Mathematics for Fusion Power part 6

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Weak Isodrasticity

Strong isodrasticity

Beyond omnigenity

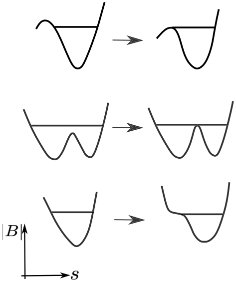
- Good conservation of $L = \int_\gamma p_\parallel b^\flat$ is not guaranteed when bounce period T is large, so omnigenity might not guarantee small $\langle\dot\psi\rangle$ near marginal cases.
- Also, change of ZGCM class can produce a large change in region visited, or transition into a class whose trajectories are not bounded (like ripple bouncers).
- Equilibria with anisotropic pressure (but no flow) might not have a flux function. Closest is $i_b dp_\perp + |B| i_b d(\frac{p_\parallel - p_\perp}{|B|}) = 0$.
- But for confinement, don't need all GC motion to stay close to flux surfaces. Indeed, don't need flux surfaces: for $B \in C^3$, approximate integrability of B suffices to confine circulating GCs, and approximate conservation of L suffices for bouncing GCs away from marginal cases.
- Idea: Drop requirement for a flux function, prevent transitions between classes of GC motion, and make some KAM tori in each relevant class.

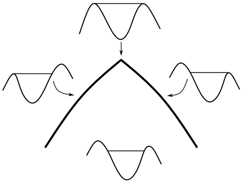
Isodrastic fields

- Magnetic fields for which transitions in FGCM between different types of ZGCM are prevented. Assume $B \in C^r$, $r \geq 2$.
- Can formulate without assuming a flux function.
- Say B is *weak isodrastic* if marginal cases are never reached from non-marginal ones by L-reduced FGCM (Cary & Shasharina call it omnigenity for marginally trapped particles).
- L-reduced dynamics in scaled variables $h = E/\mu, j = L/\sqrt{m\mu}$: reduced phase space F_j at given $j \in \mathbb{R}^+$ is the space of possible segments γ for ZGCM with $\int_\gamma \sqrt{2(h - |B|)} = j$ for some $h \in \mathbb{R}$ with $|B| = h$ at the ends of γ and $|B| < h$ in between. The reduced Hamiltonian H_j is the value of h. The symplectic form is β in any transverse section to the segments (it gives the same value in any transverse section). Gives reduced vector field X in scaled time $\tau = \frac{\mu}{e} t$ by $i_X \beta = -dH_j$.
- Reduced dynamics not valid near marginal cases, but continue nonetheless; later, treat transitions exactly (strong isodrasticity), discover reduced dynamics gives correct 1st order answer.

continued

- In general, F_j consists of several C^r surfaces, limited by curves of non-degenerate marginal cases (local maximum at an end or interior point of γ), which in turn possibly meet in doubly marginal points (non-quadratic critical point or heteroclinic).





continued

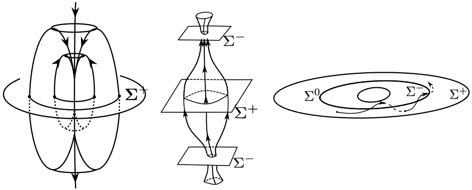
- H_j is differentiable at all non-marginal points of F_j: given vector field on F_j, extend to GC phase space by any smooth vector field X that takes B-lines to B-lines with $\frac{1}{2}u_\parallel^2 + |B| = H_j$. Then a calculation gives

$$i_X dH_j = \frac{1}{T}\int (i_{X_\perp} d|B| + 2(H_j - |B|)\Omega(X, b, c))\ dt$$

  with $T = \int dt$ for ZGCM.
- So $i_X dH_j = \langle i_{X_\perp}(d|B| + 2(H_j - |B|)\kappa^\flat)\rangle$, with κ curvature vector of fieldlines.
- In particular, note that as a segment approaches (single) marginality then $dH_j \to d|B|$ at the point with $i_B d|B| = 0$, because the period is dominated by time near that point.

Critical points of $|B|$ along B

- To address marginality, need the set Σ of critical points of $|B|$ along fieldlines, i.e. the zeroes of $|B|' = i_b d|B|$.
- Subdivide Σ into $\Sigma^+ \cup \Sigma^0 \cup \Sigma^-$ according to the sign of $|B|''$.



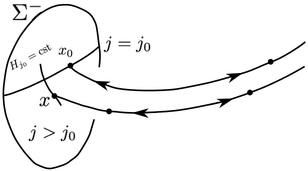
- Marginality consists of having an endpoint in $\Sigma^{-0} = \Sigma^- \cup \Sigma^0$.
- Bouncing segments have a point of $\Sigma^+ \cup \Sigma^0$ in their interior.
- Σ is a C^{r-1} surface as long as $d(|B|') \neq 0$, which is generic on Σ. In particular, it is guaranteed on $\Sigma^\pm$ (where $i_b d(|B|') \neq 0$), so $\Sigma^\pm$ are always C^{r-1} surfaces.
- For $r \geq 3$, Σ^0 is generically a C^{r-2} curve forming common boundary of $\Sigma^\pm$: defined by $|B|' = 0, |B|'' = 0$, so fails only if $d|B|', d|B|''$ parallel, which sums to 4 conditions in 3 variables.

Reformulation

- B is weak isodrastic iff marginal segments remain marginal.
- Define functions $\mathcal{H}$ and $\mathcal{J}^{\pm}$ on Σ^- by $\mathcal{H} = |B|_{|\Sigma^-}$ and $\mathcal{J}^{\sigma} = \int_{\gamma^{\sigma}} \sqrt{2(\mathcal{H} - |B|)}\,|ds|$ for the segment γ^{σ} from the chosen point of Σ^- in direction $\sigma \in \{\pm\}$ to the first point at which $|B| = \mathcal{H}$ again (if exists).
- Note that for $x \in \Sigma^-$, $H_{\mathcal{J}^{\sigma}(x)}(x) = \mathcal{H}(x)$, and as a segment endpoint approaches Σ^-, $dH_j \to d\mathcal{H}$.
- For $A \subset \Sigma^{-0}$ let A^{σ} be the subset without heteroclinic connection in direction σ.
- **Theorem**:
  1. If B weak isodrastic & $\sigma \in \{\pm\}$ then $d\mathcal{H}, d\mathcal{J}^{\sigma}$ are linearly dependent at each point of ${\Sigma^-}^{\sigma}$;
  2. If B weak isodrastic and ${\Sigma^0}^{\sigma}$ smooth then $\mathcal{H}, \mathcal{J}^{\sigma}$ are constant on connected components of ${\Sigma^0}^{\sigma}$;
  3. If $\mathcal{J}^{\pm}$ constant on components of level sets of $\mathcal{H}$ then B weak isodrastic.
- Informally, B weak isodrastic iff contours of $\mathcal{H}, \mathcal{J}^{\pm}$ on Σ^{-0} coincide.

Proof

1. If $d\mathcal{J}^\sigma, d\mathcal{H}$ indpt at $x_0 \in \Sigma^{-\sigma}$, let $j_0 = \mathcal{J}^\sigma(x_0)$. $\mathcal{J}^{-1}(j_0)$ is locally a smooth curve and the boundary of F_{j_0}. $d\mathcal{H}(x_0) \neq 0$ and tangent to ∂F_{j_0} not in $\ker d\mathcal{H}$. $dH_{j_0} \to d\mathcal{H}$ as x_0 is approached from $\mathrm{Int} F_{j_0}$. So $\ker dH_{j_0}$ is transverse to the boundary near x_0. So trajectories of the reduced dynamics reach the boundary in finite positive time for one sign of e. So B is not weak isodrastic.



2. Σ^{0^σ} smooth curve and $\mathcal{H}$ not constant along it implies $\exists\, x_0 \in \Sigma^{0^\sigma}$ where $d\mathcal{H} v \neq 0$ for a tangent v to Σ^0. So $v \neq \ker d\mathcal{H}$. Let $j_0 = \mathcal{J}^\sigma(x_0)$. Then $dH_{j_0} \to d\mathcal{H}$ as x_0 is approached from Σ^-. Thus $\ker dH_{j_0}$ is transverse to Σ^0 near x_0. So trajectories of the reduced dynamics reach Σ^0 in finite positive time for one sign of e. So B is not weak isodrastic. Thus weak isodrastic implies $\mathcal{H}$ constant along smooth components of Σ^{0^σ}. Since we proved in 1. that it also implies $d\mathcal{H} \wedge d\mathcal{J}^\sigma = 0$ up to the boundary of F_{j_0} then $\mathcal{J}^\sigma$ is also constant along the boundary.

continued

3. Suppose $\mathcal{J}^\sigma$ constant on level sets of $\mathcal{H}$. Let γ be a trajectory of reduced dynamics for segments on side σ of Σ^- s.t. $\gamma(0)$ is not marginal. If $\gamma(t)$ is marginal for some positive time, there is a first such t_0, because the set of non-marginal cases is open. The initial value problem for $x_0 = \gamma(t_0)$ has unique solution because the reduced dynamics is a factor of the full GC dynamics. x_0 is not an equilibrium point else it was not possible to reach it in finite time from $\gamma(0)$. So for $j = \mathcal{J}^\sigma(x_0)$, $dH_j(x_0) \neq 0$. Thus $d\mathcal{H}(x_0) \neq 0$, so the level set of $\mathcal{H}$ containing x_0 is locally a smooth curve Γ. By hypothesis, $\mathcal{J}^\sigma$ is constant along it, so H_j is constant along it. So Γ is locally invariant. But that implies that t_0 was not the first time that $\gamma(t)$ is marginal.

   The case that a segment is split by an interior local maximum of $|B|$ can be treated the same way, replacing $\mathcal{J}^\sigma$ by $\mathcal{J}^+ + \mathcal{J}^-$. $\square$

Transition flux

- Quantify failure of weak isodrasticity by $d\mathcal{H} \wedge d\mathcal{J}^\sigma$. Can write it as $\mathcal{M}^\sigma \beta$ and quantify by the function $\mathcal{M}^\sigma$ on Σ^-.

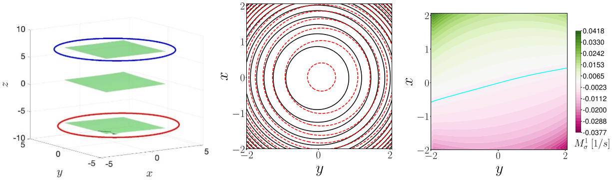


Figure: The parts of Σ above $[-2,2]^2$ for a non-axisymmetric mirror field of two circular coils with weaker field at top neck, contours of $\mathcal{H}, \mathcal{J}$ on upper Σ^-, and $\mathcal{M}$ there.

- Liouville volume $\Lambda = \frac{1}{2}\omega \wedge \omega = e\beta \wedge d(p_\| b^\flat) = e\tilde{B}_\| \Omega \wedge dp_\|$ on GC phase space.
- **Theorem**: The transition flux-form for GCM corresponding to segments in direction σ from Σ^- becoming marginal is $2\sigma m^{1/2}\mu^{3/2}\mathcal{M}^\sigma \beta$.

Proof & Consequences

- Phase-space volume-flux for a 2DoF Hamiltonian system $i_X\Lambda = dH \wedge \omega$. So given an area A on Σ^- and a choice of direction σ from it, the corresponding transition flux is $\int_{\phi(A)} dH \wedge \omega$, where $\phi(A)$ denotes the 3D volume produced by flowing $A \times \{p_\| = 0\}$ along ZGCM in direction σ and back.
- $dH = \mu d\mathcal{H}$ and integrating ω along each homoclinic of ZGCM gives $2\sigma d \int_\gamma p_\| b^\flat = 2\sigma dL = 2\sigma\sqrt{m\mu}\, d\mathcal{J}$. So $\int_{\phi(A)} dH \wedge \omega = 2\sigma\mu\sqrt{m\mu} \int_A d\mathcal{H} \wedge d\mathcal{J}$, and the reduced flux-form is $2\sigma m^{1/2}\mu^{3/2}\mathcal{M}\beta$. □
- For the rate of splitting of segments into two by Σ^-, add the two transition fluxes.
- Liouville volume in the 3DoF phase space converts to $m\tilde{B}_\|\Omega \wedge dp_\| \wedge d\mu \wedge d\phi = \frac{m}{e}\Lambda \wedge d\mu \wedge d\phi$ in gyro-coordinates.
- For particle number density ρ in the full 3DoF phase space, obtain gyro-averaged density $2\pi\frac{m}{e}\bar{\rho}$ wrt $\Lambda \wedge d\mu$.
- Then number flux transitioning is $4\pi\frac{m^{3/2}}{e}\sigma\langle\bar{\rho}\rangle\mathcal{M}\beta \wedge \mu^{3/2}d\mu$ in $\Sigma^- \times \{\mu\}$, for bounce-average $\langle\bar{\rho}\rangle$ along marginal segment.

Perturbed tokamak example

- In cylindrical polars, $B^R = -\frac{z}{R} - \varepsilon \cos\phi$, $B^\phi = \frac{C}{R^2}$, $B^z = 1 - \frac{1}{R} + \varepsilon \frac{z}{R} \cos\phi$.

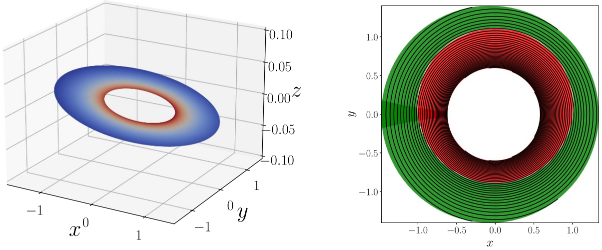


Figure: Σ, contours of $\mathcal{H}$ on Σ and green for Σ^+, red for Σ^-.

- Remains to compute $\mathcal{J}^\pm$. Many branches, depending on number of poloidal turns before bouncing. Separated by heteroclinic cases.
- Note that contours of $\mathcal{H}$ crossing from green to red indicate short bouncers that become unstable to long bouncing (left or right or over the top).

Σ for stellarators

- Distinguish three types for Σ in toroidal configurations.
  1. ΣT: The main component of Σ is an annulus making no poloidal turns for one **toroidal** turn, subdivided into $\Sigma^{\pm}$ by a closed curve of Σ^0; there may be additional components corresponding to ripple.
  2. ΣP: The main components are **poloidal** disks in a toroidal sequence, alternating between $\Sigma^{\pm}$.
  3. ΣH: The main component is a **helical** annulus making N poloidal turns for 1 toroidal turn, subdivided into $\Sigma^{\pm}$ by a closed curve of Σ^0. Could allow $M \neq 1$ toroidal turns with self-intersection along Σ^0, but not a generic case.
- Omnigenous fields are a special case of weak isodrastic.

Realisability of weak isodrasticity

- We can make non-axisymmetric isodrastic mirror fields, including ones that are not omnigenous, e.g.
- Choose $\mathcal{B}(s,u,v) = c + r^2 - a(u,v)s^2 + b(u,v)s^3$, where $r^2 = u^2 + v^2$, $a, b > 0$, on an open neigbourhood of $r \leq r_0$, $0 \leq s \leq \frac{b}{a}$, and assume $c + r^2 > \frac{4a^3}{27b^2}$ so that $\mathcal{B} > 0$.
- Field $\mathcal{B}\partial_s$ has $\Sigma^- = \{s = 0\}$, $\mathcal{H}(u,v) = c + r^2$, $\mathcal{J}(u,v) = \int_0^{a/b} \sqrt{2(as^2 - bs^3)}\, ds = \frac{4\sqrt{2}a^{5/2}}{15b^2}$. So it is isodrastic if $a^{5/2}b^{-2}$ is a function of r only.
- The field has a flux function r^2, but local minimum of $\mathcal{B}$ at $s_m = \frac{2a}{3b}$ has $\mathcal{B}(s_m) - \mathcal{H} = -\frac{4a^3}{27b^2}$, so if we choose this not to be a function of only r then the field is not omnigenous.
- To realise in Euclidean space (x,y,z), want a diffeomorphism $\phi : (s,u,v) \mapsto (x,y,z)$ such that $|\phi_*\partial_s|^2 = 1$ and $\operatorname{div} B = 0$ for $B = \phi_*(\mathcal{B}\partial_s)$. Or let $\beta = \phi_*(du \wedge dv)$ (closed), ask for $\phi^*|\beta| = \mathcal{B}$ & $|\phi_*\partial_s|^2 = 1$ and define B by $i_B(dx \wedge dy \wedge dz) = \beta$. The conditions for ϕ are 2 PDEs in first derivatives. If $\mathcal{B}$ is analytic then $\exists$ local solution by Cauchy-Kovalevskaya theorem. Hence a solution on the required domain if c is large enough.

Illustration

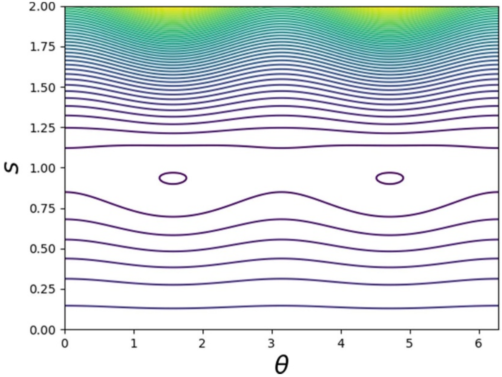


Figure: $\mathcal{B}$-contours on a flux surface $r =$ constant in polar angle θ and arclength s, for an isodrastic field that is not omnigenous.

Isodrastic stellarator fields?

- Want to make weak isodrastic stellarator fields that are not omnigenous. In particular, try to make with all marginal segments heteroclinic. Then $\mathcal{H}$ extends to a flux function. If can specify $|B|$ as function of arclength s from Σ^- then can make $\mathcal{J}$ a function of only $\mathcal{H}$, and $\mathcal{B}_{\mathsf{min}}$ not. But a priori don't know the length from one piece of Σ^- to the next.
- Using $\mathcal{H}$ as flux function, have local coordinate $\int i_B \mathcal{A}$ on flux surfaces that is preserved by B, so can make a coordinate system in which B-lines are straight.
- But what to specify in these coordinates and how to realise it as image of a divergence-free field in Euclidean space?

Strong isodrastic fields

- There is an exact version of isodrasticity for FGCM: Σ^- is a submanifold consisting of saddles for ZGCM, so persists to a normally hyperbolic submanifold (NHS) for FGCM.
- Say field is (strong) *isodrastic* if forwards & backwards contracting submanifolds of NHS coincide in relevant directions.

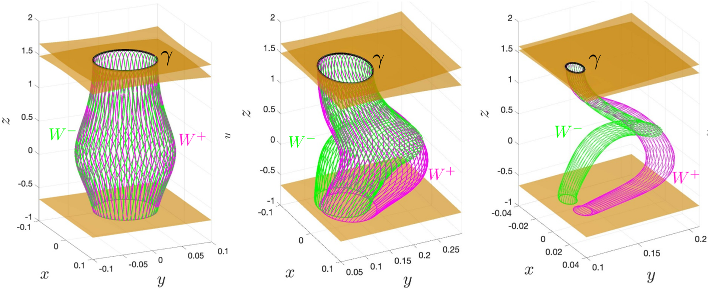


Figure: Projection to physical space of a hyperbolic periodic orbit γ of FGCM with one direction of contracting manifolds $W^{\pm}$ up to first bounce, for (a) an axisymmetric field, (b) a non-axisymmetric one, (c) E/μ close to minimum of $|B|$ on Σ^-. Brown is $|B| = E/\mu$.

Normal hyperbolicity theory

- Informally, an invariant submanifold N of a C^r $(r \geq 1)$ vector field V on manifold M is *normally hyperbolic* if any forwards "normal" contraction onto N is at faster rate than any forwards contraction tangent to N, and same for backwards.
- e.g. $\dot{x} = -y, \dot{y} = x$, $\dot{u} = u, \dot{s} = -s$ in $\mathbb{R}^4$, $N = \{u, s = 0\}$.
- Contracting submanifolds: For $\sigma = \pm$, let $W^\sigma(N) =$ set of points in M whose trajectory in direction σ of time converges to N. They are (injectively immersed) submanifolds of M, containing N. Furthermore, they are the unions of submanifolds $W^\pm(x)$ (Arnol'd's contracting whiskers) of points whose trajectories converge together with that of a point $x \in N$.
- Persistence theorem: For C^r-small change to V, an NHS persists and is C^s for any $s \leq r$ and $<$ ratios of normal to tangential contraction rates in forwards and backwards time (technical conditions if not compact or only locally invariant).
- So an approximately invariant submanifold with suitable contraction estimates implies a true NHS nearby.

Scaled FGCM

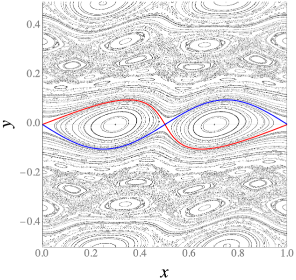
- Recall scaling to make $\dot{Q} = \tilde{B}_{\parallel}^{-1}(u_{\parallel}\tilde{B} + \delta\, b \times \nabla|B|)$, $\dot{u}_{\parallel} = -\frac{\tilde{B}}{\tilde{B}_{\parallel}} \cdot \nabla|B|$, with $\tilde{B} = B + \delta\, u_{\parallel}\mathrm{curl}\, b$, $\tilde{B}_{\parallel} = \tilde{B} \cdot b$, Hamiltonian with $H = \frac{1}{2}u_{\parallel}^2 + |B|, \omega = \frac{\beta}{\delta} + d(u_{\parallel}b^{\flat})$.
- Then for $\delta = 0$, $N = \Sigma^- \times \{u_{\parallel} = 0\}$ is invariant. It consists entirely of equilibrium points. The linearised normal dynamics is hyperbolic: $\dot{s} = u_{\parallel}, \dot{u}_{\parallel} = -|B|''s$. So it is an NHS.
- So it persists to an NHS for all small enough δ.
- Dynamics on the NHS is Hamiltonian because $\omega_{|N}$ is non-degenerate. 1 DoF & to leading order $H = |B|$. So to leading order dynamics looks like vector field X given by $i_X\beta = \delta\, d|B|$ on Σ^-, i.e. periodic orbits along the closed level sets of $|B|_{|\Sigma^-}$.
- Can compute it to any desired accuracy as a "symplectic slow manifold". RS MacKay, Slow manifolds, in T Dauxois, A Litvak-Hinenzon, RS MacKay, A Spanoudaki (eds), Energy localisation and transfer (World Sci, 2004), 149–192.

Contracting whiskers

- For $\delta = 0$, the contracting whiskers are the marginal trajectories of ZGCM approaching $x \in \Sigma^-$ in time-direction σ.
- Those that bounce approach the same point of Σ^- in both directions of time, so their union over $x \in \Sigma^-$ forms a *separatrix*: a closed invariant submanifold (but not smooth at N) of codimension one. It separates two classes of ZGCM.
- For $\delta > 0$, the local whiskers move smoothly, but in general the separatrices are broken.
- Say B is *isodrastic* if the unions of the whiskers continue to form separatrices for all $\delta > 0$ (perhaps too much to ask?).

Examples of separatrices without integrability

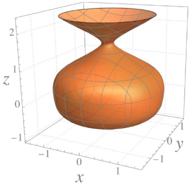
- de la Llave map: $y' = y + h(x), x' = x + y'$ with $h(x) = g(x) + g^{-1}(x) - 2x$ for invertible degree-1 circle map g.

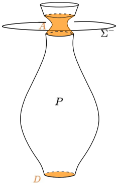


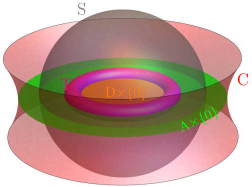
- Given $S : \mathbb{R}^2 \to \mathbb{R}$ with $S(x+m, y+n) = S(x,y) + mA + nB$ then $H(q,p) = \frac{1}{2}|p|^2 - \frac{1}{2}|\nabla S(q)|^2$ on $T^*\mathbb{T}^2$ has invariant graph $p = \nabla S(q)$ in $H^{-1}(0)$.

Visualisation of energy level

- $W^{\pm}(N) \cap H^{-1}(E)$ is typically one or more periodic orbits. $H^{-1}(E)$ is a double cover of $\{|B| \leq E/\mu\}$ glued along the boundary ($p_{\|}^2 = 2m(E - \mu|B|)$). Projection to physical space identifies $\pm p_{\|}$.
- Better to choose coordinates (X, Y, W) so that $|B| = E/\mu$ is flattened to $W = 0$ and $\pm p_{\|}$ correspond to $\pm W$.
- e.g. if accessible region is the inside of an amphora, centre (x, y) on the lowest point and find functions s, t such that writing $(x, y) = t(z)(X, Y)$ and $p_{\|} = s(z)W$ then $\partial_z \left(\frac{s(z)^2}{2m} W^2 + \mu|B|(t(z)X, t(z)Y, z) \right) \neq 0$ on $H^{-1}(E)$. Then $H^{-1}(E)$ is a graph $z = Z(X, Y, W)$, so eliminate z & plot (X, Y, W).

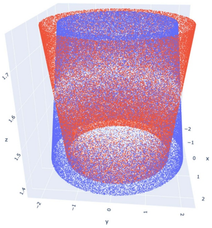




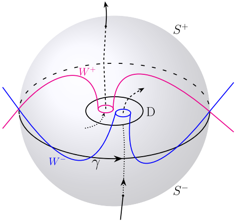


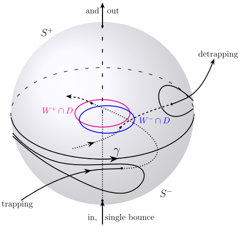
Dynamics

- In particular, get a hyperbolic periodic orbit near neck, with contracting submanifolds (plotted using Wazewski principle).



And sketches of how they might continue:

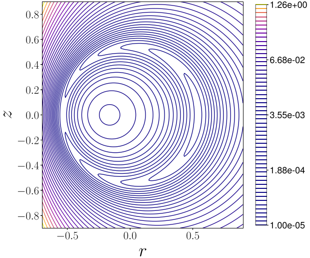


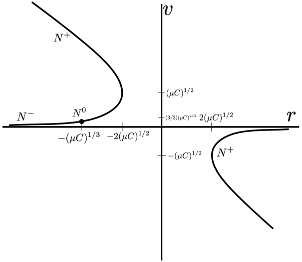


- Isodrastic is the case where the contracting submanifolds join.

Perturbed tokamak example

- Take scaling $H = \frac{1}{2}v^2 + \mu|B|$, $\omega = \beta + d(vb^\flat)$.
- Axisymmetric case conserves $p_\phi = \psi + \frac{vC}{|B|}$, where $\psi = \frac{1}{2}(r^2 + z^2)$ with $r = R - 1$.
- Contours of reduced Hamiltonian H_{p_ϕ} for some value of p_ϕ, and set of critical points of H_{p_ϕ} in (r, v) $(z = 0)$.

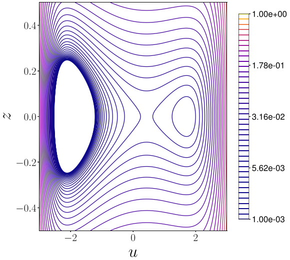


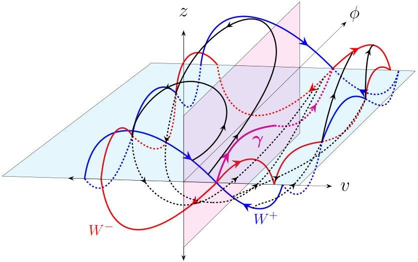


- $\Sigma^- \times \{v = 0\}$ perturbs to NHS N^-. Get a normally elliptic submanifold N^+ too from Σ^+ and a transverse submanifold of circular periodic orbits in $z = 0$ whose vertical components of parallel velocity and curvature drift cancel. [CHECK]

continued

- On breaking axisymmetry, p_ϕ no longer conserved, so best to change view to level sets of p_ϕ, given E. Axisymmetric case:





- So expect righthand picture on perturbation.
- MacKay RS, On guiding centre motion, in: Transport, chaos and plasma physics, eds Benkadda S, Doveil F, Elskens Y (World Sci, 1994) 96–101.

Splitting of separatrices

- It is convenient to examine the splitting of the separatrices by following the contracting submanifolds to the first bounce.
- Let $W^{\pm}(x)$ be the contracting whiskers of $x \in N^-$ and $\Xi^{\pm}(x)$ be fieldline labels of their first points with $v = 0$. Can use intersection with Σ^- as fieldline label. SKETCH
- Isodrastic requires $\Xi^+(x) = \Xi^-(\tau(x))$ for some $\tau(x) \in N^-$ in the same orbit as x.
- Generically, x is on a periodic orbit $\gamma \subset N^-$ and $\Xi^{\pm}(x)$ trace out closed curves $\gamma^{\pm}$ in Σ^-.
- $\gamma^{\pm}$ enclose the same magnetic flux, equal to $-\frac{1}{e}\int_\gamma \omega$ (use flux of energy-surface volume $= \omega$).
- Isodrastic requires $\gamma^{\pm}$ to coincide. Equivalent to $\exists\, \tau$ such that $\Xi^+(x) = \Xi^-(\phi_\tau(x))$ for $x \in \gamma$.
- Otherwise, they may intersect, forming lobes of transitioning flux; or miss each other, forming disks of transitioning flux.
- The flux for a lobe equals the difference in actions $\int eA^\flat + p_{\|} b^\flat$ between homoclinic orbits from the intersections.

Case of equilibria

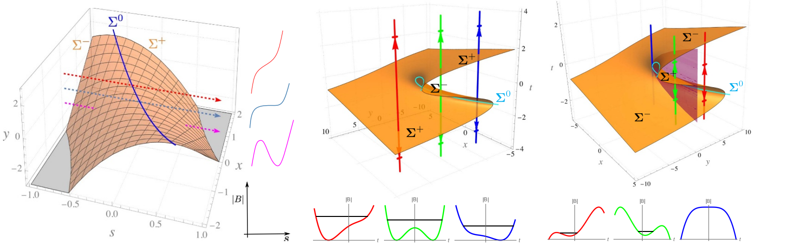
- If $x \in N^-$ is an equilibrium point then in general $\Xi^\pm(x)$ miss each other. It is codimension-2 to coincide.
- If they miss then so do $\gamma^\pm$ for all small enough periodic orbits γ around it.

Melnikov analysis

- Can compute the curves $\gamma^{\pm}$ to first order in δ: given a field-line label ξ (e.g. $\mathcal{H}$ extended along field from Σ^-) and a point $x \in \Sigma^-$, let $n(x)$ be the point of N^- on the same fieldline, & $a(s)$ be the covector at arclength s along ZGCM from x with $a(0) = d\xi$ on $T\Sigma^-$, $a(0)b = 0$ and $\frac{da_i}{ds} = -a_j \partial_i b^j$, $\frac{dx}{ds} = b$.
- Then the first order change $\Delta\xi$ along $W^{\pm}(n(x))$ to the first bounce is given by $\pm\delta\,\mathcal{M}_\xi$, where Poincaré-Melnikov function $\mathcal{M}_\xi = \int \frac{u}{|B|} ac_\perp + \frac{1}{u}(a\eta - K)\, ds$ along the ZGCM from x, with $\eta = \frac{b}{|B|} \times \nabla|B|$, $K = a\eta$ at x and $u = \sqrt{2(\mathcal{H} - |B|)}$.
- So the first-order difference between ξ on $\Xi^{\pm}$ is $2\delta\,\mathcal{M}_\xi$.
- Tidier to compute a and $\mathcal{M}$ wrt fieldline time instead of s.
- For $\xi = \mathcal{H}$ it simplifies to $\mathcal{M}_{\mathcal{H}} = \int ak \frac{ds}{|B|}$ where $k = \mathrm{curl}(ub)$.
- Can show $d\mathcal{H} \wedge d\mathcal{J} = \mathcal{M}_{\mathcal{H}}\beta$, so deduce that weak isodrasticity is the first-order condition for strong isodrasticity.
- For equilibrium points on N^-, choose two independent field-line labels to compute first-order displacement between $\Xi^{\pm}$.

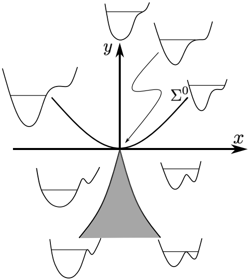
Behaviour near generic Σ^0

- For a C^4-generic point of Σ^0 there are fieldline labels x, y nearby for which $|B| = f(x, y) + ys + xs^2 + ks^3 + as^4 + o(s^4)$ for $(k, a) \neq (0, 0)$ and some function f.
- Then $|B|' = y + 2xs + 3ks^2 + 4as^3 + o(s^3)$, so Σ is locally $y = -2xs - 3ks^2 - 4as^3 + o(s^3)$.
- $|B|'' = 2x + 6ks + 12as^2 + o(s^2)$, so Σ^0 is locally $x = -3ks - 6as^2 + o(s^2)$, $y = 3ks^2 + 8as^3 + o(s^3)$.
- $k \neq 0$; $k = 0, a > 0$; $k = 0, a < 0$:



Melnikov near Σ^0

- Define c by $\beta = c(x,y)dx \wedge dy$. Take $k \neq 0$. Use x, s as coordinates on Σ^- $(x + 3ks < 0)$. For short marginal bouncers to Σ^-, $\mathcal{M} = -\frac{\sqrt{-x-3ks}}{3ck^2}(2xs + 3ks^2 + 2xf_{,y} + 3kf_{,x})$, evaluated at $(x, -2xs - 3ks^2)$, to leading order in s.
- So weak isodrastic requires $kf_{,x} = 0$, which is equivalent to $\mathcal{H}$ constant along Σ^0.
- For long marginal bouncers, generic F_j for $k \neq 0$ is



Short bouncers around general Σ^+

- Bouncing GCs with short segment oscillate around Σ^+.
- The limit $j \to 0$ of the reduced motion X is given by $i_X \beta = -dH_0$ with $H_0 = |B|$ on Σ^+.
- So bounded components of level sets of $|B|$ on Σ^+ can be used for approximate confinement.
- There is a true periodic orbit of FGCM near to such a level set.
- The short bouncers nearby have linear approximation $H_j = |B| + \frac{j}{\pi}\sqrt{|B|''}$ on Σ^+.
- Generically for all small enough $j > 0$, there is a set of invariant tori for FGCM.
- If level set of $|B|$ on Σ crosses from Σ^+ to Σ^- then short bouncers lose stability and become long bouncers to one side or the other or both.