Mathematics for Fusion Power part 7

R.S.MacKay

February 26, 2024

KAM theory

Converse KAM theory

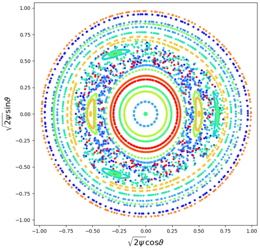
Divertors

KAM tori

- Bouncing GCs have approximate invariant $L = \int_\gamma p_\| b^\flat$. If its level sets have bounded components, can use them to approximately confine.
- Because the FGCM is Hamiltonian, if B is C^3 and some non-degeneracy, then KAM theory provides some true invariant tori of FGCM near to some level sets of L on each energy level. They provide true confinement of GCM.
- Similarly, for circulating GCs, if B has an approximate flux function ψ with some bounded components of level sets, then KAM theory provides some true invariant tori near to some level sets of ψ.

Practicalities

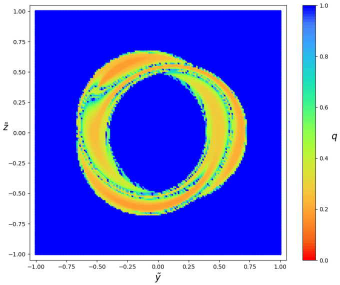
- KAM theory is technical and it is hard work to get KAM tori for reasonable perturbation sizes.
- Can look numerically and see fairly clearly whether there are invariant tori, e.g. Poincaré section for sample magnetic field.



- To be more sure, can apply tori v chaos tests:
  1. Fractal dimension estimators
  2. Lyapunov exponent estimators
  3. My version of Lyapunov estimator using trace at recurrences
  4. Gottwald & Melbourne (2004) test
  5. DESCRIBE THEM

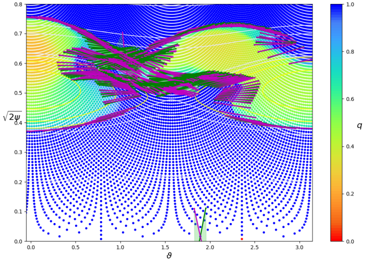
Converse KAM method

- Best is probably to use Converse KAM method: it finds the complement of the union of invariant tori of given class. It is much easier than KAM theory. Begin with use for magnetic fields.
- N Kallinikos, RS MacKay, D Martinez-del-Rio, Regions without flux surfaces of given class for toroidal magnetic fields, Plasma Phys & Contr Fusion 65 (2023) 095021; Erratum, PPCF 65 (2023) 129602
- Choose class of tori by specifying a direction field ξ to which they are transverse, e.g. gradient of an approximate flux function.
- Basic method for 3D vector field v: flow ξ with the linearised dynamics of v, if $\phi_{t*}\xi$ becomes $a\xi + bv$ with $a < 0$ then that trajectory does not lie on an invariant torus transverse to ξ.



Conefields

- Can refine to produce "conefields": upper and lower bounds on the slopes of possible invariant tori of the given class (basic method corresponds to finding where the cones are empty).



[Martinez]

- In general, the conefields can be used to eliminate more points: those from which every possible invariant torus would have to enter a non-existence region ("killends" extension).
- In principle, can apply to GCM: just need to choose an appropriate direction field on each energy level that allows both circulating and banana tori.

Other uses of Converse KAM for fusion programme

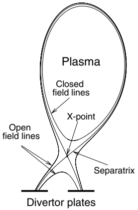
- Can apply Converse KAM also to obtain constraints on geodesic foliations on tori (as for isodynamic fields) and on pressure jumps across current sheets.
- In both of these, the relevant class of tori is just the closed 1-forms on the torus.
- Isodynamic fields: recall that they are defined to be integrable ones with $\dot{\psi} = 0$ for FGCM, and that this implies $b \times \kappa$ is perpendicular to the flux surfaces, i.e. b is a geodesic field on each flux surface, using the surface metric. Converse KAM can rule out certain ι on a given flux surface (all if the surface has a 'big bump'). Maybe all rational values in many cases?

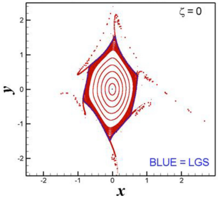
Pressure-jump Hamiltonian

- For a toroidal interface Σ between force-free regions ($J \times B = 0$), assumed tangent to the limiting $B^{\pm}$ on each side, a current sheet on Σ supports a pressure jump $P = p^- - p^+$ if the jump in $\frac{1}{2}|B|^2$ is P.
- Given one limiting field B^-, then $B^+ = \nabla f$ for a multivalued potential f on Σ s.t. $\frac{1}{2}|\nabla f|^2 = \frac{1}{2}|B^-|^2 - P$, using surface metric. It is Hamilton-Jacobi equation for an invariant torus $p = \nabla f(q)$ of $H(q,p) = \frac{1}{2}|p|^2 - V(q)$ with energy P, where $V = \frac{1}{2}|B^-|^2$.
- No solutions if $P < -V_{\min}$, so take $P + V \geq 0$. Then reformulate as seeking a geodesic field of Maupertuis metric $(P + V(q))g$ on Σ.
- Converse KAM can constrain Σ for solutions [Kaiser & Salat, Phys Plasma 1 (1994) 281] (also P and ι^+).
- Can analyse P near $-V_{\min}$ by perturbation of the singular case and I think get no solutions for some interval of $P \geq -V_{\min}$.
- V negligible for $P > 0$ large, so outcome depends only on Σ.
- Could extend use of Converse KAM numerically and thereby justify observations by Qu et al, PPCF 63 (2021) 125007?

Divertors

- Want a clear gap between plasma and vessel wall, a transition from confinement to not.
- So design the magnetic field to have an outermost flux surface S in the interior of the vessel and put the plasma inside S.
- e.g. for axisymmetric field, make a hyperbolic periodic fieldline, then its separatrix can bound the plasma. Not robust to breaking axisymmetry, but find a last invariant torus slightly inside, e.g. Abdullaev et al, Nucl Fusion 46 (2006) S113-26





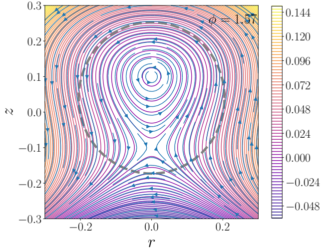
- or usual mix of tori & chaos makes a last torus, e.g. Boozer&Punjabi, Phys Plasma 25 (2018) 092505
- Can find it by 1-sided residue criterion: JM Greene, RS MacKay, J Stark, Boundary circles for area-preserving maps, Physica D 21 (1986) 267–95

Strike points

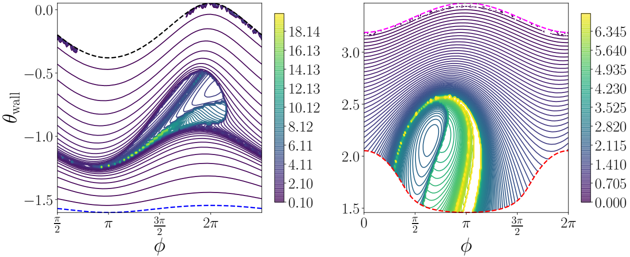
- If GCs diffuse across their last invariant tori (roughly corresponding to S), where do they hit the vessel wall? Simplify by looking at fieldline flow (in both directions).
- Answer can be obtained by working backwards from the wall. SKETCH
- Decompose the wall W (assumed smooth) into $W^+ \cup W^0 \cup W^-$, according to sign of $B \cdot n$ where n is the unit outward normal to W. Modulo W^0, forward fieldlines hit W^+, backward ones hit W^-.
- Decompose $W^0 = W^0_i \cup W^0_o \cup W^0_d$ according as fieldline makes non-degenerate inner or outer tangency or a degenerate one. Modulo W^0_d, forward fieldlines hit $W^+ \cup W^0_i$, backward ones hit $W^- \cup W^0_i$.
- Let $W^\pm(\tau)$ be the sets of strike points reachable by $\pm$ fieldline flow from the interior of W in time at most τ; increasing sequences.
- Can do same for length instead of time; the resulting sets are those with "connection length" at most the given value.
- Endow W with measure $|B \cdot dS|$ and let $m(\tau) = \mu(W^\pm(\tau)$ (equal). Then $\operatorname{div} B = 0$ implies $m(\tau) \le V_{\mathrm{acc}}/\tau$, where V_{acc} is the accessible volume along the fieldlines from the wall; and $V_{\mathrm{acc}} = \int \tau dm(\tau)$.
- If use length instead of time, use $di_b(|B|\Omega) = 0$ and replace V_{acc} by $\int_{V_{\mathrm{acc}}} |B|\Omega$.

Perturbed tokamak example

- Poloidal section



- Contours of connection length on the two parts of W^+ [Naik]



- Much more to do here!