_{u_0} \Omega = (\text{div} \, u_0) \Omega \), and then \( i_b \) in turn to arrive at \( (i_b L_{u_0} b^b) \beta = |B| (\text{div} \, u_0) i_b \Omega \) and hence \( i_b L_{u_0} b^b = \text{div} \, u_0 \) since \( B \neq 0 \). Thus, all the conditions of Thm 5 are met, with (26) trivially satisfied. Therefore \( u \) generates an approximate Hamiltonian symmetry of FGCM, and since \( V_0 \) and \( \psi_1 \) are independent of \( p_{\parallel} \), it is a weak quasisymmetry.
Remarks

- For general approximate symmetries, given $u_0$ and $\psi_1$, we can construct $u_1$, as we can see from Thm 5.

- This becomes an advantage, in particular, for weak quasisymmetry, because in this case $u_0$ and $\psi_1$ decouple. Thus, as we see from Thm 6, the conditions for weak quasisymmetry to zeroth-order are completely uncoupled from the first-order ones. Moreover, the latter amount to simply building $u_1$ once $u_0$ is known.

- On this ground, the existence of weak quasisymmetry (but not weak quasisymmetry itself) is rightfully identified with a v.f. $u_0$ such that $L_{u_0} \beta = 0$, $\text{div } u_0 = 0$, $L_{u_0} |B| = 0$, as the last condition of Thm 6 is merely a construction (assuming flux function $\psi_1$).

- This allows to compare the part $u_0$ of weak quasisymmetry with quasisymmetry $u$, despite their different nature. From their conditions respectively, we see then that $\text{div } u_0 = 0$ relaxes $L_u b^b = 0$. 
Approximate Symmetries of FGCM (continued)

**Theorem 7 (Rodríguez, Helander & Bhattacharjee)**

Let \( f = B \cdot \nabla |B| \neq 0 \). A weak quasisymmetry exists if and only if \( \nabla \psi_0 \times \nabla |B| \cdot \nabla f = 0 \) and \( \psi_0 + \varepsilon \psi_1 \) is a flux function.

**Proof.** From Thm 6, if \( u = u_0 + \varepsilon u_1 \) generates a weak quasisymmetry, then \( L_{u_0} \beta = 0 \), \( \text{div} \, u_0 = 0 \), \( L_{u_0} |B| = 0 \), and there’s a flux function \( \psi_1 \). Repeating (23), the first condition gives \( B \times u_0 = \nabla \psi_0 \), where \( \psi_0 \) is a flux function. Crossing with \( \nabla |B| \) and using the third condition we get

\[
 u_0 = \nabla \psi_0 \times \nabla |B|/f \quad (30)
\]

Applying then the second condition, we find \( \nabla \psi_0 \times \nabla |B| \cdot \nabla f = 0 \), as any \( \nabla \psi_0 \times \nabla |B| \) has zero divergence.

In the other direction, given flux function \( \psi_0 \), define \( u_0 \) from (30). Then \( L_{u_0} |B| = 0 \). Also \( \text{div} \, u_0 = 0 \), because \( \nabla \psi_0 \times \nabla |B| \cdot \nabla f = 0 \).

Thirdly, crossing (30) with \( B \) gives \( B \times u_0 = \nabla \psi_0 \), since \( B \cdot \nabla \psi_0 = 0 \).

Finally, take \( u_1 = b \times (p_\parallel V_0 - \nabla \psi_1)/|B| \) given flux function \( \psi_1 \). Thus, \( u = u_0 + \varepsilon u_1 \) generates a weak quasisymmetry from Thm 6. \qed
Approximate Symmetries of FGCM (continued)

**Theorem 8**

For an MHS magnetic field with $dp \neq 0$ almost everywhere on $M$ and density of irrational surfaces, an approximate symmetry of FGCM on $N$ implies an approximate quasisymmetry.

---

**References**


Food for thought

- How does MHS (or at least vacuum) combine with (exact) velocity symmetries?
- Does circle action of quasisymmetry extend to an analogue for velocity symmetries?
- Need to impose boundedness. What’s the role of magnetic curvature for approximate symmetries?
- Are there special $\mu$-dependent symmetries? Need to study GCM as a Hamiltonian reduction of charged particle motion.
- How far is weak QS from isometries compared to QS? Note that $\text{tr}_g A = 0$ but $\det A \neq 0$, where $A = L_{u_0} g$ for weak QS $u = u_0 + \varepsilon u_1$.